A SIMPLE MODEL FOR PRICING SECURITIES WITH
EQUITY, INTEREST RATE, AND DEFAULT RISK

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Abstract

We develop a model for pricing derivative and hybrid securities whose value may depend on different sources of risk, namely, equity, interest-rate, and default risks. In addition to valuing such securities the framework is also useful for extracting probabilities of default (PD) functions from market data. Our model is not based on the stochastic process for the value of the firm [which is unobservable], but on the stochastic process for interest rates and the equity price, which are observable. The model comprises a risk-neutral setting in which the joint process of interest rates and equity are modeled together with the default conditions for security payoffs. The model is embedded on a recombining lattice which makes implementation of the pricing scheme feasible with polynomial complexity. We present a simple approach to calibration of the model to market observable data. The framework is shown to nest many familiar models as special cases. The model is extensible to handling correlated default risk and may be used to value distressed convertible bonds, debt-equity swaps, and credit portfolio products such as CDOs. We present several numerical and calibration examples to demonstrate the applicability and implementation of our approach.

1   Introduction

A vast class of financial securities depends on more than just one source of risk. Prominent among these are corporate bonds (which depend on interest rate risk and on credit risk of the issuing firm) and convertible bonds (which depend, in addition, on equity risk). In this paper, we offer and implement a simple, tractable model for the pricing of securities whose values may depend on one or more of three sources of risk: equity risk, credit risk, and interest-rate risk.

Our framework stitches together three standard building blocks. We begin with a Heath-Jarrow-Morton (1992) (henceforth, HJM) term-structure model to describe interest-rate movements; of course, any other standard interest-rate model could also be used in its stead. Then, as in the reduced-form approach to credit-risk modelling (e.g., Duffie and Singleton, 1999), we append a hazard-rate process to this model to capture the likelihood of default. This gives us a "defaultable HJM" model along the lines of Schönbucher (1998) and Das and Sundaram (2000). Finally, we overlay on this structure the Cox-Ross-Rubinstein (1979) (henceforth, CRR) process for the evolution of equity prices generalized to allow for default of the firm issuing the equity.

The resulting framework combines, in a single parsimonious model and accounting for correlations, the three major sources of risk. Default information in the model is extracted from *both* equity- and debt-market information rather than from just debt-market information (as in the reduced-form credit-risk models) or from just equity-market information (as in structural credit-risk models). In particular, default probabilities may be jointly calibrated to market prices of equity and risky debt. This allows valuation, in a single consistent framework, of hybrid debt-equity securities such as convertible bonds that are vulnerable to default, as well as of derivatives on interest rates, equity and credit. Our model can also serve as a basis for valuing credit portfolios where correlated default risk is an important source of risk.

We develop this theoretical framework and identify conditions under which the model may be implemented on a recombining lattice. We then provide illustrative implementations of the model. Numerical examples show that the model is able to produce credit spread term structures of any level, slope and curvature, capturing equity and interest rate effects. We show how default risk impacts the prices of callable-convertible bonds, which contain a conversion option subject to equity risk, and a callable feature dependent on interest rates. As the level of default risk increases, both these options decline in value and the change in the value of the callable-convertible bond depends on which option declines faster. If for instance, the callable feature declines faster than the conversion option, the convertible bond may in fact increase in value. It is shown that eventually default risk extinguishes both options, leaving the convertible bond equivalent to a defaultable plain bond.

Our framework uses the prices of traded derivative securities to extract default probability functions, which may then be used to price various other contracts subject to default risk. We show how the term structure of credit default swap (CDS) spreads may be used to calibrate these implicit default functions and demonstrate that the framework is flexible enough to generate CDS spread curves of various shapes. We present examples by fitting the spread curves for IBM (an industrial firm) and AMBAC (a financial firm) to different dates. We also fit the model to the spread curves on a daily basis for a two and a

half year period (Jan 2000 to July 2002) for IBM and UAL. This exercise enables us to examine the stability of the parameters of the default function from day to day, and we find that the functions are fairly stable, thereby suggesting that they may be used for forecasting. Using the fitted functions we compute one-year default probabilities, and find that they are strongly correlated to market variables in expected ways: negatively to stock prices, and positively to stock volatility. The times series of extracted risk-neutral probabilities of default from our model tracks the default probabilities provided by CreditMetrics for the same period with a correlation in excess of 80%.

Relationship to the Literature As the description above notes, our framework may be regarded as a generalization of the models in Schönbucher (1998) and Das and Sundaram (2000): each of these models admits interest-rate and credit-risk, but not equity risk. Equally, our framework may also be viewed as a generalization of Amin and Bodurtha (1995). The Amin-Bodurtha model combines interest rate risk and equity risk but does not incorporate credit risk. Since there is no default, equity in their model is necessarily infinitely-lived and never gets "absorbed" in a post-default value.

Several other frameworks too are nested within our model. For example, if the equity and hazard-rate processes are switched off, we obtain the HJM model, while if the interest-rate and hazard-rate processes are switched off, we obtain the CRR model.

Our framework is intimately linked to the two standard approaches to credit-risk modeling: the class of reduced-form models (e.g., Duffie and Singleton (1999), Madan and Unal (2000)) and the class of structural models (Merton (1974), Black-Cox (1976), and others). As in the reduced-form approach, we represent default likelihood using a hazard-rate process. From the structural model approach, we borrow the boundary conditions on equity in the event of default, specifically, the idea that default is identified with zero equity value.

There are, however, important differences. The typical reduced-form model only considers interest-rate and default processes; thus, implementation is achieved solely using debt-market information. Our framework also incorporates the equity process, so default information may be extracted from both equity- and debt-market information. This is significant, given that equity markets are more liquid than corporate debt markets.[1]

Structural models begin with a process for the value of the firm; equity and debt are viewed as contingent claims on this value. Since the firm value process is latent, its parameters must be inferred from observed variables. Equity-market information is typically used for this purpose, but implementation in practice uses involves making restrictive simplifying assumptions on the debt-structure of the firm (e.g., the Moody's KMV model or the Credit Grades model). These restrictions are unnecessary in our model since we work directly with observable equity prices rather than firm values.

Most closely related to our framework are the recent papers on "hybrid" models that also focus on jointly modeling the three sources of risk on which we focus. An early paper on these lines is Davis and Lischka (1999). Their set-up, like ours, employs a bivariate tree in interest rates and stock prices and

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[1]Incorporation of equity risk into reduced form models has also been examined in Jarrow (2001) and Mamaysky (2002). The approach in these papers is very different from ours; it is based on deriving equity values through modeling the dividend process.

a process for default intensitys. We examine the various consistent default intensity specifications that may be used to implement this model. There are some important operational differences between the Davis-Lischka framework and ours. One is the choice of interest-rate process: Davis-Lischka use the Hull-Whilte model while we use the more flexible HJM model. Second, the specification of the default intensity process in Davis-Lischka is restrictive; the process is perfectly correlated with the equity process. We allow the default intensity process to depend more generally on both equity returns and interest-rates as also other information. Finally, we also provide an algorithm for handling correlated default risk as well.

The Davis-Lischka idea has also been developed in other papers in the literature including Jarrow (2001), Takahashi, Kobayashi and Nakagawa (2001), and Carayannopoulos and Kalimipalli (2001). Carayannopoulos and Kalimipalli, for instance, use it to price convertible bonds, but in a model which does not account fully for interest rate risk.

Our lattice design allows recombination, which is an essential feature in making the implementation of the model highly efficient. Our technique is a modification of the approach developed in Amin and Bodurtha (1995), with the additional feature that it also accounts for default risk. A finite-differencing approach using the Fokker-Planck equations is presented in Andersen and Buffum (2002). Their paper points out the various niceties in calibrating such models for the pricing of convertible bonds. In this paper also we develop a calibration approach that depends entirely on observables, with the additional feature that we obtain implicit *endogenous* default intensity functions (not just default intensitys) from the prices of traded credit default swaps. Moreover, our approach uses a tree approach which is simpler to implement than the finite-difference model.

2 Modeling the Lattice

Representing the stochastic processes on a lattice permits valuation by dynamic programming via backward recursion. We employ a parsimonious model that can be embedded on a bivariate lattice, on which we model the joint risk-neutral evolution of equity prices and the forward interest rate curve. Our model accommodates the correlation between interest rates and equity prices, and resembles the work of Amin and Bodurtha [1995]. The method by which we represent the joint distribution differs from that paper, and we are also able to show that our approximation converges to an exact bivariate process as the time interval, represented by h, shrinks to zero on the lattice. More importantly, we show how to embed credit risk in the model.

2.1 A simple motivating example

In this subsection, we present a simplified version of our model, based purely on equity values, assuming that interest rates are not stochastic. Consider the case of equity option valuation in the presence of unanticipated default. In this case, defaultable equity prices follow a stochastic process which embeds a sudden jump to default. Hence, the prices of call options may be determined using Merton's [1976] jump-diffusion option model. The prices of calls are analogous to a model where the firm may default,

with a corresponding zero recovery rate for equity. Samuelson [1972] had provided the solution to this problem, which is as follows:

$$
\begin{aligned}
\text{Call on defaultable equity} \ &= \ \exp(-\xi T) \ BS[S_0 e^{\xi T}, K, T, \sigma, r] \\
&= \ BS[S_0, K, T, \sigma, r + \xi]
\end{aligned}
$$

where $BS[.]$ is the standard Black-Merton-Scholes (BSM) option pricing model, with an initial stock price S_0, interest rate r, stock volatility σ, maturity T, and exercise price K. ξ is the default intensity, or the instantaneous rate of default. Notice that the price of the call is exactly priced by the BSM model with an adjusted risk-neutral interest rate $(r + \xi)$.

