Fixed Income Pricing

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Contents

1 Introduction

This chapter surveys the literature on fixed-income pricing models, including dynamic term structure models (*DTSM*s) and interest rate sensitive, derivative pricing models. This literature is vast with both the academic and practitioner communities having proposed a wide variety of models and model-selection criteria. Central to all pricing models, implicitly or explicitly, are: (i) the *identity* of the state vector: whether it is latent or observable and, in the latter case, which observable series; (ii) the law of motion (conditional distribution) of the state vector under the pricing measure; and (iii) the functional dependence of the short-term interest rate on this state vector. A primary objective, then, of research on fixed-income pricing has been the selection of these ingredients to capture relevant features of history, given the objectives of the modeler, while maintaining tractability, given available data and computational algorithms. Accordingly, we overview alternative conceptual approaches to fixed-income pricing, highlighting some of the tradeoffs that have emerged in the literature between the complexity of the probability model for the state, data availability, the pricing objective, and the tractability of the resulting model.

A pricing model may be "monolithic" in the sense that it prices both bonds (as functions of a set of underlying state variables or "risk factors" – i.e., is a "term structure model") and fixed-income derivatives (with payoffs expressed in terms of the prices or yields on these underlying bonds). Alternatively, a model may be designed to price fixed-income derivatives, taking as given the current shape of the underlying yield curve. The former modeling strategy is certainly more comprehensive than the latter. However, researchers have often found that the latter approach offers more flexibility in calibration and tractability in computation when pricing certain derivatives.

Initially, taking the monolithic approach, we overview a variety of models for pricing default-free bonds and associated derivatives written on these (or portfolios of these) bonds. Basic issues in pricing fixed-income securities (*FIS*) for the case where the state vector follows a diffusion are discussed in Section 2. "Yield-based" *DTSM*s are reviewed in Section 3. Extensions of these pricing models to allow for jumps or regime shifts are explored in Sections 4 and 5, respectively.

Then, in Section 6, we turn to the case of defaultable securities. Here we start by considering a quite general framework in which there are multiple credit classes (possibly indexed by rating) and deriving pricing relations for the case where issuers may transition between classes according to a Markov process. Several of the most widely studied models for pricing defaultable bonds are compared by specializing to the case of a single credit class.

The pricing of fixed-income derivatives is overviewed in Section 7. Initially, we continue our discussion of *DTSM*s and overview recent research on the pricing of derivatives using yield-based term structure models. Then we shift our focus from monolithic models to models for pricing derivatives in which the current yield curve, and possibly the associated yield volatilities, are taken as inputs into the pricing problem. These include models based on forward rates (both for default-free and defaultable securities), and the LIBOR and Swaption Market models.

To keep our overview of the literature manageable we focus, for the most part, on term structure models and fairly standard derivatives on zero-coupon and coupon bonds (both default-free and defaultable), plain-vanilla swaps, caps, and swaptions. In particular, we do not delve deeply into many of the complex structured products that are increasingly being traded. Of particular note, we have chosen to side-step the important issue of pricing securities in which correlated defaults play a central role in valuation.[1] Additionally, we focus almost exclusively on pricing and the associated "pricing measures." Our companion paper Dai and Singleton [2002] explores in depth the specifications of the market prices of risk that connect the pricing with the actual measures, as well as the empirical goodness-of-fit of models[2] under alternative specifications of the market prices of risks.

2 Fixed-income Pricing in a Diffusion Setting

A standard framework for pricing *FIS* has the riskless rate r_t being a deterministic function of an $N \times 1$ vector of risk factors Y_t,

$$r_t = r(Y_t, t), \tag{1}$$

[1]Musiela and Rutkowski [1997b] discuss the pricing of a wide variety of fixed-income products, and Duffie and Singleton [2001] discuss pricing of structured products in which correlated default is a central consideration.

[2] See also Chapman and Pearson [2001] for another surveys of the empirical term structure literature.

and the *risk-neutral* dynamics of Y_t following a diffusion process,[3]

$$dY_t = \mu(Y_t, t)\, dt + \sigma(Y_t, t)\, dW_t^{\mathbb{Q}}. \tag{2}$$

Here, $W_t^{\mathbb{Q}}$ is a $K \times 1$ vector of standard and independent Brownian motions under the risk-neutral measure $\mathbb{Q}$, $\mu(Y, t)$ is a $N \times 1$ vector of deterministic functions of Y and possibly time t, and $\sigma(Y, t)$ is a $N \times K$ matrix of deterministic functions of Y and possibly t.

2.1 The Term Structure

Central to the pricing of *FIS* is the term structure of zero-coupon bond prices. The time-t price of a zero-coupon bond with maturity T and face value of \$1 is given by

$$D(t, T) = E^{\mathbb{Q}}\left[e^{-\int_t^T r_s\, ds} \middle| \mathcal{F}_t \right], \tag{3}$$

where $\mathcal{F}_t$ is the information set at time t, and $E^{\mathbb{Q}}[\,\cdot\,|\mathcal{F}_t]$ denotes the conditional expectation under the risk-neutral measure $\mathbb{Q}$. Since a diffusion process is Markov, we can take $\mathcal{F}_t$ to be the information set generated by Y_t. Thus, the discount function $\{D(t, T) : T \geq t\}$ is completely determined by the risk-neutral distribution of the riskless rate and Y_t.[4]

As an application of the Feynman-Kac theorem, the price of a zero-coupon bond can alternatively be characterized as a solution to a partial differential equation (*PDE*). Heuristically, this *PDE* is obtained by applying Ito's lemma to the pricing function $D(t, T)$, for some fixed $T \geq t$:

$$\begin{aligned} dD(t, T) &= \mu(Y_t, t; T)\, dt + \sigma(Y_t, t; T)'\, dW_t^{\mathbb{Q}}, \\ \mu(Y, t; T) &= \left[\frac{\partial}{\partial t} + \mathcal{A} \right] D(t, T),\ \ \sigma(Y, t; T) = \sigma(Y, t)' \frac{\partial D(t, T)}{\partial Y}, \end{aligned}$$

where $\mathcal{A}$ is the infinitesimal generator for the diffusion Y_t:

$$\mathcal{A} = \mu(Y, t)' \frac{\partial}{\partial Y} + \frac{1}{2} \mathrm{Trace}\left[\sigma(Y, t)\, \sigma(Y, t)' \frac{\partial^2}{\partial Y \partial Y'} \right].$$

[3] See Duffie [1996] for sufficient technical conditions for a solution to (2) to exist.

[4] Here we assume that sufficient regularity conditions conditions (that may depend on the functional form of $r(Y, t)$) have been imposed to ensure that the conditional expectation in (3) is well-defined and finite.

No-arbitrage requires that, under $\mathbb{Q}$, the instantaneous expected return on the bond be equal to the riskless rate r_t. Imposing this requirement gives

$$\left[\frac{\partial}{\partial t} + \mathcal{A}\right] D(t,T) - r(Y,t)\, D(t,T) = 0, \tag{4}$$

with the boundary condition $D(T,T) = \$1$ for all Y_T.