Since the defaultable call option, like the value of equity, has a zero recovery rate, it is tempting to intuit that the price of a defaultable call option should be the price of a non-defaultable call option, i.e. $BS[S_0, K, T, \sigma, r]$, multiplied by the risk-neutral probability of survival, i.e. $\exp(-\xi T)$. However, this simple intuition would be wrong (notice this from a simple comparison with Samuelson's formula above). Not only does the discount rate need to be adjusted for the probability of default, but the drift of the risk-neutral equity process is also impacted by the jump to default compensator.[2] Therefore, special care should be taken to ensure that the correct risk-neutral processes are used for pricing defaultable securities.

This intuition may be further clarified in a discrete-time setting. Defaultable equity may be represented by the following tree, which embodies a single period of length h, wherein the stock price moves from $S(t)$ to a stochastic value $S(t + h)$. When jump to default is allowed for, the value of $S(t + h)$ is assumed to take one of three values.

$$
s(t+h) = \begin{cases} uS(t) & \text{w/prob} & q \ \exp(-\xi h) \\ dS(t) & \text{w/prob} & (1-q) \ \exp(-\xi h) \\ 0 & \text{w/prob} & 1 - \exp(-\xi h) \end{cases}
$$

Here u, d are the respective "upshift" and "downshift" parameters for the changes in the stock price over interval h. Given the constant risk-neutral default intensity ξ, the probability of survival in the interval h is $\exp(-\xi h)$. Since this risk-neutral setting requires that the normalized stock price is a martingale, it is easy to solve for the value of the risk-neutral probability q. Hence,

$$
\exp(rh) = uq \exp(-\xi h) + d(1-q)\exp(-\xi h),
$$

implying that

$$
q = \frac{\exp(rh) - d\exp(-\xi h)}{u\exp(-\xi h) - d\exp(-\xi h)} = \frac{\exp[(r+\xi)h] - d}{u - d}
$$

For illustrative purposes, we set $u = \exp(\sigma h)$ and $d = 1/u$, to mimic the Cox, Ross and Rubinstein (CRR) model. Suppose $r = 0.10$, $\sigma = 0.20$, $\xi = 0.01$, and $h = 0.25$. Then, the risk-neutral probability $q = 0.766203$. If there were no defaults, i.e. $\xi = 0$, then $q = 0.740548$. Hence, notice that the drift upwards tends to occur with greater probability in the presence of default, corresponding to the fact that in the risk-neutral setting, the jump to default is compensated.

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[2]For an excellent exposition of default jump compensators, see Giesecke [2001].

In the ensuing sections, we generalize this model to apply to the case with stochastic interest rates, and stochastic default processes, so that ξ is no longer constant, nor uncorrelated with equity prices and interest rates.

2.2 Equity Model

The model for the evolution of equity prices is based on the branching process outlined in the previous section, with σ_s as the parameter governing the volatility of the equity process, and $X_s(t)$ is a random variable, taking values in the set $\{+1, -1, -\infty\}$. Under this specification, the trinomial movement in the stock price at time t over the next period is set to be $S(t+h) = S(t) \exp[\sigma_s X_s(t)]$.

A probability measure is chosen such that the expected return on equity in each period, is set to $r(t)h$ and the variance of the return is $\sigma_s^2 h$. Allowing $X_s(t) = -\infty$ embeds default risk in the model, of the sort envisaged in the Duffie-Singleton [1999] model. The firm suddenly defaults in which case its stock price goes to zero, when $X_s(t) \to -\infty$. Setting the expected return to $r(t)h$ is equivalent to normalizing the equity prices by a money market account numeraire, and ensuring that the normalized prices are martingales. Since the same numeraire is also used in the case of bonds, we generate a lattice that is arbitrage-free in bond and equity markets. The choice of probability measure to satisfy the martingale requirements in the model will be taken up shortly in a subsequent section.

2.3 Term-Structure Model

Our lattice adopts the discrete-time, recombining form of the Heath-Jarrow-Morton (HJM) [1990] model, which it defaults to if there is no equity component in the derivative security being priced. We quickly review this here, before moving on to the description of the joint lattice, and readers may examine the original HJM paper for comprehensive details. Initially, we prepare the univariate HJM lattice for the evolution of the term structure, and subsequently stitch on an equity process.

The model is based on a time interval $[0, T^*]$. Periods are of fixed length $h > 0$; thus, a typical time-point t has the form kh for some integer k. At all times t, zero-coupon bonds of all maturities are available. Assuming no arbitrage, there exists an equivalent martingale measure Q for all assets. For any given pair of time-points (t, T) with $0 \leq t \leq T \leq T^* - h$, $f(t, T)$ denotes the forward rate on the default-free bonds applicable to the period $(T, T+h)$. The short rate is $f(t, t) = r(t)$. Forward rates follow the stochastic process:

$$f(t+h, T) = f(t, T) + \alpha(t, T)h + \sigma(t, T)X_f\sqrt{h}, \tag{1}$$

where α is the drift of the process and σ the volatility; and X_f is a random variable. Both α and σ are taken to be only functions of time, and not other state variables. This is done to preserve the computational tractability of the model. Relaxing this assumption will make the model non-recombining, though technically feasible nevertheless.

We denote by $P(t, T)$ the time–t price of a default-free zero-coupon bond of maturity $T \geq t$. As

usual,

$$P(t,T) = \exp\left\{-\sum_{k=t/h}^{T/h-1} f(t,kh) \cdot h\right\} \tag{2}$$

The well-known recursive representation of the drift term α of the forward-rate and spread processes, is required to complete the risk-neutral lattice. Let $B(t)$ be the time–t value of a "money-market account" that uses an initial investment of \$1, and rolls the proceeds over at the default-free short rate:

$$B(t) = \exp\left\{\sum_{k=0}^{t/h-1} r(kh) \cdot h\right\}. \tag{3}$$

The equivalent martingale measure Q is defined with respect to $B(t)$ as numeraire; thus, under Q all asset prices in the economy discounted by $B(t)$ will be martingales. Let $Z(t,T)$ denote the price of the default-free bond discounted using $B(t)$:

$$Z(t,T) = \frac{P(t,T)}{B(t)}. \tag{4}$$

which is a martingale under Q, for any $t < T$, i.e. $Z(t,T) = E^t[Z(t+h,T)]$:

$$E^t\left[\frac{Z(t+h,T)}{Z(t,T)}\right] = 1. \tag{5}$$

It follows that $Z(t+h,T)/Z(t,T) = (P(t+h,T)/P(t,T)) \cdot (B(t)/B(t+h))$. Extensive, though well-known algebra leads to a recursive expression relating the risk-neutral drifts α to the volatilities σ at each t:

$$\sum_{k=t/h+1}^{T/h-1} \alpha(t,kh) = \frac{1}{h^2}\ln\left(E^t\left[\exp\left\{-\sum_{k=t/h+1}^{T/h-1} \sigma(t,kh)X_f h^{3/2}\right\}\right]\right). \tag{6}$$

2.4 The Joint Process

We now connect the two processes for the term structure and the equity price together on a bivariate lattice. There are two goals here. First, we set up the probabilities of the joint process so as to achieve the correct correlation between equity returns and changes in the spot rate. Second, our lattice is set up so as to be recombining, allowing for polynomial computational complexity, providing for fast computation of derivative security prices.

Specification of the joint process requires a probability measure over the random shocks $[X_f(t), X_s(t)]$. This probability measure is chosen to (i) obtain the correct correlations, (ii) ensure that normalized equity prices and bond prices are martingales, and (iii) makes the lattice recombining. Our lattice model is hexanomial, i.e. from each node, there are 6 emanating branches or 6 states. The following table depicts the states:

X_f	X_s	Probability
1	1	$\frac{1}{4}(1+m_1)[1-\lambda(t)]$
1	-1	$\frac{1}{4}(1-m_1)[1-\lambda(t)]$
-1	1	$\frac{1}{4}(1+m_2)[1-\lambda(t)]$
-1	-1	$\frac{1}{4}(1-m_2)[1-\lambda(t)]$
1	$-\infty$	$\frac{\lambda(t)}{2}$
-1	$-\infty$	$\frac{\lambda(t)}{2}$

where $\lambda(t)$ is the probability of default at each node of the tree. We also associate $\lambda(t)$ with a default intensity process $\xi(t)$, such that the survival probability in time interval h is:

$$1 - \lambda(t) = \exp[-\xi(t)h]. \tag{7}$$

We are now able to solve for the correct values of m_1 and m_2 to provide a default consistent martingale measure, with the appropriate correlation between the equity and interest rate processes, ensuring too, that the lattice recombines.

In order for the lattice to be recombining, it is essential that the drift of the process for equity prices be zero. Hence, we write the modified stochastic process for equity prices as follows:

$$\ln\left[\frac{S(t+h)}{S(t)}\right] = \sigma_s X_s(t)\sqrt{h} \tag{8}$$

and then we adjust the probability measure over $X_s(t)$ such that $E[\exp(\sigma_s X_s(t)\sqrt{h})] = \exp[r(t)h]$. In addition, for the HJM model to be recombining, we require that two conditions be met by the probability measure over the random variable X_f. The mean value of this random variable must be zero, and its variance should be 1. These properties are verified as follows.

$$
\begin{aligned}
E(X_f) &= \frac{1}{4}[1+m_1+1-m_1-1-m_2-1+m_2](1-\lambda(t)) \\
&\quad + \frac{\lambda(t)}{2}[1-1] \\
&= 0 \\
Var(X_f) &= \frac{1}{4}[1+m_2+1-m_1+1+m_2+1-m_2](1-\lambda(t)) \\
&\quad + \frac{\lambda(t)}{2}[1+1] \\
&= 1
\end{aligned}
$$

This confirms that the term structure random variable X_f is mean zero with variance one.

Now, we compute the two conditions required to determine m_1 and m_2. Use the expectation of the equity process to determine one equation. We exploit the fact that under risk-neutrality the equity return must equal the risk free rate of interest. This leads to the following:

$$E\left[\frac{S(t+h)}{S(t)}\right] = E[\exp(\sigma_s X_s(t)\sqrt{h})]$$

$$
\begin{aligned}
&= \frac{1}{4}(1 - \lambda(t))[e^{\sigma_s \sqrt{h}}(1 + m_1) + e^{-\sigma_s \sqrt{h}}(1 - m_1) \\
&\quad + e^{\sigma_s \sqrt{h}}(1 + m_2) + e^{\sigma_s \sqrt{h}}(1 - m_2)] + \frac{\lambda(t)}{2}[0] \\
&= \exp(rh)
\end{aligned} \tag{9}
$$

Hence the stock return is set equal to the riskfree return. This implies the following from a simplification of equation (9):

$$
m_1 + m_2 = \frac{\frac{4e^{r(t)h}}{1 - \lambda(t)} - 2(a + b)}{a - b} = A \tag{10}
$$

$$
a = \exp(\sigma_s \sqrt{h}) \tag{11}
$$

$$
b = \exp(-\sigma_s \sqrt{h}) \tag{12}
$$

Our second condition comes from the correlation specification. Let the correlation (coincident with covariance for unit valued variables) between the shocks $[X_f(t), X_s(t)]$ be equal to ρ, where $-1 \le \rho \le 1$. A simple calculation follows:

$$
Cov[X_f(t), X_s(t)] = \frac{1}{4}(1 - \lambda(t))[1 + m_1 - 1 + m_1 - 1 - m_2 + 1 - m_2] \tag{13}
$$

$$
+ \frac{\lambda(t)}{2}[-\infty + \infty] \tag{14}
$$

$$
= \frac{m_1 - m_2}{2}(1 - \lambda(t)). \tag{15}
$$

Setting this equal to ρ, we get the equation

$$
m_1 - m_2 = \frac{2\rho}{1 - \lambda(t)} = B. \tag{16}
$$

Solving the two equations (10) and (16) leads to the following solution:

$$
m_1 = \frac{A + B}{2} \tag{17}
$$

$$
m_2 = \frac{A - B}{2} \tag{18}
$$

These values may now be substituted into the probability measure in the table above. Notice that since the interest rate $r(t)$ only enters the probabilities and not the random shock $X_s(t), \forall t$, the equity lattice will also be recombining, just as was the case with the HJM model for the term structure. Hence, the product space of the equity and interest rates will also be recombining. As interest rates change, the probability measure will also change, but this will not impact the recombining property of the lattice. It is also necessary that the solutions for m_1 and m_2 be such that the resultant probabilities do not become negative or greater than 1. From the table above, we see that the necessary condition is $-1 \le m_i \le +1, \ i = 1, 2$. As $h \to 0$, this condition is satisfied.

2.5   Analysis of the approximation of the stock variance

Finally, there remains one small detail to be considered. The variance of the equity return may not be exactly what is parameterized, on account of the discretization of the stock variance.[3] We need to check that $Var[\sigma_s X_s(t)\sqrt{h}] = \sigma_s^2 h$, conditional on no default. We show that this condition does not hold exactly, but only in approximation. However, this approximation error goes to zero as the lattice interval h goes to zero. Moreover, we show that for realistic values of $h > 0$, the approximation error is exceedingly small. To see this, we look at the following calculation:

$$Var[\sigma_s X_s(t)\sqrt{h}\|\text{no default}] = \frac{\sigma_s^2 h Var[X_s(t)]}{1 - \lambda(t)} \tag{19}$$

$$= \frac{\sigma_s^2 h[E(X_s^2) - E(X_s)^2]}{1 - \lambda(t)} \tag{20}$$

$$= \sigma_s^2 h\left[1 - (1 - \lambda(t))\left(\frac{m_1 + m_2}{2}\right)^2\right] \tag{21}$$

We focus in on the term $\left[\frac{m_1+m_2}{2}\right]^2$, which should be as small as possible, so as to minimize the bias from the approximation in the probability measure. Substituting in the appropriate variables, this term may be represented in detail as follows:

$$\left[\frac{m_1 + m_2}{2}\right]^2 = \left[\frac{\frac{2e^{rh}}{1-\lambda(t)} - (e^{\sigma_s\sqrt{h}} + e^{-\sigma_s\sqrt{h}})}{e^{\sigma_s\sqrt{h}} - e^{-\sigma_s\sqrt{h}}}\right]^2 \tag{22}$$

Notice that this term is pretty small for the usual values of r, h, σ_s. For example, suppose the interest rate is 4%, the stock volatility is 20%, and default probability is 0%, then if $h = 0.25$, then the term is 0.0025, which is a small number. In general, as $h \to 0$, the term $[1 - (1 - \lambda(t))\left(\frac{m_1+m_2}{2}\right)^2] \to 1$. Hence, the lattice recombines subject to a manageable approximation. Note that the approximation error declines in h, and also declines in r.

3   Credit Risk

Accounting for credit risk is achieved by adding the process for default probability $[\lambda(t)]$ to the lattice. Rather than add an extra dimension to the lattice model by embedding a separate $\lambda(t)$ process, we define one-period default probability functions at each node on the bivariate lattice, by making default a function of equity prices and interest rates at each node. There are two reasons for this. First, equity prices already reflect credit risk, and hence there is a connection between $\lambda(t)$ and equity prices. Second, default probabilities are empirically known to be connected to the term structure, and hence, may be modeled as such. Therefore, our approach entails modeling the default risk at each node as a function of the level of equity and the term structure at each node.

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[3]The same issue arises in the discretization in the original CRR [1979] paper as well. With the additional feature of default, we recheck the same convergence condition here.

Our approach specifies a *conditional* $\lambda(t)$ at each node, i.e. rather than add a separate default probability process, we simply make $\lambda(t)$s a function of the state variables of equity and interest rates. We refer to this as an *endogenous* default approach.[4] If in fact, default probabilities were added as a separate stochastic process (which we denote the *exogenous* approach, as in David and Lischka [1999] or Andersen and Buffum [2002]), the question of consistency conditions between $\lambda(t)$, equity and interest rates would arise, a complex situation to resolve. By positing a functional relationship of $\lambda(t)$ to the other variables, we are able to obtain a consistent lattice as well as a more parsimonious one. As noted before, $\lambda(t) = 1 - e^{-\xi(t)h}$, and we express the default intensity $\xi(t)$ as:

$$\xi[\mathbf{f}(t), S(t), t; \theta] \in [0, \infty) \tag{23}$$

i.e. a function of the term structure of forward rates $\mathbf{f}(t)$, the stock price $S(t)$ at each node, and time t. This function may be as general as possible. We impose the condition that is required of default intensities, i.e. $\xi(t) \geq 0$. θ is a parameter set that defines the function. This is not a new approach. A similar *endogenous* default intensity extraction has been implemented in Das and Sundaram [2000], Carayannopoulos and Kalimipalli [2001], and Acharya, Das and Sundaram [2002]. However, the settings in those papers were less general than in this one.

Of course, in addition to the probability of default of the issuer, a recovery rate is required. In the two states in which default occurs, this recovery rate is applied. The recovery rates may be treated as constant, or as a function of the state variables in this model. It may also be pragmatic to express recovery as a function of the default intensity, supported by the empirical analysis of Altman, Brooks, Resti and Sironi [2002].

Various possible parameterizations of the default intensity function may be used. For example, the following model (subsuming the parameterization of Carayannopoulos and Kalimipalli [2001]) prescribes the relationship of the default intensity $\xi(t)$ to the stock price $S(t)$, short rate $r(t)$, and time on the lattice $(t - t_0)$.

$$
\begin{aligned}
\xi(t) &= h(y) \exp[a_0 + a_1 r(t) - a_2 \ln S(t) + a_3(t - t_0)] \\
&= h(y) \frac{\exp[a_0 + a_1 r(t) + a_3(t - t_0)]}{S(t)^{a_2}}
\end{aligned} \tag{24}
$$

For $a_2 \geq 0$, we get that as $S(t) \to 0$, $\xi(t) \to \infty$, and as $S(t) \to \infty$, $\xi(t) \to 0$. Further, we also specify the function $h(y)$, based on a state variable y (such as the debt-equity ratio) through which other influences on the default intensity function may be imposed. This function must satisfy consistency conditions depending on its choice of state variable. For example, if y were the debt-equity ratio, then we might require that (a) $h(0) = 0$, (b) $h(\infty) = \infty$, and (c) $h'(y) > 0, \forall y$.

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[4]After many presentations of this model to Wall Street firms, practitioners have called our model the "two and a half" model to stand for the fact that equity, interest rate and default are modeled, leading to a 3-dimensional model. But since default is endogenously modeled, it is denoted as a half dimension, leading therefore, to a $2\frac{1}{2}$ dimension model.