2.2 *FIS* with Deterministic Payoffs

The price of a security with a set of deterministic cash flows $\{C_j : j = 1, 2, \ldots, n\}$ at some given *relative* payoff dates τ_j ($j = 1, 2, \ldots, n$) is given by

$$P(t; \{C_j, \tau_j : j = 1, 2, \ldots, n\}) = \sum_{j=1}^{n} C_j D(t, t + \tau^j).$$

In particular, the price of a coupon-bond with face value F, semi-annual coupon rate of c, and maturity $T = J \times .5$ years (where J is an integer) is

$$P(t; \{c, T\}) = \sum_{j=1}^{J} F \times \frac{c}{2} \times D(t, t + .5j) + F \times D(t,T).$$

It follows that the *par yield* – i.e., the semi-annually compounded yield on a par bond (with $P_t = F$) – is given by

$$\mathrm{PY}(t,T) = \frac{2\,[1 - D(t,T)]}{\sum_{j=1}^{J} D(t, t + .5j)}. \tag{5}$$

2.3 *FIS* with State-dependent Payoffs

The price of a *FIS* with coupon flow payment h_s, $t \le s \le T$, and terminal payoff g_T is

$$\begin{aligned} &P\left(t; \{h_s : t \le s \le T; g_T\}\right) \\ =&E^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r_u\, du}\, h_s\, ds \,\middle|\, \mathcal{F}_t\right] + E^{\mathbb{Q}}\left[e^{-\int_t^T r_u\, du}\, g_T \,\middle|\, \mathcal{F}_t\right]. \end{aligned} \tag{6}$$

When $r_u = r(Y_u, u)$, $h_u = h(Y_u, u)$, and $g_u = g(Y_u, u)$ are deterministic functions of the state vector Y_u, this price is obtained as a solution to the

PDE

$$\left[\frac{\partial}{\partial t} + \mathcal{A}\right] P(t) - r(Y,t)\, P(t) + h(Y,t) = 0, \tag{7}$$

under the boundary condition $P(T; \{h_T; g_T\}) = g(Y_T, T)$, for all Y_T. (Equation (4) is obtained as the special case of (7) with $h_u \equiv 0$ and $g_T = \$1$.)

A mathematically equivalent way of characterizing the price $P(t; \{h_s : t \le s \le T; g_T\})$ is in terms of the *Green's* function. Let $\delta(x)$ denote the Dirac function, with the property that $\delta(x) = 0$ at $x \neq 0$, $\delta(0) = \infty$, and $\int dx\, \delta(x-y)\, f(x) = f(y)$ for any continuous and bounded function $f(\cdot)$. The price, $G(Y_t, t; Y, T)$, of a security with a payoff $\delta(Y_T - Y)$ at T, and nothing otherwise, is referred to as the Green's function. By definition, the Green's function is given by

$$G(t, Y_t; T, Y) = E^{\mathbb{Q}}\left[e^{-\int_t^T r_u\, du}\, \delta(Y_T - Y)\right].$$

It is easy to see that G solves the *PDE* (7) with $h(Y,t) \equiv 0$ under the boundary condition $G(Y_T, T; Y, T) = \delta(Y_T - Y)$. If G is known, then any *FIS* with payment flow $h(Y_t, t)$ and terminal payoff $g(Y_T, T)$ is given by

$$\begin{aligned} & P\left(t; \{h_s : t \le s \le T; g_T\}\right) \\ &= \int_t^T ds \int dY\, G(Y_t, t; Y, s)\, h(Y, s) + \int dY\, G(Y_t, t; Y, T)\, g(Y, T). \end{aligned} \tag{8}$$

Essentially, the Green's function represents the set of Arrow-Debreu prices for the case of a continuous state space. When the Green's function is known, equation (8) is often convenient for the numerical computation of the prices of a wide variety of *FIS* (see Steenkiste and Foresi [1999] for applications of the Green's function for affine term structure models).

In the absence of default risk, some fixed-income derivative securities with state-dependent payoffs can be priced using the discount function alone, because they can be perfectly hedged or replicated by a (static) portfolio of spot instruments. These include:

- **Forward Contracts**: a forward contract with settlement date T and forward price F on a zero-coupon bond with par \$100 and maturity date $T+\tau$ can be replicated by a portfolio of spot instruments consisting of long a zero-coupon bond with maturity $T+\tau$ and par \$100 and short a zero-coupon bond with maturity T and par F. Thus, the market value of the forward contract is $\$100 \times D(t, T+\tau) - F \times D(t, T)$. Consequently the forward price is given by $F = \$100 \times \frac{D(t,T+\tau)}{D(t,T)}$.

- **A Floating Payment**: a floating payment indexed to a riskless rate with tenor τ, with coupon rate reset at T and payment made at $T + \tau$, can be replicated by a portfolio of spot instruments consisting of long a zero-coupon bond with maturity T and par \$100 and short a zero-coupon bond with maturity $T + \tau$ and par \$100. Thus, the price of the floating payment is $\$100 \times [D(t, T) - D(t, T + \tau)]$. This implies immediately that a floating rate note with payment in arrears is always priced at par on any reset date.

- **A Plain Vanilla Interest Rate Swap:** a plain-vanilla interest rate swap with the tenor of the floating index matching the payment frequency can be perfectly replicated by a portfolio of spot instruments consisting of long a floating rate note with the same floating index, payment frequency, and maturity and short a coupon bond with the same maturity and payment frequency, and with coupon rate equal to the swap rate. It follows that, at the inception of the swap, the swap rate is equal to the par rate:

$$s(t, T) = \frac{1 - D(t, T)}{\sum_{j=0}^{N-1} \delta_j D(t, T_j)},$$

  where $t \equiv T_0 < T_1 < \ldots < T_N \equiv T$, $\delta_j = T_{j+1} - T_j$ is the length of the accrual payment period indexed by j, $0 \leq j \leq N - 1$, based on an appropriate day-count convention, N is the number of payments, and T is the maturity of the swap.

In the presence of default risk, the above pricing results may not hold except under specific conditions (see, e.g., Section 6.5 for pricing of Eurodollar swaps).

2.4 *FIS* with Stopping Times

For some fixed-income securities, including American options and defaultable securities, the cash flow payoff dates are also random. A random payoff date is typically modeled as a *stopping time*, that may be exogenously given or endogenously determined (in the sense that it must be determined jointly with the price of the security under consideration).

The optimal exercise policy of an American option can be characterized as an endogenous stopping time. Valuation of American options in general, and valuation of fixed-income securities containing features of an American option in particular, is challenging, because closed-form solutions are rarely available and numerical computations (finite-difference, binomial-lattice, or Monte Carlo simulation) are typically very expensive (especially when there are multiple risk factors). As a result, approximation schemes are often used (see, e.g., Longstaff and Schwartz [2001]), and considerable attention has been given to establishing upper and lower bounds on American option prices (e.g., Haugh and Kogan [2001] and Anderson and Broadie [2001]). In the light of these complexities in pricing, some have questioned whether the optimal exercise strategies implicit in the parsimonious models typically used in practice are correctly valuing the American option feature of many products (e.g., Andersen and Andreasen [2001] and Longstaff, Santa-Clara, and Schwartz [2001]). Of course, characterizing the optimal exercise policy itself can be challenging, particularly in the case of mortgage backed securities, because factors other than interest rates may influence the prepayment behavior (e.g., Stanton [1995]).

In "reduced-form" pricing models for defaultable securities (e.g., Jarrow, Lando, and Turnbull [1997], Lando [1998], Madan and Unal [1998], and Duffie and Singleton [1999]), the default time is typically modeled as the exogenous arrival time of an autonomous counting process. The claim to the recovery value of a defaultable security with maturity T is the present value of the payoff $q_\tau = q(Y_\tau, \tau)$ (recovery upon default) at the default arrival time τ whenever $\tau \leq T$:

$$P(t; \{q(Y_\tau, \tau)\}) = E^{\mathbb{Q}} \left[e^{-\int_t^\tau r_u \, du} q_\tau 1_{\{\tau \leq T\}} \middle| Y_t \right]. \tag{9}$$

This expression simplifies if τ is the arrival time of a *doubly stochastic* Poisson process with state-dependent intensity $\lambda_t = \lambda(Y_t, t)$. At date t, the cumulative distribution of arrival of a stopping time before date s, conditional on $\{Y_u : t \leq u \leq s\}$ is $\Pr(\tau \leq s; t | Y_u : t \leq u \leq s) = 1 - e^{-\int_t^s \lambda_u du}$. It follows that (see, e.g., Lando [1998])

$$
\begin{aligned}
P(t;\{q(Y_\tau,\tau)\}) &= E^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r_u\,du}\, q_s\, d\,\mathrm{Pr}(\tau \le s; t | Y_u : t \le u \le s)\,\middle|\, Y_t\right] \\
&= E^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s (r_u+\lambda_u)\,du}\, \lambda_s\, q_s\, ds \,\middle|\, Y_t\right].
\end{aligned}
$$

This pricing equation is a special case of (6) with $h_s = \lambda_s q_s$, $g_T = 0$, and an "effective riskless rate" of $r_s + \lambda_s$.