3.1 Calibration with credit default swaps

The increasing amount of trading in default swaps now offers a source of empirical data for calibrating the model. Other models, such as CreditGrades[5], also use default swap data. Hence, the term-structure of default swaps is now available for cross-sectional fitting of our model parameters. As an illustration of the lattice computations that may be employed for pricing, we consider the simplest form of a default swap, i.e. that written on a zero-coupon bond. A recent paper by Longstaff, Mithal and Neis [2002] undertakes an empirical comparison of default swap and bond premia in a parsimonious closed-form model.

Assume that we have "pure" default swap spreads for a range of maturities, $t = 1, 2, 3...T$ years. The pure premium on a default swap is the present value of insurance payments on a defaultable zero-coupon bond. The premium is equal to the expected present value of payouts on default of the underlying zero-coupon instrument. Expectations are taken under the default-risk based martingale measure described in this paper. Given any four maturities, we can calibrate the four parameters $\{a_0, a_1, a_2, a_3\}$ in the function in equation (24) by exact fitting of four default swap premia. If more than four maturities for default swap spreads are available, the parameters may be fitted using a least squares criterion.

We denote the recovery rate on default as ϕ, which may be specified in this case as constant, without loss of generality. Applying the recovery of market value (RMV) assumption on default, the pure default swap rate is the continuous stream of payments expressed in basis points that equates the present value of these payments to the expected present value of the payoffs on the default swap. On the lattice, these values may be computed via backward recursion. We define the following quantities as recursive expressions on the pricing lattice.

First, we define the price of a defaultable zero-coupon bond. We denote the price of this bond at time t as $ZCCB(t)$. The pricing recursion under the RMV condition is as follows:

$$ZCCB(t) = e^{-r(t)h} \left\{ \sum_{k=1}^{4} p_k(t) ZCCB_k(t+h) \right\} [1 - \lambda(t)(1-\phi)], \quad ZCCB(T) = 1.0. \quad (25)$$

Here, $p_k(t), k = 1..4$ are the four probabilities for the non-default branches of the lattice, *conditional* on no default occurring, and k indexes the four states of non-default.

Second, we compute the expected present value of all payments in the event of default of the zero-coupon bond. Again, the lattice-based recursive expression is:

$$\begin{aligned} CDS(t) &= e^{-r(t)h} \left\{ \sum_{k=1}^{4} p_k(t) CDS_k(t+h) \right\} [1 - \lambda(t)] \\ &+ \lambda(t) ZCCB(t)(1-\phi), \quad CDS(T) = 0.0. \end{aligned} \quad (26)$$

In this implementation of the model, it is assumed that the insurance premiums on the CDS are paid at each step on the tree. Hence, this approximates a continuous insurance payment. It is more accurate

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[5]This is a model developed by RiskMetrics, who use default swaps to calibrate a Merton-type model to obtain probabilities of default.

for the payments to be based on the same frequency as the coupon payments on the underlying bond, in line with current practice.

Third, we calculate the expected present value of a \$1 payment at each point in time conditional on no default occurring. This is defined as follows:

$$G(t) = \left[e^{-r(t)h} \left\{ \sum_{k=1}^{4} p_k(t) G_k(t+h) \right\} + 1 \right] [1 - \lambda(t)(1-\phi)], \ \ G(T) = 0.0. \tag{27}$$

In order to get the annualized basis points spread (s) for the premium payments on the default swap, we equate the quantities $s \times h \times G(0) = CDS(0)$, and the premium spread is:

$$s = \frac{CDS(0)}{h \times G(0)} \times 10,000. \tag{28}$$

We multiply by 10,000 and divide by the time interval h in order to convert the amount into annualized basis points. We use this calculation in the numerical examples that are provided in the sequel.

4   Numerical Examples

4.1   Pricing Credit Default Swaps

Default swaps are easy to use to calibrate the model. A default swap is a contract between two parties, whereby the buyer of the default swap pays a flat stream of insurance payments to the seller, who makes good any loss on default of a reference bond. The seller's payment is contingent upon default. The price of a default swap is quoted as a spread rate per annum. Therefore, if the default swap rate is 100 bps, paid quarterly, then the buyer of the insurance in the default swap would pay 25 bps of the notional each quarter to the seller of insurance in the default swap. The present value of all these payments must equal the expected loss on default anticipated over the life of the default swap. In the event of default, the buyer of protection in the default swap receives the par value of the bond less the recovery on the bond. In many cases, this is implemented by selling the bond back to the insurance seller at par value.

The way the lattice is set up in our model makes it very simple to compute the default swap spread s (stated as a rate). Since the probability of default is known at each node on the tree, we can compute the expected cashflow from the default swap at each node, which is just $[\lambda(t)(1-\phi)]$. We accumulate these values at each node and discount them back along the tree to obtain the expected present value of loss payments by the writer of the default swap. The buyer then pays in a constant spread s each period, such that the present value of these payments equals the present value of expected loss on default. The present value of all spread payments under the default swap, conditional on no default is obtained by discounting the default swap rate (s) payment at each node, in the event that there is no default at that node. This value may be computed on the lattice. Hence it is easy to solve for s given any maturity T of the default swap.

In the plots in Figure 1 we present the term structure of default swap spreads for maturities from 1 to 10 years. The figure has 2 graphs, each containing 2 plots each. The default intensity is written as $\xi(t) = \exp[a_0 + a_1 r(t) + a3(t - t_0)]/S(t)^{a_2}$. Keeping a_0 fixed, we varied parameters a_1 (impact of the short rate), a_2 (impact of the equity price) and a_3 (impact of time) over two values each. Four plots are the result, 2 in each graph. The other inputs to the model, such as the forward rates and volatilities, stock price and volatility, etc., are provided in the description of the figure. Comparison of the plots provides an understanding of the impact of the parameters.

When $a_3 > 0$, the term structure of default swap spreads is upward sloping, as would be expected. When $a_3 < 0$, i.e. default spreads first rise with horizon, and are then driven down as maturity increases, the term structure is flatter than when $a_3 > 0$. The parameter a_3 may be used to tune the model for different credit ratings. It is known that higher quality credits have a tendency to deteriorate in quality over time, hence $a_3 > 0$ would be plausible. On the other hand, poorer quality credits, conditional on survival, tend to upgrade, and hence $a_3 < 0$ may be appropriate. Comparison of the plots also shows the effect of parameter a_2, the coefficient of the equity price $S(t)$. As $a_2 > 0$ increases, default spreads decline as the stock price lies in the denominator of the default intensity function. The impact of parameter a_1, the coefficient on interest rates, has a level effect on the spread curve.

4.2 Calibrating the model to default swaps

The ability to calibrate no-arbitrage models like the one presented in this paper is increasing. The market for default swaps is steadily expanding, which provides observable data on pure credit spreads for many issuers. Default swap spreads are preferred to bond credit spreads since they do not necessarily embed significant premia for liquidity and taxes. Hence, they are purer data sources. Firms such as RiskMetrics now make available credit spread term structures, which are calibrated to default swaps. We can use this data directly in our models.

In this subsection, we present an illustrative calibration of the model to the term structure of default swap spreads of IBM (ticker symbol: IBM). We chose two dates for the calibration, 02-Jan-2002 and 28-Jun-2002. The stock price on the 2 dates was $72.00 and $121.10 respectively. Stock return volatility was roughly 40% on both dates. Recovery rates on default were assumed to be 40% and the correlation between short rates (i.e. 3 month Treasury bills) and the stock return of IBM was computed over the period January 2000 to June 2002; it was found to be almost zero, i.e. 0.01528. The yield curves for the chosen dates were extracted from the historical data pages provided by the Federal Reserve Board. We converted these into forward rates required by our model. Forward rate volatilities were set to the average historical volatility over the periods January 2000 to June 2002. Our goal in this exercise is to examine how easily our model fits its four default parameters $\{a_0, a_1, a_2, a_3\}$ to default spreads of various maturities. For this exercise, we searched over the four parameters to best fit spreads of 1,2,3,4 year maturities. Hence, using the lattice model as a numerical equation, we have to solve four equations in four unknowns. However, there is no way to show that an exact solution exists, since the lattice comprises a highly non-linear function of the inputs and the parameters. By examining how well the calibrated model reproduces the spread curve, we get an idea of how difficult it is to fit our model. As

Figure 1: **Term Structure of Default Swap Spreads**

This figure presents the term structure of default swap spreads for maturities from 1 to 10 years. The figure has 2 graphs, each containing 2 plots each. The default intensity is written as $\xi(t) = \exp[a_0 + a_1 r(t) + a3(t - t_0)]/S(t)^{a_2}$. Keeping all the other parameters fixed, we varied parameters a_1, a_2 and a_3. Hence, the 4 plots are the result. Periods in the model are quarterly, indexed by i. The forward rate curve is very simple and is just $f(i) = 0.06 + 0.001i$. The forward rate volatility curve is $\sigma_f(i) = 0.01 + 0.0005i$. The initial stock price is 100, and the stock return volatility is 0.30. Correlation between stock returns and forward rates is 0.30, and recovery rates are a constant 40%. The default function parameters are presented on the plots.
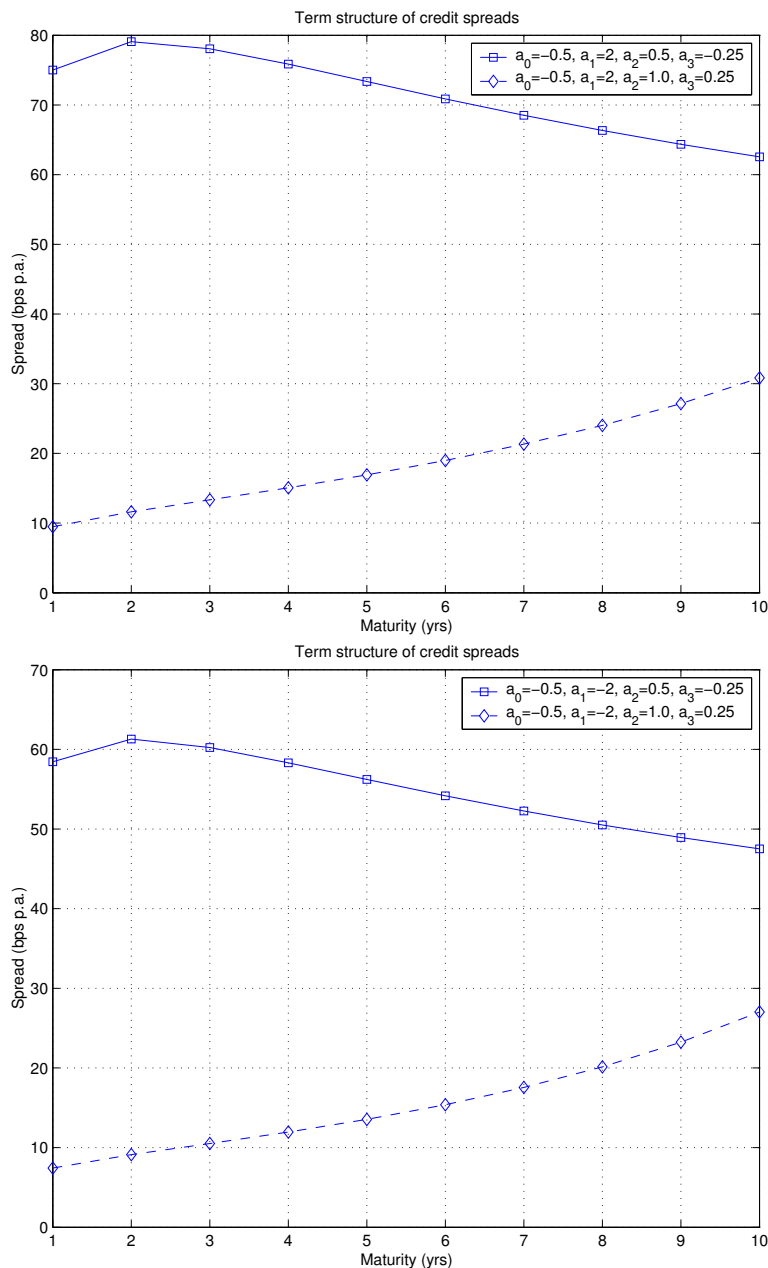
it turns out, the model fits the data well, as can be seen from the overlapping plots in Figure 2. The figure contains two plots, one for each of the dates we chose for calibration, and the fit is excellent for both dates.

Credit risk increased from January 2002 to June 2002, and can be seen in the higher spreads on the second graph, on account of worsening economic conditions in the U.S. economy. A comparison of the parameters of the default intensity function on each date provides some intuition for the impact of increasing credit risk. Notice that a_0 has increased, since the default intensity has gone up from January 2002 to June 2002. Also, a_1 has declined, making default risk less sensitive to interest rates. Since a_2 has declined, the firm's default intensity now increases faster as the stock price falls. Finally, a_3 has become more negative in June 2002, which signifies that, conditional on survival in the short-run, the probability of default will be lower in the long-run. This would be an intuitive outcome in an environment where short-run survival is less likely. Hence, the model calibrates well, and also provides useful economic intuition.

We extended the same analysis to the default swaps of a financial company, namely AMBAC Inc (ticker symbol: ABK). It has often been postulated that default processes in the finance sector are different because firms have extreme leverage. It has been alleged that fitting spread curves for the financial sector is therefore more complicated. However, our model calibrates just as easily to the default swap rates for AMBAC as it did in the case of IBM. For comparison, we calibrated the model on the same dates as we did for IBM. The results in Figure 3 portray the plots of the empirical default swap spreads and the fitted ones. It is seen that these are very close to each other.

It is interesting to note that the spreads for AMBAC have fallen from January to June 2002. This is possibly on account of declining interest rates, which usually bodes well for the finance industry. The coefficient a_3, which is negative, is less negative in June versus January, signaling that, though spreads have declined, the slope of the term structure has become a little steeper, indicating that the market has only indicated better credit quality in the short-run. This coefficient a_2 has also increased from January to June, implying that PDs became more sensitive to changes in the stock price.

4.3 The importance of default factors in pricing convertible debt

The model may be easily used to price callable-convertible debt. One aspect of considerable interest is the extent to which default risk impacts the pricing of convertible debt, through an impact on the values of the call feature (related to interest rate risk) and the convertible feature (related to equity price risk). We set up an initial set of parameters to price convertible debt, and examined to what extent changing levels of default risk impacted a plain vanilla bond versus a convertible bond.

The parameters used for the convertible debt are as follows. To keep the model simple, we assumed a flat forward curve of 6%. We also assumed a flat curve for forward rate volatility of 20 basis points. The maturity of the bonds is taken to be 5 years, and interest is assumed paid quarterly on the bonds at an annualized rate of 6%. Default risk is based on default intensities which come from the model in equation (24). The base parameters for this function are chosen to be $a_0 = 0$, $a_1 = 0$, $a_2 = 2$, and $a_3 = 0$. Under these base parameters default risk varies only with the equity price. In our numerical experiments

Figure 2: **Fitted Default Swap Spreads for IBM**

This figure presents the fitted term structure of default swap spreads for maturities from 1 to 4 years, plotted against the original default spreads. The figure has 2 graphs, each containing 2 plots each. The default intensity is written as $\xi(t) = \exp[a_0 + a_1 r(t) + a3(t - t_0)]/S(t)^{a_2}$. The fitted parameters of this function are provided in the figures below. The first graph is for the spreads on 02-Jan-2002, and the second for 28-Jun-2002. The stock price on the 2 dates was \$72.00 and \$121.10 respectively. Stock return volatility was roughly 40% on both dates. Recovery rates on default were assumed to be 40% and the correlation between short rates (i.e. 3 month tbills) and the stock return of IBM was computed over the period January 2000 to June 2002; it was found to be almost zero, i.e. 0.01528. The yield curves for the chosen dates were extracted from the historical data pages provided by the Federal Reserve Board.
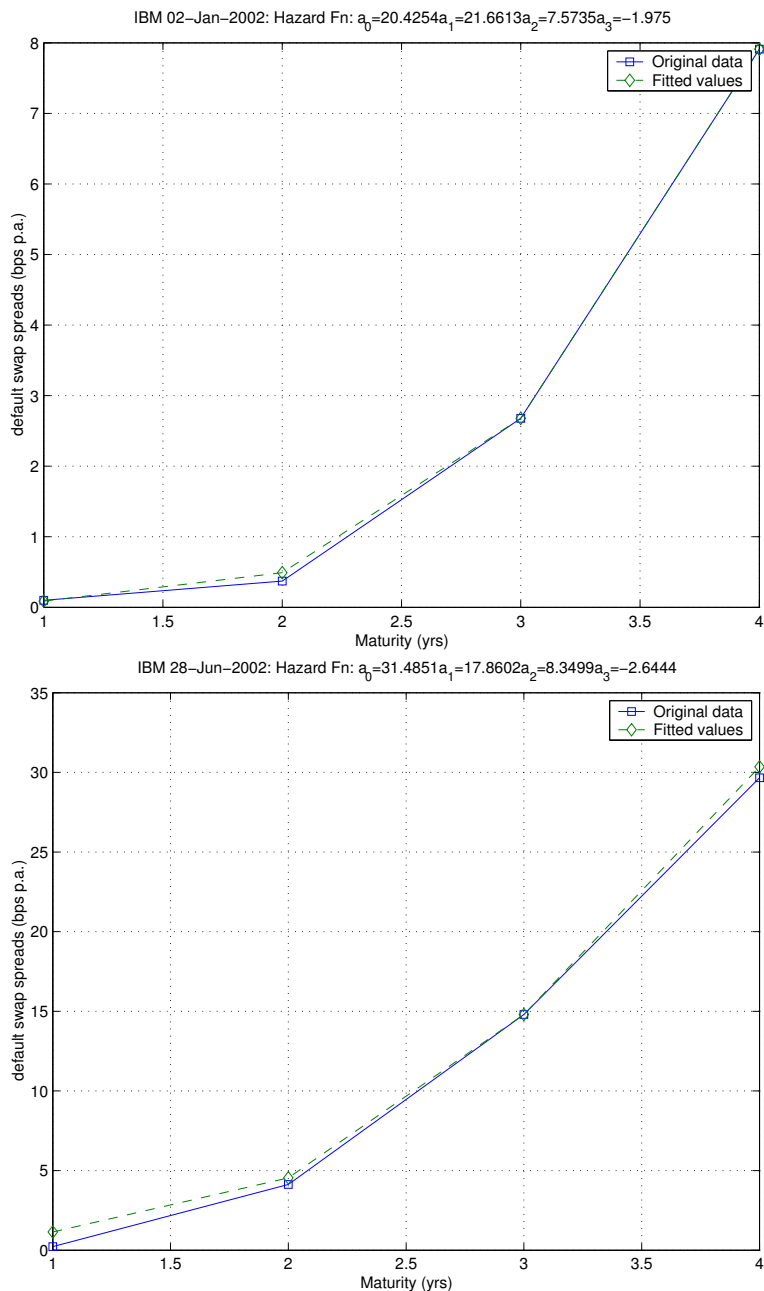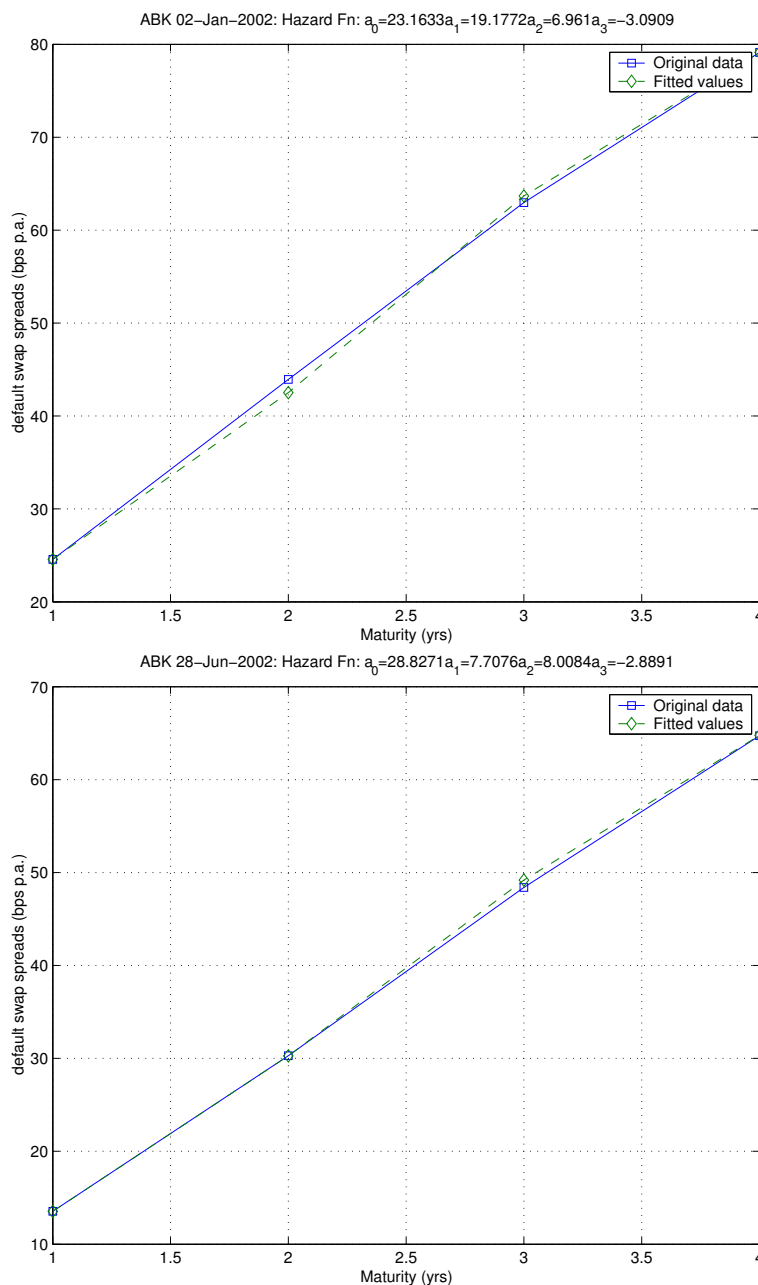