In "structural" pricing models of defaultable securities, the default time is typically modeled as the first passage time of firm value below some *default boundary*. With a constant default boundary and exogenous firm value process (e.g., Merton [1974], Black and Cox [1976], and Longstaff and Schwartz [1995]), the pricing of the default risk amounts to the computation of the first-passage probability under the forward measure. With an endogenously determined default boundary (e.g., Leland [1994] and Leland and Toft [1996]), the probability of the first passage time and the value of the risky debt must be jointly determined.[5]

3 *DTSM*s for Default-free Bonds

In this section we overview the pricing of default-free bonds within *DTSM*s. We begin with an overview of one-factor models ($N = 1$) and then turn to the case of multi-factor models.

3.1 One-factor *DTSM*s

Some of the more widely studied one-factors models are:

- **Nonlinear *CEV* Model** r follows the one-dimensional Feller [1951] process

$$dr(t) = (\kappa\theta r(t)^{2\eta-1} - \kappa r(t))\,dt + \sigma r(t)^\eta dW^{\mathbb{Q}}(t). \tag{10}$$

In this model, the admissible range for η is $[0, 1)$, and the zero boundary is entrance (cannot be reached from the interior of the state space) if

[5] Similar to an American option, the price of the risky debt can be characterized as the solution to a PDE with a "free boundary", with the boundary conditions given by the so-called "value-matching" and the "smooth-pasting" conditions.

$\kappa\theta > \sigma^2/2$ (the so called "Feller condition"). Further, the distribution of r_t conditional on r_{t-1} is known to be a generalized Bessel process (Eom [1998]).

The solution to (10) has $r > 0$ for all $\eta \in [0, 1)$, including $\eta = 0$, so long as the Feller condition is satisfied. However, we are not aware of closed-form solutions for $D(t, T)$ in this model outside of the case of $\eta = .5$.

- **Square-root model** (Cox, Ingersoll, and Ross [1985]) For this special case of (10) with $\eta = .5$ the discount function is given by

$$D(t,T) = A(\tau)\, e^{-B(\tau)\, r_t},\ T \geq t,\ \tau \equiv (T-t),$$

where, with $\gamma \equiv \sqrt{\kappa^2 + 2\sigma^2}$,

$$A(\tau) = \left[\frac{2\gamma e^{(\kappa+\gamma)\tau/2}}{(\kappa+\gamma)\,(e^{\gamma\tau}-1)+2\gamma}\right]^{2\kappa\theta/\sigma^2},\ B(\tau) = \frac{2\,(e^{\gamma\tau}-1)}{(\kappa+\gamma)\,(e^{\gamma\tau}-1)+2\gamma}.$$

The Green's function for the CIR model is given by

$$G(r_t, t; r, T) = D(t,T)\, 2c\, \chi^2(2c\, r; \nu, \phi),$$

where $\chi^2(\cdot, \nu, \phi)$ is the non-central chi-square density with the degrees of freedom ν and the parameter of noncentrality ϕ, defined by

$$c = \frac{\kappa + \gamma - (\kappa - \gamma)\ e^{-\gamma\,(T-t)}}{\sigma^2\ \left[1 - e^{-\gamma\,(T-t)}\right]},\ \nu = \frac{4\kappa\theta}{\sigma^2},$$

$$\phi = \frac{8\,\gamma^2\, e^{-\gamma(T-t)}\, r_t}{\sigma^2\,(1-e^{-\gamma(T-t)})\,[2\gamma + (\kappa-\gamma)\,(1-e^{-\gamma(T-t)})]}.$$

- **Log-normal model** (Black, Derman, and Toy [1990]). As $\eta \to 1$ in (10), the process for r converges to that of a log-normal process. Though widely used in the financial industry (often in this one-factor formulation with time-dependent parameters), we are not aware of closed-form solutions for discount curves in this model.

- **Three-halves model** (Cox, Ingersoll, and Ross [1980]). r follows the process

$$dr(t) = \kappa(\theta - r(t))r(t)\, dt + \sigma r(t)^{1.5} dW^{\mathbb{Q}}(t). \qquad (11)$$

This process is stationary and zero is entrance if κ and σ are greater than 0. $D(t,T)$ is given by (see Ahn and Gao [1999])

$$D(t,T) = \frac{\Gamma(\beta-\gamma)}{\Gamma(\beta)} M(\gamma,\beta,-x(t))\, x(t)^{\gamma},$$

where $\Gamma(\cdot)$ is the "gamma" function, $M(\cdot)$ is a confluent hypergeometric function (computed through a series expansion), $x(t) = \frac{-2b}{\sigma^2\left(e^{b(T-t)}-1\right)r(t)}$, and

$$\gamma = \frac{1}{\sigma^2}\left[\sqrt{(.5\sigma^2-a)^2+2\sigma^2} - (.5\sigma^2-a)\right], \quad \beta = \frac{2}{\sigma^2}\left[-a+(1+\gamma)\sigma^2\right].$$

- **Gaussian Model** (Vasicek [1977]). r follows the diffusion with linear drift $\kappa(\theta - r(t))dt$ and constant diffusion coefficient σ. In this case, the discount function is given by

$$\begin{aligned} D(t,T) &= e^{-A(T-t)-B(t,T)\,r_t}, \ T \geq t, \\ \text{where, } A(\tau) &= \left(\delta - \frac{\sigma^2}{2\kappa^2}\right)[\tau - B(\tau)] + \frac{\sigma^2}{4\kappa}B(\tau)^2, \quad B(\tau) = \frac{1-\kappa\tau}{\kappa}. \end{aligned}$$

  The Green's function for this model is given by (see Jamshidian [1989]):

$$G(r_t,t;t,T) = D(t,T)\,\frac{e^{-\frac{(r-f(t,T))^2}{2v(t,T)}}}{\sqrt{2\pi v(t,T)}},$$

  where $f(t,T) = e^{-\kappa(T-t)}r + (1-e^{-\kappa(T-t)})\theta - \frac{\sigma^2}{2\kappa^2}\left(1-e^{-\kappa(T-t)}\right)^2$ is the instantaneous forward rate and $v(t,T) = \frac{\sigma^2}{2\kappa}\left(1-e^{-2\kappa(T-t)}\right)$ is the conditional volatility of the spot short rate.

An alternative means of constructing tractable one-factor models is to maintain a simpler representation of the state Y and to let $r_t = g(Y_t,t)$, for some nonlinear function g. For example, the one-factor Quadratic Gaussian (QG) model (see Beaglehole and Tenney [1991]) is obtained by letting $g(Y_t,t) = \alpha + \beta Y_t + \gamma Y_t^2$ and Y_t following a Gaussian diffusion. The discount function in this model is given by

$$D(t,T) = e^{-A(\tau)-B(\tau)\,Y_t - C(\tau)\,Y_t^2},$$

where $B(\tau) = C(\tau)\left[\frac{2\kappa(\theta+\frac{\beta}{2\gamma})}{\Gamma}\frac{e^{\Gamma\tau}-1}{e^{\Gamma\tau}+1} + \frac{\beta}{\gamma}\right]$, and $C(\tau) = \frac{\gamma(e^{2\Gamma\tau}-1)}{(\Gamma+\kappa)(e^{2\Gamma\tau}-1)+2\Gamma}$, and $\Gamma = \sqrt{\kappa^2 + 2\gamma\sigma^2}$. ($A(\tau)$ is also known as a relatively complicated function of the underlying parameters.)