IBM 02–Jan–2002: Hazard Fn: a_0=20.4254a_1=21.6613a_2=7.5735a_3=−1.975

IBM 28–Jun–2002: Hazard Fn: a_0=31.4851a_1=17.8602a_2=8.3499a_3=−2.6444

Figure 3: **Fitted Default Swap Spreads for AMBAC**

This figure presents the fitted term structure of default swap spreads for maturities from 1 to 4 years, plotted against the original default spreads. The figure has 2 graphs, each containing 2 plots each. The default intensity is written as $\xi(t) = \exp[a_0 + a_1 r(t) + a3(t - t_0)]/S(t)^{a_2}$. The fitted parameters of this function are provided in the figures below. The first graph is for the spreads on 02-Jan-2002, and the second for 28-Jun-2002. The stock price on the 2 dates was \$58.31 and \$67.20 respectively. Stock return volatility was roughly 40% on both dates. Recovery rates on default were assumed to be 40% and the correlation between short rates (i.e. 3 month tbills) and the stock return of AMBAC was computed over the period January 2000 to June 2002; it was found to be statistically zero. The yield curves for the chosen dates were extracted from the historical data pages provided by the Federal Reserve Board.

ABK 02–Jan–2002: Hazard Fn: a_0=23.1633 a_1=19.1772 a_2=6.961 a_3=−3.0909

ABK 28–Jun–2002: Hazard Fn: a_0=28.8271 a_1=7.7076 a_2=8.0084 a_3=−2.8891

we will vary a_0 to examine the effect of increasing default risk. The stock price is $S(0) = 100$, and stock volatility is 20% per annum. The recovery rate on default is 0.4, and the correlation between the stock return and term structure is 0.25. If the bond is callable, the strike price is 100. Conversion occurs at a rate of 0.3 shares for each bond. The dilution rate on conversion is assumed to be 0.75.

Given this base set of parameters, we varied a_0 from 0 to 4. As a_0 increases, the level of default risk increases too. For each increasing level of default risk, we plot the prices of a defaultable plain vanilla coupon bond with no call or convertible features. We also plot the prices of (a) a callable-only bond, (b) a convertible-only bond, and (c) a callable and convertible bond. Note that this numerical experiment has been kept simple in the default risk case by setting $a_1 = a_3 = 0$, so that there are no interest-rate and term effects on the default probabilities.

The results comparing the plain coupon bond with a callable coupon bond are presented in Figure 4. The value of a_0 is varied from 0 (no default risk) to 4 (higher risk). The remaining parameters are as per the base case described above in this section. We gain the following insights from Figure 4. First, note that the values of bonds decline as default risk (a_0) increases, and that the callable bond price is lower than that of the non-callable bond, as it should be, given no other difference between the two bonds. Second, a comparison of callable bonds with vanilla coupon bonds shows that the difference from the call feature is greatest when default risk is the lowest. As default risk increases, the difference in price between the callable and vanilla bonds declines rapidly and eventually goes to zero. Since default risk effectively shortens the duration of the bonds, it also reduces the value of the call option. Hence, the price difference between the non-callable vanilla bond and the callable bond declines as a_0 increases. Third, the conversion and dilution ratios were chosen to render convertible bond prices as close as possible to those of vanilla bonds. The second graph in Figure 4 shows that default risk does not appear to impact the differences between the vanilla bond and the convertible bonds. Hence, default risk may not be of serious consequence for the convertibility aspect of bonds. Fourth, in the last graph in Figure 4 we see that the convertible-callable bond does show the same differences from the vanilla bond price as does the pure callable bond from which it derives its properties.

The same analysis was undertaken exactly as before with a single change, i.e. equity volatility was increased from 20% per year to 40% per year. The results are plotted in three graphs in Figure 5. Similar results to those seen in Figure 4 are noticed. In particular, in this case, we find that at even lower levels of default risk, the price difference between the vanilla bond and the callable bond goes to zero. Hence, increasing stock volatility amplifies the impact of default risk on the value of the call feature, because the default intensity function is assumed to be dependent of the level of the stock price, as coefficient $a_2 > 0$.

Therefore, default risk systematically impacts the commingled values of interest rate calls and equity convertible features in debt contracts. First, increasing default risk reduces the values of call and convertible options, to the extent that these features become of less consequence for junk debt. Second, the rate at which increasing default risk reduces the values of embedded options in bonds increases rapidly as can be seen from the exponentially declining price curves in the Figures 4 and 5. Third, increasing equity volatility is seen to enhance the first two effects mentioned above. Therefore, we may conclude that care is required when analyzing the impact of default risk on equity and interest rate derivatives,

Figure 4: **Impact of default risk on vanilla and convertible-callable bonds**

This figure presents a comparison of prices for vanilla and callable bonds when the value of a_0 is varied in the default intensity function. There are 3 graphs in this figure. The first plot examines the difference in prices between the vanilla bond and a pure callable bond. The second graph shows the difference between the vanilla bond and a pure convertible bond. The third graph shows the difference between the vanilla bond and a callable-convertible bond.
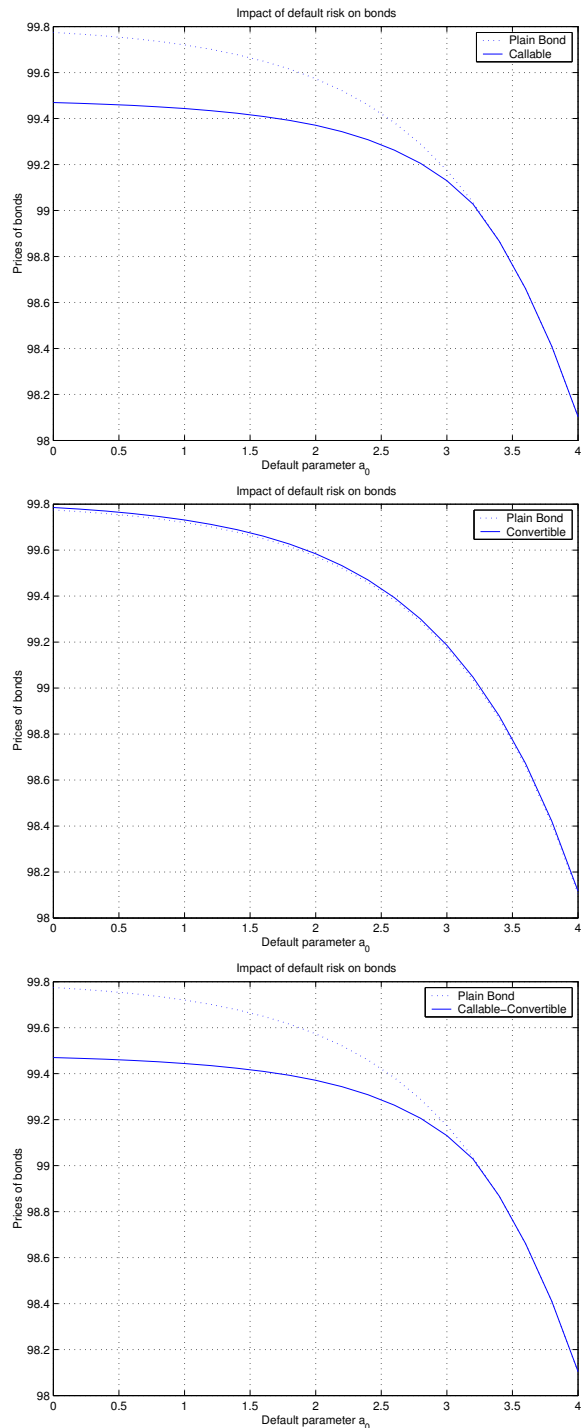
as the effects may be non-intuitive.