Although the QG model is driven by one risk factor, it can be viewed equivalently as a degenerate two-factor model (with two state variables driven by the same Brownian motion). To see this, note that, from Ito's lemma,

$$dr_t = \left[(2\kappa\alpha + \kappa\theta\beta + \gamma\sigma^2) + 2\kappa\theta\gamma Y_t - 2\kappa\gamma r_t\right]\, dt + (\beta + 2\gamma Y_t)\sigma dW_t^{\mathbb{Q}}.$$

Using the fact that r is affine in Y and Y^2, we see that the instantaneous conditional means and covariance of r_t and Y_t are affine in (r_t, Y_t).

One can build up multi-factor *DTSM*s from these one factor examples by simply assuming that the short rate is the sum of N independent risk factors, $r_t = \sum_{i=1}^N Y_t^i$, with each Y following one of the preceding one-factor models for which a solution for zero prices is known (see Cox, Ingersoll, and Ross [1985], Chen and Scott [1993], Pearson and Sun [1994], Duffie and Singleton [1997], and Jagannathan, Kaplan, and Sun [2001] for multi-factor versions of the square-root model). In this case, the discount function is given by $D(t,T) = \prod_{i=1}^N D(t,T)^i$, where $D(t,T)^i$ is the discount function in a single-factor model with the short rate given by $r_t = Y_t^i$. This approach leads, however, to rather restrictive formulations of multi-factor models, particularly with regard to the assumption of zero correlations among the risk factors. We turn next to multi-factor models with correlated risk factors.

3.2 Multi-factor *DTSM*s

A quite general formulation of multi-factor models has $r_t = g(Y_t, t)$, where $Y_t = (Y_t^i : 1 \leq i \leq N)'$ and these risk factors may be mutually correlated. Specifications of the function $g(\cdot, t)$ and the dynamics of Y_t are constrained only by the so-called admissibility conditions which stipulate that (i) Y_t must be a well-defined stochastic process; and (ii) the conditional expectation in (3) exists and is finite (equivalently, the *PDE* (4) has a well-defined and finite solution). In practice, however, model specifications are often influenced by their computational tractability in pricing *FIS*.

Two classes of diffusion-based multi-factor models have been the focal points of much of the literature on pricing default-free bonds: affine models (see, e.g., Duffie and Kan [1996] and Dai and Singleton [2000]) and quadratic

Gaussian models (see, e.g., Beaglehole and Tenney [1991], Ahn, Dittmar, and Gallant [2002] and Leippold and Wu [2001]). These models have the common feature that the discount function has the exponential form:

$$D(t,T) = e^{-G(Y_t,t;T)},\ T \geq t, \tag{12}$$

where $G(Y_T, T; T) = 0$ for all Y_T, and $\lim_{T\to t} G_T(Y_t, t; T)$ exists and is finite for all Y_t and $t \leq T$.

Heuristically, the affine and quadratic Gaussian models are "derived" from the requirement that $G(Y, \cdot; \cdot)$ be, respectively, an affine and quadratic function of the state vector Y. Naturally, such a requirement restricts the functional form of $g(Y_t, t)$ and the $\mathbb{Q}$-dynamics of Y_t. By definition, $r_t = -\lim_{T\to t} \frac{\log D_{t,T}}{T-t}$ from which it follows that

$$r_t = g(Y_t, t) = \lim_{T\to t} \frac{G(Y_t, t;\, T)}{T-t}. \tag{13}$$

Thus, r_t must be affine in Y in affine models and quadratic in Y in quadratic Gaussian models. Furthermore, substituting (12) into (4) yields

$$G_t + \mu(Y,t)' G_Y + \frac{1}{2}\mathrm{Trace}\left[\sigma(Y,t)\sigma(Y,t)'\left(G_{YY'} - G_Y G_{Y'}\right)\right] + g(Y,t) = 0, \tag{14}$$

which may be viewed as a restriction on the risk-neutral drift $\mu(Y,t)$ and diffusion $\sigma(Y,t)$ for the state vector Y.

Affine Models

Affine term structure models are characterized by the requirement that $G(Y_t, t; T)$ be affine in Y_t; i.e.,

$$G(Y_t, t;\, T) = A(T-t) + B(T-t)' Y_t.$$

In this case, r_t must also be affine in Y_t: $r_t = \alpha + \beta' Y_t$, where $\alpha = A'(0)$ and $\beta = B'(0)$. Furthermore, equation (14) reduces to

$$\dot{A}(\tau) + \dot{B}(\tau) Y_t = \mu(Y_t,t)' B(\tau) - \frac{1}{2}\left[B(\tau)'\sigma(Y_t,t)\sigma(Y_t,t)' B(\tau)\right] + (\alpha + \beta' Y_t), \tag{15}$$

where $\tau \equiv T-t$, $\dot{A}(\tau) = \frac{\partial A(\tau)}{\partial \tau} = -\frac{\partial A(T-t)}{\partial t}$, and $\dot{B}(\tau)$ is similarly defined. Duffie and Kan [1996] show that, in order for (15) to hold for any Y_t, it is sufficient that[6]

1. $\mu(Y_t, t)$ be affine in Y_t: $\mu(Y_t, t) = a + bY_t$, where a be a $N \times 1$ vector and b be a $N \times N$ matrix.

2. $\sigma(Y_t, t)\sigma(Y_t, t)'$ be affine in Y_t: $\sigma(Y_t, t)\sigma(Y_t, t)' = h_0 + \sum_{j=1}^{N} h_1^j Y_t^j$, where h_0 and h_1^j, $j = 1, 2, \ldots, N$, are $N \times N$ matrices.

3. $A(\tau)$ and $B(\tau)$ satisfy the following ODEs:

$$\dot{A} = \alpha + a'B(\tau) - \frac{1}{2}B(\tau)'h_0 B(\tau), \tag{16}$$

$$\dot{B} = \beta + b'B(\tau) - \frac{1}{2}v(\tau), \tag{17}$$

   where $v_j(\tau) \equiv B(\tau)'h_1^j B(\tau)$.

For suitable choices of $(\alpha, \beta; a, b; h_0, h_1^j : 1 \le j \le N)$, the Ricatti equations (16)–(17) admit a unique solution $(A(\tau), B(\tau))$ under the initial conditions $A(0) = 0$ and $B(0) = 0_{N\times 1}$. It is easy to verify that the solution has the property $A'(0) = \alpha$ and $B'(0) = \beta$.

Dai and Singleton [2000] examine multi-factor affine models with the following structure:

$$\begin{aligned} r_t &= \delta_0 + \delta' Y_t, \\ dY_t &= \mathcal{K}\,(\Theta - Y_t)\,dt + \Sigma\sqrt{S_t}\,dW_t^{\mathbb{Q}}, \end{aligned}$$

where S_t is a diagonal matrix with $[S_t]_{ii} = \alpha_i + Y_t'\beta_i$. Letting $\mathbb{B}$ be the $N \times N$ matrix with i^{th} column given by β_i, they construct admissible affine models– models that give unique, well-defined solutions for $D(t, T)$– by restricting the parameter vector $(\delta_0, \delta, \mathcal{K}, \Theta, \Sigma, \alpha, \mathbb{B})$. Specifically, for given $m = rank(\mathbb{B})$, they introduce the canonical model with the structure

$$\mathcal{K} = \begin{bmatrix} \mathcal{K}^{\mathrm{BB}}_{m\times m} & 0_{m\times(N-m)} \\ \mathcal{K}^{\mathrm{DB}}_{(N-m)\times m} & \mathcal{K}^{\mathrm{DD}}_{(N-m)\times(N-m)} \end{bmatrix},$$

[6] A more general mathematical characterization of affine models is presented in Duffie, Filipovic, and Schachermayer [2001]. See Gourieroux, Monfort, and Polimenis [2002] for a formal development of multi-factor affine models in discrete time.

for $m > 0$, and $\mathcal{K}$ is either upper or lower triangular for $m = 0$,

$$\begin{aligned}
\Theta &= \begin{pmatrix} \Theta^{\mathrm{B}}_{m\times 1} \\ 0_{(N-m)\times 1} \end{pmatrix}, \quad \Sigma = I, \\
\alpha &= \begin{pmatrix} 0_{m\times 1} \\ 1_{(N-m)\times 1} \end{pmatrix}, \quad \mathbb{B} = \begin{bmatrix} I_{m\times m} & B^{\mathrm{BD}}_{m\times(N-m)} \\ 0_{(N-m)\times m} & 0_{(N-m)\times(N-m)} \end{bmatrix},
\end{aligned}$$

with the following parametric restrictions imposed:

$$\begin{aligned}
\delta_i &\geq 0, \; m+1 \leq i \leq N, \\
\mathcal{K}_i\Theta &\equiv \sum_{j=1}^{m} \mathcal{K}_{ij}\Theta_j > 0, \; 1 \leq i \leq m, \; \mathcal{K}_{ij} \leq 0, \; 1 \leq j \leq m, \; j \neq i, \\
\Theta_i &\geq 0, \; 1 \leq i \leq m, \;\; \mathbb{B}_{ij} \geq 0, \; 1 \leq i \leq m, \; m+1 \leq j \leq N.
\end{aligned}$$

Then the sub-family $\mathbb{A}_m(N)$ of affine term structure models is obtained by inclusion of all models that are *invariant* transformations of this canonical model or nested special cases of such transformed models. For the case of N risk factors, this gives $N+1$ non-nested sub-families of admissible affine models.[7] Members of the families $\mathbb{A}_m(N)$ include, among others, Vasicek [1977], Langetieg [1980], Cox, Ingersoll, and Ross [1985], Longstaff and Schwartz [1992], Chen and Scott [1993], Pearson and Sun [1994], Duffie and Singleton [1997], Balduzzi, Das, Foresi, and Sundaram [1996], Balduzzi, Das, and Foresi [1998], Duffie and Liu [2001], and Collin-Dufresne and Goldstein [2001a].

Quadratic Gaussian Models

If $G(Y_t, t; T)$ is quadratic in Y_t, i.e.,

$$G(Y_t, t; T) = A(T-t) + B(T-t)'Y_t + Y_t'C(T-t)Y_t,$$

then it must be the case that $r_t = \alpha + \beta' Y_t + Y_t'\gamma Y_t$, where $\alpha = A'(0)$, $\beta = B'(0)$, and $\gamma = C'(0)$. Without loss of generality, we can assume that $C(\tau)$ is symmetric. Thus γ must also be symmetric and (14) becomes

$$\begin{aligned}
\dot{A} + Y_t'\dot{B} + Y_t'\dot{C}Y_t &= (\alpha + Y_t'\beta + Y_t'\gamma Y_t) + \mu(Y_t,t)'[B(\tau) + 2CY_t] \qquad (18) \\
&+ \mathrm{Trace}\left[\sigma(Y_t,t)'C\sigma(Y_t,t)\right] - \frac{1}{2}\left[[B + 2CY_t]'\,\sigma(Y_t,t)\sigma(Y_t,t)'\,[B + 2CY_t]\right].
\end{aligned}$$

[7] Although the classification scheme was originally used by Dai and Singleton [2000] to characterize the state dynamics under the actual measure, it is equally applicable to the state dynamics under the $\mathbb{Q}$-measure. Note that not all admissible affine *DTSM*s are subsumed by this classification scheme (i.e., not all admissible models are invariant transformations of a canonical model).

In order for (18) to hold for any Y_t, it is sufficient that

1. $\mu(Y_t, t) = a + bY_t$, where the $N \times 1$ vector a and $N \times N$ matrix b are constants;

2. $\sigma(Y_t, t) = \sigma$, where σ is a $N \times N$ constant matrix;

3. $A(\tau)$, $B(\tau)$, and $C(\tau)$ satisfy the following ODEs:

$$\dot{A} = \alpha + a'B - \frac{1}{2}B'\sigma\sigma'B + \mathrm{Trace}\left[\sigma'C\sigma\right], \tag{19}$$

$$\dot{B} = \beta + b'B - 2C'\sigma\sigma'B + 2C'a, \tag{20}$$

$$\dot{C} = \gamma + [b'C + C'b] - 2C'\sigma\sigma'C. \tag{21}$$

For suitable choices of $(\alpha, \beta, \gamma; a, b; \sigma)$, the Ricatti equations (19)–(21) admit a unique solution $(A(\tau), B(\tau), C(\tau))$ under the initial conditions $A(0) = 0$, $B(0) = 0_{N\times 1}$, and $C(0) = 0_{N\times N}$. It is easy to verify that the solution has the property that $A'(0) = \alpha$, $B'(0) = \beta$, and $C'(0) = \gamma$.

The canonical representation of the QG models is simpler than in the case of affine models, because shocks to Y are homoskedastic. To derive their canonical model, Ahn, Dittmar, and Gallant [2002] normalize the diagonal elements of γ to unity, set $\beta = 0$, have $\mathcal{K}$ (the mean reversion matrix for Y) being lower triangular, and have Σ diagonal. They show that the QG models in Longstaff [1989], Constantinides [1992], and Lu [1999] are restricted special cases of their most flexible canonical model.

4 *DTSM*s with Jump Diffusions

Suppose that $r_t = r(Y_t, t)$ is a function of a jump-diffusion process Y with risk-neutral dynamics

$$dY_t = \mu(Y_t, t)\, dt + \sigma(Y_t, t)\, dW_t^{\mathbb{Q}} + \Delta Y_t\, dZ_t, \tag{22}$$

where Z_t is a Poisson counter with risk-neutral intensity λ_t, and the jump size ΔY_t is drawn from a risk-neutral distribution $\nu_t(x) \equiv \nu(x; Y_t, t)$.

No arbitrage implies that the zero-coupon bond price $D(t, T)$ satisfies

$$\left[\frac{\partial}{\partial t} + \mathcal{A}\right] D(t,T) - r(Y,t)\, D(t,T) = 0, \tag{23}$$

where $\mathcal{A}$ is the risk-neutral infinitesimal generator defined by

$$\mathcal{A}f(Y,t) = \mu_t' f_Y + \frac{1}{2}\text{Trace}\left[\sigma_t\sigma_t' f_{YY'}\right] + \lambda_t \int \left[f(Y+x,t) - f(Y,t)\right] d\nu_t(x)$$

for by any test function $f(Y_t, t)$. If $f(Y,t)$ is exponential in Y, i.e., $f(Y,t) = e^{A(t)+B(t)'Y}$, and $\nu(x; Y, t)$ is independent of Y, then

$$\int \left[f(Y_t + x, t) - f(Y_t, t)\right] d\nu_t(x) = C(B(t), t)\, f(Y_t, t),$$

where $C(u,t) = \int e^{u'x} d\nu_t(x)$ is the Laplace transform of the jump distribution. This observation underlies many of the analytic pricing relations that have been derived.

Specifically, for the case of affine jump-diffusions, analytic expressions for zero-coupon bond prices are obtained under the following assumptions: r_t, μ_t, $\sigma_t\sigma_t'$, and λ_t are affine in Y_t, and the Laplace transform of the distribution $\nu_t(x)$, $\theta(u,t) = \int e^{ux} d\nu_t(x)$, depends at most on u and t. With these assumptions, $D(t,T) = e^{A(t,T)+B(t,T)Y_t}$, with the coefficients $A(t,T)$ and $B(t,T)$ again determined by a set of *ODE*s (Duffie, Pan, and Singleton [2000]).