4.4 Time series calibration of the default function

In order to undertake a more extensive numerical exercise, we calibrated the CreditGrades based default swap spreads to our model. This was undertaken for the two and a half year period from January 2000 to June 2002. The period spanned covers a total of 655 trading days. This period is of interest because it spans the transition from a time in which the economy experienced low defaults to a period characterized by many corporate failures.

For each date, we fit the four parameters $\{a_0, a_1, a_2, a_3\}$ of the default function in equation (24) to the cross-sectional data on default swap spreads, the stock price and volatility, as well as the current term structure of interest rates. This generates a default function for each of the 655 days in the data set. Setting $h = 0.5$, and using the stock price and current short rate of interest, we computed the value of the function $\xi(t)$ in equation (24).[6] We compare these outputs to the calculated default probabilities provided by CreditMetrics.

The empirical analysis was conducted for two stocks, IBM and UAL, as typical examples of high and low credit quality firms respectively. For IBM, we present the output graphically in Figure 6. The figure contains 2 graphs. The first graph presents the time series of the parameters $\{a_0, a_1, a_2, a_3\}$ for the period spanning 655 days. The second graph plots the value of ξ against the probability of default from the CreditMetrics model. As is to be expected the function ξ (which is proportional to the time $t = 0$ default intensity), tracks the default probabilities very closely. The same outputs are also presented for UAL in Figure 7.

Of greater interest, however, is the relationship between the parameters and observable market variables. We computed some of these correlations, and summarize many of these findings below.

- The parameter values a_0, a_2, a_3 are fairly stable over time, with an occasional shift occurring with very low frequency. This suggests that the default function is time-homogeneous, or at least somewhat stable over time. The only parameter that varies much is a_1, the coefficient of the short rate in the default intensity function. For both firms, IBM and UAL, a_1 increases over time, and may be in response to the steady decline of the short rate over this period, resulting in a constant effect of the term structure on default probabilities.

- The parameter a_0 (baseline default intensity parameter) is positive for both firms, as is to be expected. Likewise, a_2 is also positive for both firms, implying, as it should, that increases in stock price lower the probability of default. The parameter a_1 (coefficient on the interest rate)

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[6]Since these calculations involve a root finding exercise over a numerically generated function, the computational overhead is severe. However, optimization of the program code does much to make the numerical complexity low. The fitting exercise was implemented in Matlab, and each cross-sectional fit takes on average a quarter to half a minute of run time. The majority of the numerical effort is expended on the root finding procedure. The actual computations on the lattice take but a fraction of the total run time. Lattice computations using Java instead of Matlab yielded far better results (in terms of compute times). For a tree with 250 time steps, the run time is approximately 4 seconds on a 700Mhz processor with 256MB of RAM memory.

Figure 5: **Impact of default risk on vanilla and convertible-callable bonds**

This figure presents a comparison of prices for vanilla and callable bonds when the value of a_0 is varied in the default intensity function. There are 3 graphs in this figure. The first plot examines the difference in prices between the vanilla bond and a pure callable bond. The second graph shows the difference between the vanilla bond and a pure convertible bond. The third graph shows the difference between the vanilla bond and a callable-convertible bond. In contrast to Figure 4, the stock return volatility was changed from 20% to 40%.
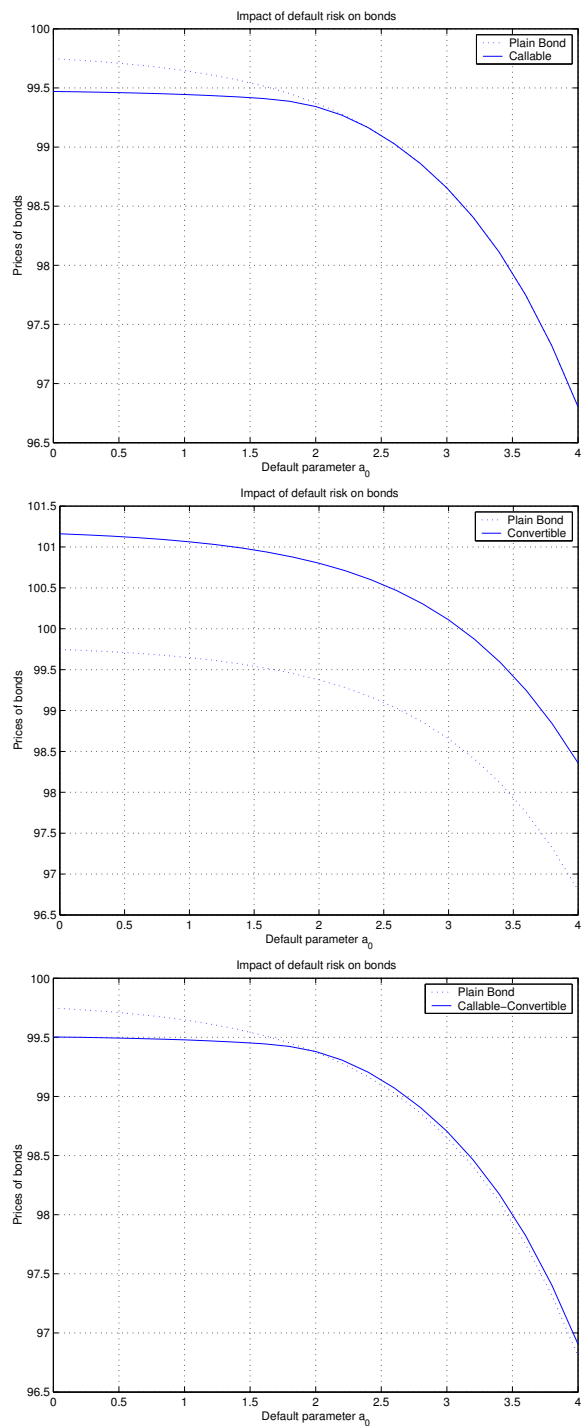
Figure 6: **Impact of default risk on vanilla and convertible-callable bonds for IBM**

The figure contains 2 graphs depicting the time series of default parameters and the probability of default. The first graph presents the time series of the parameters $\{a_0, a_1, a_2, a_3\}$ for the period spanning 655 days. The second graph plots the value of ξ against the computed probability of default from the CreditMetrics model. The CreditGrades PD is labeled as "PD" and the computed ξ from the model is labeled as "lambda".
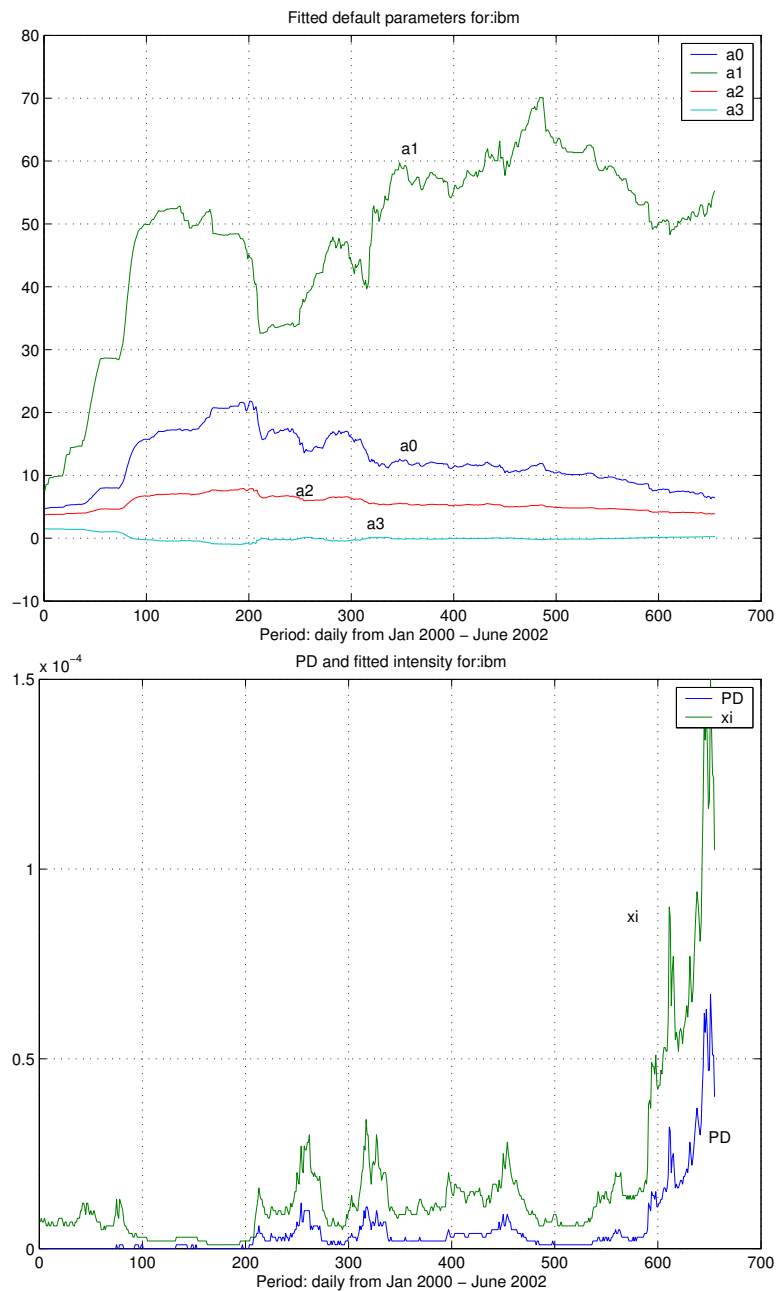
Figure 7: **Impact of default risk on vanilla and convertible-callable bonds for UAL**