Ahn and Thompson [1988] extend the equilibrium framework of Cox, Ingersoll, and Ross [1985] to the case of Y following a square-root process with jumps. Brito and Flores [2001] develop an affine jump-diffusion model, and Piazzesi [2001] develops a mixed affine-QG model, in which the jumps are linked to the resetting of target interest rates by the Federal Reserve.

5 *DTSM*s with Regime Shifts

The "regime switching" framework was introduced by Hamilton [1989] to model business cycle fluctuations in real variables, and was subsequently adapted by Gray [1996] to model short-term interest rates with state-dependent regime switching probabilities. Only recently has Hamilton's framework been extended to bond pricing (see, e.g., Naik and Lee [1997], Evans [2000], Landen [2000], and Bansal and Zhou [2002].) Following Dai and Singleton [2002], we present a continuous-time formulation of fixed-income pricing with regime shifts.

The evolution of "regimes" is governed by an $(S+1)$-state continuous-time *conditionally Markov chain* $s_t : \Omega \to \{0, 1, \ldots, S\}$ with a $(S+1) \times (S+1)$

rate or generator matrix R_t with the property that all rows sum to zero.[8] Intuitively, $R_t^{ij}\, dt$, $i \neq j$, which may be state-dependent (i.e., $R_t = R(Y_t, t)$), represents the probability of moving from regime i to regime j over the next interval dt, and $1 + R_t^{ii}\, dt$ is the probability of staying in regime i in the next interval dt.

The relation between r and Y may be indexed by regime in that $r_t \equiv r(s_t; Y_t, t)$. Additionally, The state vector Y_t is a solution to

$$dY_t = \mu^j(Y_t, t)\, dt + \sigma^j(Y_t, t)\, dW_t^{\mathbb{Q}},$$

with the conditional moments of Y_t indexed by the regime j. Though these moments may change across regimes, the sample path of Y remains continuous. For simplicity, we assume that regime shifts and Brownian shocks are mutually independent.

To facilitate analytical development of bond pricing under regime switching, we introduce $(S + 1)$ regime indicator functions: $z_t^j = 1_{s_t=j}$, $j = 0, 1, \ldots, S$. Clearly, $E[dz_t^j | s_t, Y_t] = R_t^j\, dt$, therefore $m_t^j \equiv z_t^j - \int_0^t R_u^j\, du$ is a Martingale. A useful property of these random variables is that, any regime-dependent variable $\Phi(s_t; Y_t, t)$ can be written as

$$\Phi(s_t; Y_t, t) \equiv \sum_{j=0}^{S} z_t^j\, \Phi^j(Y_t, t), \tag{24}$$

where $\Phi^j(Y_t, t) \equiv \Phi(s_t = j; Y_t, t)$. Conversely, given a set of $(S+1)$ functions $\Phi^j(Y_t, t)$, $j = 0, 1, \ldots, S$, a regime-dependent random variable $\Phi(s_t; Y_t, t)$ can be defined through equation (24). In particular, each column of the matrix R_t defines a regime-dependent random variable $R_t^j \equiv R^j(s_t; Y_t, t) = \sum_{i=0}^{S} z_t^i R_t^{ij}$, $j = 0, 1, \ldots, S$. Furthermore, the drift and diffusion functions of the state vector under the $(S + 1)$ different regimes can be represented by two regime-dependent random variables: $\mu(s_t; Y_t, t) \equiv \sum_{j=0}^{S} z_t^j \mu^j(Y_t, t)$ and $\sigma(s_t; Y_t, t) \equiv \sum_{j=0}^{S} z_t^j \sigma^j(Y_t, t)$.

Writing $D(t, T) \equiv \sum_{j=0}^{S} z_t^j D^j(t, T)$, where $D^j(t, T) \equiv D(s_t = j; Y_t, t; T)$,

[8] See Bielecki and Rutkowski [2001] for a formal definition of a "conditionally Markov chain", where R_t is also referred to as the "conditional infinitesimal generator" of s_t under a proper extension of the probability measure $\mathbb{Q}$, given the σ-field $\mathcal{F}_t$.

Ito's lemma implies that

$$\begin{aligned}
\frac{dD(t,T)}{D(t,T)} &= \mu_{t,T}^D\,dt + \sigma_{t,T}^{D'}\,dW_t^{\mathbb{Q}} - \sum_{j=0}^{S}\left[1 - \frac{D^j(t,T)}{D(t,T)}\right]\left[dz_t^j - R_t^j\,dt\right], \\
\mu_{t,T}^D &= \frac{1}{D(t,T)}\left[\frac{\partial}{\partial t} + \mathcal{A}\right]D(t,T) + \sum_{j=0}^{S} R_t^j \frac{D^j(t,T)}{D(t,T)}, \\
\sigma_{t,T}^D &= \sigma(s_t;Y_t,t)'\frac{\partial}{\partial Y_t}\log D(t,T), \quad \mathcal{A} = \sum_{j=0}^{S} z_t^j \mathcal{A}^j, \\
\mathcal{A}^j &= \mu^j(Y_t,t)'\frac{\partial}{\partial Y} + \frac{1}{2}\text{Trace}\left[\sigma^j(Y_t,t)\,\sigma^j(Y_t,t)'\,\frac{\partial^2}{\partial Y\,\partial Y'}\right], \; 0 \le j \le S.
\end{aligned}$$

No arbitrage requires that $\mu_{t,T}^D = r_t$ for all $0 \le s_t \le S$ and all $Y_t = Y$ in the admissible state space. This implies $(S+1)$ partial differential equations:

$$\left[\frac{\partial}{\partial t} + \mathcal{A}^i\right]D^i(t,T) + \sum_{j=0}^{S} R_t^{ij} D^j(t,T) - r_t^i D^i(t,T) = 0, \quad 0 \le i \le S,$$

where $r_t^i \equiv r(s_t = i; Y_t, t)$, $0 \le i \le S$. In general, the matrix R_t is not diagonal. Therefore the above PDEs are coupled and the $(S+1)$ functions $(D^i(t,T) : 0 \le i \le S)$ must be solved jointly. The boundary condition is $D(T,T) = 1$ for all s_T, which is equivalent to $(S+1)$ boundary conditions: $(D^i(T,T) = 1 : 0 \le i \le S)$.

Dai and Singleton [2002] derive a closed-form solution for $D(t,T)$ in this framework under two additional assumptions. First, under $\mathbb{Q}$, the state dynamics for each regime i is described by

$$\begin{aligned}
r_t^i &\equiv r(s_t = i; Y_t, t) = \delta_0^i + Y_t'\delta_Y, \\
\mu_t^i &\equiv \mu(s_t = i; Y_t, t) = \kappa(\theta^i - Y_t), \\
\sigma_t^i &\equiv \sigma(s_t = i; Y_t, t) = \text{diag}(\alpha_k^i + Y_t'\beta_k)_{k=1,2,\ldots,N}.
\end{aligned}$$

with regime dependence entering through the scalar constant δ_0^i and α_k^i and the $N \times 1$ constant vectors θ^i (the $N \times N$ constant matrix κ and the $N \times 1$ vectors δ_Y and β_k are regime independent). Second, the risk-neutral rate matrix R_t is state-independent. Under these assumptions, the discount functions are given by

$$D(i;t,T) = e^{-A^i(T-t) - Y_t'B(T-t)}, \; 0 \le i \le S,$$

where $A^i(\cdot)$ and $B(\cdot)$ are explicitly known up to a set of *ODE*s. Regime-dependence of bond prices is captured through the "intercept" term $A^i(T-t)$; the derivative of zero-coupon bond yields with respect to Y does not depend on the regime.

The one-factor, two-regime model developed in Naik and Lee [1997], with $\beta_k^j = 0$ (for $k = 1$ and $j = 0, 1$), is a special case. Evans [2000] and Bansal and Zhou [2002] develop discrete-time, regime-switching models, with regime-dependent κ^i and β_t^i. The continuous-time limit of their models are special cases of the above general pricing framework.

6 *DTSM*s with Rating Migrations

With some technically minor, but conceptually important, modifications the framework developed in the last section can be adapted to model defaultable term structures with rating migrations. To illustrate this, we consider the case of a single economic regime and $S+1$ credit rating classes. The rating history of a defaultable bond is represented by a conditionally Markov chain s_t taking values in the set of rating classes $\{0, 1, 2, \ldots, S\}$, with risk-neutral rate matrix R_t. The mathematical constructions of s_t and R_t are exactly the same as in Section 5. Without loss of generality, we will designate S as the default state. As usual, we will assume that the default state is absorbing, so that $R_t^{Sj} = 0$, $0 \leq j \leq S$.

Letting, for each rating class j, $(B^j(t,T) : T \geq t)$ denote the rating-specific discount function at time t, the price of a defaultable zero-coupon bond can expressed as

$$B(t,T) \equiv B(s_t; Y_t, t; T) \equiv \sum_{j=0}^{S-1} z_t^j B^j(t,T) + z_t^S B^S(s_{t-}; Y_t, t; T),$$

where $z_t^j \equiv 1_{\{s_t=j\}}$ is now interpreted as the rating indicator (at time t, the bond is in the rating class j if and only if $z_t^j = 1$). The bond price in the default state is treated separately in order to account for recovery. The nature of the defaultable bond pricing relations depends on the nature of the recovery assumption. We begin by developing pricing relations under the assumption of *fractional recovery of market value*, proposed by Duffie and Singleton [1999], followed by a parallel development based on the assumption of *fractional recovery of face value*, proposed in various forms by Jarrow, Lando, and Turnbull [1997], Duffie [1998], and Bielecki and Rutkowski [2000].

6.1 Fractional Recovery of Market Value

If, in the event of default, a fraction, $1 - l(k; Y_t, t)$, of pre-default market value of the bond in rating class k is recovered, then

$$B^S(s_{t-}; Y_t, t; T) = [1 - l(s_{t-}; Y_t, t)] B(s_{t-}; Y_t, t; T). \tag{25}$$

We assume that while $l(k; Y_t, t)$, the *loss rate*, may be state-dependent, it does not depend explicitly on the pre-default bond price. To characterize the defaultable discount functions $(B^j(t,T) : 0 \le j \le S-1, T \ge t)$, consider a defaultable bond rated $i \neq S$ at time t, with price $B^i(t,T)$. In the next instant, $t + dt$, the rating may change to j with probability π_t^{ij}, where $\pi_t^{ij} = R_t^{ij} dt$ for $0 \le j \neq i \le S$ and $\pi_t^{ii} = 1 + R_t^{ii} dt$. The risk-neutral instantaneous expected return on the bond is therefore given by

$$\begin{aligned} \mu^B(i; Y_t, t; T) &= \lim_{dt \to 0} \frac{1}{B^i(t,T) dt} \\ &\times \left[\sum_{j=0}^{S-1} [B^j(t+dt, T) - B^i(t,T)] \, \pi_t^{ij} + [B^S(i; Y_t, t; T) - B^i(t,T)] \, \pi_t^{iS} \right] \\ &= \frac{1}{B^i(t,T)} \left\{ \left[\frac{\partial}{\partial t} + \mathcal{A} \right] B^i(t,T) + \sum_{j=0}^{S-1} \hat{R}_t^{ij} B^j(t,T) \right\}, \end{aligned}$$

where $\hat{R}_t^{ij} = R_t^{ij}$ for $j \neq i$, $\hat{R}_t^{ii} = -\sum_{j \neq i}^{S-1} R_t^{ij} - R^{iS} l_t^i$, $l_t^i \equiv l(s_{t-} = i; Y_t, t)$, and $\mathcal{A}$ is the infinitesimal generator for the state vector Y_t. No arbitrage requires that $\mu^B(i; Y_t, t; T) = r_t$ for all $0 \le i \le S-1$. Thus, the defaultable discount functions are jointly determined by the *PDE*s:

$$\left[\frac{\partial}{\partial t} + \mathcal{A} \right] B^i(t,T) + \sum_{j=0}^{S-1} \hat{R}_t^{ij} B^j(t,T) - r B^i(t,T) = 0, \;\; 0 \le i \le S-1. \tag{26}$$

The matrix $\hat{R}_t$ in equation (26) has an intuitive interpretation:[9] it is obtained from the risk-neutral rate matrix R_t by shifting a portion, $1 - l_t$, of the risk-neutral default intensity to the risk-neutral "no-transition" intensity, $\hat{R}^{ii} = R^{ii} + R^{iS}(1 - l_t^i)$. The "thinning" of the default intensity, with compensated

[9] Note that, except under full recovery, the $(S \times S)$ "modified rate matrix" $\hat{R}_t$ is not a valid transition matrix, because its rows do not sum to zero.

adjustment to the "no-transition" intensity, captures the effect of default recovery.

Letting $\mathbb{B}(t,T) = \{B^i(t,T)\}_{i=0}^{S-1}$ denote the $(S \times 1)$ vector of defaultable discount factors, (26) implies that

$$\mathbb{B}(t,T) = E_t^{\mathbb{Q}}\left[\int_t^T (\hat{R}_t - r_t)\,\mathbb{B}(u,T)\,du + \mathbb{B}(T,T)\right],$$

the solution of which, when it exists, is given by

$$\mathbb{B}(t,T) = E_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u\,du}\,\hat{\Psi}(t,T)\,\mathbb{B}(T,T)\right], \tag{27}$$

where $\hat{\Psi}(t,T)$ solves the backward differential equation

$$d\hat{\Psi}(t,T) = -\hat{R}_t\hat{\Psi}(t,T)\,dt, \quad \hat{\Psi}(T,T) = \mathbf{I}_{S\times S}. \tag{28}$$

Lando [1998] first derived (27) under the assumption of zero recovery ($l_t^i =$ 100% for $\forall i$). Li [2000] and Duffie and Singleton [2001] extended Lando's result to the case of nonzero fractional recovery of market value.

Of important practical interest is the question of under what conditions (26) or (27) admit an analytic solution. If there is only one rating class, as in Duffie and Singleton [1999], an analytical solution obtains under an affine structure. Specifically, in this case, $\hat{R}_t = -R_t l_t$ is a scalar, where R_t is the default intensity and l_t is the loss rate upon default. It follows that $\hat{\Psi}(t,T) = e^{-\int_t^T R_u l_u du}$, and the defaultable discount function is given by

$$\mathbb{B}(t,T) = E_t^{\mathbb{Q}}\left[e^{-\int_t^T (r_u + R_u l_u)\,ds}\,\mathbb{B}(T,T)\right].$$

Duffie and Singleton [1999] show that if the "risk-adjusted" short rate $r_t + R_t l_t$ is an affine function of an affine diffusion Y, then $\mathbb{B}(t,T)$ is exponential affine in Y. This result is easily extended to the case where there are multiple rating classes, but there is no migration across non-default ratings (i.e., an issuer can only migrate to default).