The figure contains 2 graphs depicting the time series of default parameters and the probability of default. The first graph presents the time series of the parameters $\{a_0, a_1, a_2, a_3\}$ for the period spanning 655 days. The second graph plots the value of ξ against the computed probability of default from the CreditMetrics model. The CreditGrades PD is labeled as "PD" and the computed ξ from the model is labeled as "lambda".
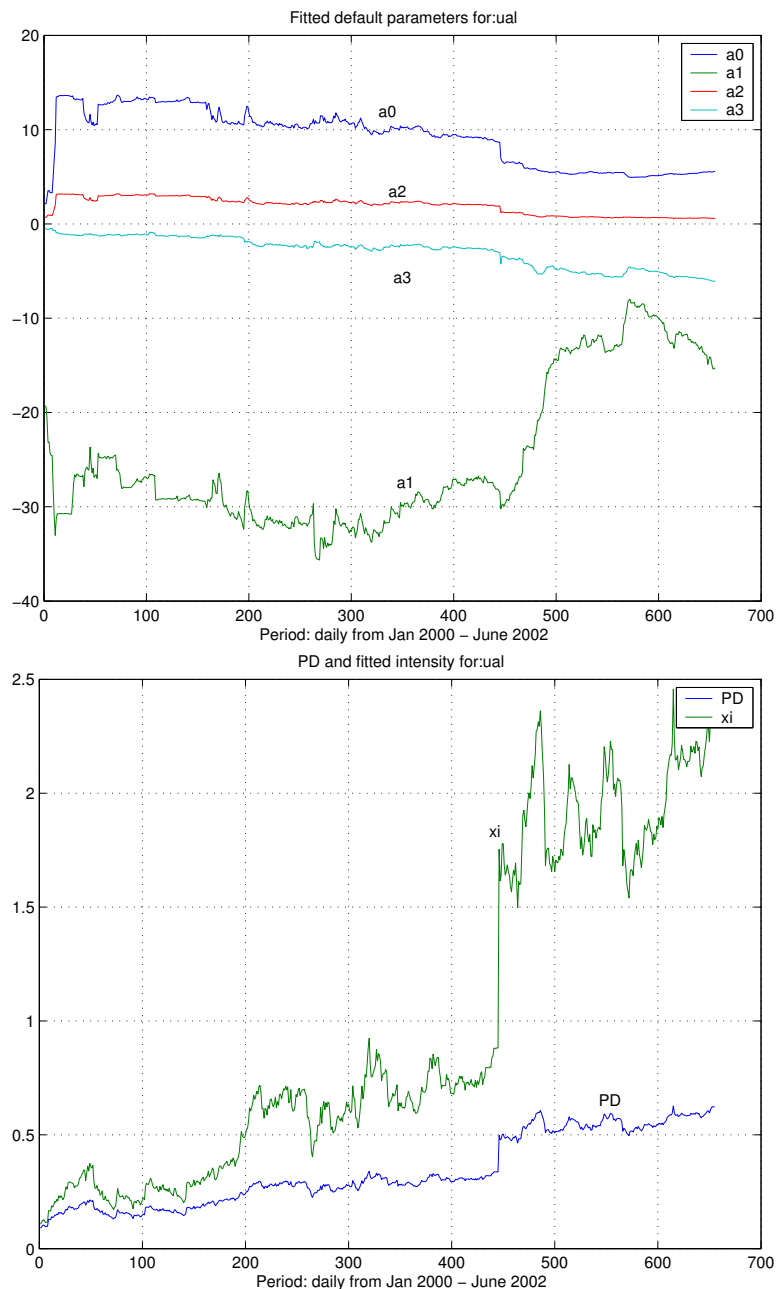
is positive for IBM but negative for UAL. Therefore, dropping interest rates have reduced IBM's probability of default but increased UAL's. One reason for this may the trade-off between falling interest rates as an indicator of recession, and the lower the cost of financing for good quality firms. Finally, a_3 is almost zero for IBM, i.e. there is no maturity impact on spreads. For UAL, a_3 is negative, reflecting the inverted spread curve.

- The function ξ is highly correlated with the CreditMetrics default probability, evidence that the hybrid model is generating probabilities that are comparable to those from a structural model.

- The derived default rate ξ is positively correlated to equity volatility for UAL (96%), but is weaker for IBM (40%). This is consistent with the fact that default probabilities are driven more by equity volatility when the firm is of poor credit quality than when the firm has good credit standing.

- ξ is negatively correlated with the stock price as would be expected. The correlation for IBM is -79% and for UAL it is -95%. Hence, the equity price is more influential in the change in the default function for stocks of weaker credit quality.

- The hazard is negatively correlated with the short interest rate, i.e. higher interest rates lower the probability of default.

Thus, we are able to demonstrate that the model calibrates well to data, and is numerically easy to work with. The fact that the model easily accommodates UAL's inverted spread curve shows that higher risk, complex spread term structures are feasibly characterized in this framework.

5   Concluding Comments

This paper presents a simple model that embeds major forms of security risk, enabling the pricing of complex, hybrid derivatives. The model addresses two distinct classes of objectives: (a) economic and (b) technical.

The following *economic* objectives are met:

- We develop a pricing model with multiple risks, which enables security pricing for *hybrid* derivatives with default risk.

- The extraction of *stable* default probability functions for state-dependent default.

- Using observable market inputs from the equity *and* bond markets, so as to value complex securities via *relative pricing* in a no-arbitrage framework, e.g.: debt-equity swaps, distressed convertibles.

- Managing *credit portfolios* and baskets, e.g. collateralized debt obligations (CDOs), via a simple extension as described in the Appendix.

In addition, the following *technical* objectives are met:

- A hybrid defaultable model combining the features of both, *structural* and *reduced-form* approaches.

- A risk-neutral setting in which the joint process of interest rates and equity are modeled together with the boundary conditions for security payoffs, *after* accounting for default.

- The model is embedded on a recombining lattice, providing fast computation with *polynomial complexity* for run times.

- Cross-sectional spread data permits calibration of an *implied default probability* function which dynamically changes on the state space defined by the pricing lattice.

The model is easily extended to handling correlated default as well, and this is explained in the Appendix. Further research, directed at parallelizing the algorithms in this paper, and improving computational efficiency is predicated and under way. On the economic front, the model's efficacy augurs well for empirical work.

A   Applying the model to correlated default analysis

The model may be used to price a credit basket security. There are many flavors of these securities, and some popular examples are n^{th} to default options, and collateralized debt obligations (CDOs). These securities may be priced using Monte Carlo simulation, under the risk-neutral measure, based on the parameters fitted on the lattice described previously in this paper.

Given a basket of N bonds of distinct issuers, we may simulate default times (τ_i) for each issuer $(i = 1...N)$ based on their stock price correlations. Thus, we first compute a stock price or stock return covariance matrix, denoted $\Sigma_S \in R^{N \times N}$. Under the risk-neutral measure, the return for all stocks is $r(t)$. The procedure is as follows:

1. Fix a time step h for the simulation period.

2. Augment the stock covariance matrix Σ_S to include the covariances with the term structure, estimated from historical data. Call the augmented covariance matrix Σ.

3. With the initial values of stock prices and interest rates, compute $\xi_i(t), i = 1...N$. These are obtained from the previously fitted functions computed from the lattice method. Correspondingly, given h, compute individual one-period default probabilities for every bond, i.e. $\lambda_i(t), \forall i$.

4. Draw N iid uniform random numbers and use them to determine which bonds will default.

5. Repeat this procedure for each period in the simulation.

This section relates closely to the work of Zhou [2001], who shows that it is possible to derive default correlations amongst firms within the structural model framework, by relating the correlations to

the default barriers of each firm. Our approach differs from Zhou's in the following way. First, Zhou's approach assumes knowledge of the firm value process for each issuer and their default boundaries. In our approach, we work off equity correlations, and hence can rely on observables. Second, Zhou's model only accommodates anticipated default, whereas our model contains components of both anticipated and unanticipated defaults, hence, the correlations are also based on the same.

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