With multiple ratings and migration across rating classes, the backward differential equation (28) typically does not admit an analytic solution. This is because the matrices $\hat{R}_t$ and $\hat{\Psi}(t,T)$ do not commute in general. To circumvent this difficulty, Lando [1998] assumed zero recovery and that (i) the risk-neutral rate matrix R_t admits an eigen-value decomposition

$R_t = JG_tJ^{-1}$, where J is a constant $(S+1)\times(S+1)$ matrix and G_t is a diagonal $(S+1)\times(S+1)$ matrix (henceforth we refer to this type of decomposition as a *Lando decomposition*); (ii) the state vector Y follows an affine process under the $\mathbb{Q}$; and (iii) the riskless rate r_t and the diagonal elements of the matrix of G_t are affine functions of Y. Under these assumptions,

$$B^i(t,T) = \sum_{j=0}^{S-1} [J^{-1}]^{ij}\, e^{(-\gamma_0^j - Y_t'\gamma_Y^j)}, \tag{29}$$

where γ_0^j and γ_Y^j are explicitly known up to a set of *ODE*s. This follows from the observation that $\mathbb{A}(t,T) = J^{-1}\mathbb{B}(t,T)$ satisfies

$$\frac{\partial \mathbb{A}(t,T)}{\partial t} + \tilde{\mu}_t' \frac{\partial \mathbb{A}(t,T)}{\partial Y} + \frac{1}{2}\text{Trace}\left[\sigma_t\sigma_t' \frac{\partial^2 \mathbb{A}(t,T)}{\partial Y \partial Y'}\right] + G_t\mathbb{A}(t,T) - r_t\mathbb{A}(t,T) = 0, \tag{30}$$

with boundary conditions $\mathbb{A}(T,T) = J^{-1}\mathbb{B}(T,T)$. Since these equations are decoupled, each element of $\mathbb{A}(t,T)$ can be solved individually. Furthermore, under the assumed affine structure, $\mathbb{A}(t,T)^j = e^{-\gamma_0^j - Y_t'\gamma^j}$, where γ_0^j and γ^j depend in general on j, because the diagonal elements of G_t need not be the same across different rating classes.

Inspired by Lando [1998], Li [2000] shows that the pricing formula (29) obtains for the nonzero-recovery case under the following assumptions: (i) the defective rate matrix $\hat{R}_t$ admits a Lando-decomposition: $\hat{R}_t = \hat{J}\hat{G}_t\hat{J}^{-1}$, where $\hat{J}$ is a constant $S\times S$ matrix and $\hat{G}_t$ is a diagonal $S\times S$ matrix; (ii) Y_t is affine under the risk-neutral measure, in the sense of Duffie and Kan [1996]; and (iii) r_t and the diagonal elements of G_t are affine in Y_t. To see that Li [2000] is a direct extension of Lando [1998], note first that $R_t = JG_tJ^{-1}$ if and only if $\hat{R}_t = \hat{J}\hat{G}_t\hat{J}^{-1}$, where

$$J = \begin{bmatrix} \hat{J} & \mathbf{1} \\ \mathbf{0}' & 1 \end{bmatrix}, \quad J^{-1} = \begin{bmatrix} \hat{J}^{-1} & -\hat{J}^{-1}\mathbf{1} \\ \mathbf{0}' & 1 \end{bmatrix}, \quad G_t = \begin{bmatrix} \hat{G}_t & \mathbf{0} \\ \mathbf{0}' & 0 \end{bmatrix}.$$

It follows immediately that R_t admits a Lando-decomposition if and only if $\hat{R}_t$ admits a Lando-decomposition. Letting $\Psi(t,T)$ be the solution to the backward differential equation $d\Psi(t,T) = -R_t\Psi(t,T)\,dt$, with boundary condition $\Psi(T,T) = \mathbf{I}_{(S+1)\times(S+1)}$, it is easy to see that $\hat{\Psi}(t,T)$ is the upper-left $S\times S$ sub-matrix of $\Psi(t,T)$.

6.2 Fractional Recovery of Par, Payable at Maturity

Suppose that, in the event of default, a fraction $\omega(s_{t-}; Y_t, t)$ of face value is recovered, and that payment of $\omega(s_{t-}; Y_t, t)$ is postponed until the original maturity date of the defaultable bond. Then

$$B^S(s_{t-}; Y_t, t; T) = \omega(s_{t-}; Y_t, t)\, D(t, T), \tag{31}$$

the recovery at the default time, is simply the recovery at maturity $\omega(s_{t-}; Y_t, t)$ discounted back to t by the default-free discount factor $D(t, T)$. For this case of zero-coupon bonds, this recovery convention agrees with that proposed by Jarrow, Lando, and Turnbull [1997] in which bond holders recover, at the time of default, a fraction $(\omega(s_{t-}; Y_t, t))$ of an otherwise equivalent Treasury bond $(D(t, T))$.

Letting $\omega_t^k \equiv \omega(k; Y_t, t)$, $\forall k$, under recovery assumption (31), the defaultable discount functions solve the following *PDE*s:

$$\left[\frac{\partial}{\partial t} + \mathcal{A}\right] B^k(t, T) + \sum_{j=0}^{S-1} R_t^{kj} B^j(t, T) - (r_t + R_t^{kS}) B^k(t, T) + \omega_t^k R_t^{kS} D(t, T) = 0, \tag{32}$$

with boundary condition $B^k(T, T) = 1$, $0 \leq k \leq S - 1$. As far as we are aware, the solution to the joint *PDE*s (32) with rating migrations ($S > 1$) has yet to be developed. However, for the special case of $S = 1$ (no rating migration) and state-independent (constant) ω,

$$B(t, T) = E_t^{\mathbb{Q}}\left[e^{-\int_t^T (r_u + \lambda_u)\, du}\right] + \omega \int_t^T E_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s\, ds} e^{-\int_t^u \lambda_v dv}\, \lambda_u\, du\right], \tag{33}$$

where the default intensity is $R_t^{0S} = \lambda_t$. Each of the expectations in (33) is known in closed-form when r and λ are affine functions of an affine diffusion, so $B(t, T)$ is known up to a one-dimensional numerical integration.[10] Jarrow, Lando, and Turnbull [1997]'s model is the special case of (33) in which r_t and λ are statistically independent under $\mathbb{Q}$, in which case

$$B(t, T) = D(t, T)\left[Q(t, T) + \omega(1 - Q(t, T))\right], \tag{34}$$

where $Q(t, T) = E_t^{\mathbb{Q}}\left[e^{-\int_t^T \lambda_u du}\right]$ is the risk-neutral *survival probability*; i.e., the probability under $\mathbb{Q}$ that default occurs after T.

[10] Even with state-dependent ω, tractability need not be lost. For instance, if $\omega(Y_t, t) = e^{\gamma_0 + \gamma_Y' Y_t}$, then all of the expectations in (33) are still known in closed form in the case of an affine state process.

6.3 Fractional Recovery of Par, Payable at Default

Duffie [1998] adopted an alternative timing convention for recovery: a fraction $\omega(s_{t-}; Y_t, t)$ of par is recovered and paid at the time of default,

$$B^S(s_{t-}; Y_t, t; T) = \omega(s_{t-}; Y_t, t). \tag{35}$$

Under this assumption, the defaultable discount functions jointly solve, for $0 \le k \le S - 1$,

$$\left[\frac{\partial}{\partial t} + \mathcal{A}\right] B^k(t,T) + \sum_{j=0}^{S-1} R_t^{kj} B^j(t,T) - (r_t + R_t^{kS}) B^k(t,T) + \omega_t^k R_t^{kS} = 0, \tag{36}$$

with boundary conditions $B^k(T,T) = 1$. Again, we are not aware of any explicit solutions of (36) in the presence of rating migrations. For the special case of $S = 1$, this model gives an expression that is identical to (33), except that each of the expectations in the second term are replaced by $E_t^{\mathbb{Q}}\left[e^{-\int_t^u (r_v + \lambda_v) dv} \lambda_u \, du\right]$. Thus, as shown by Duffie [1998], this model also admits closed-form expressions for the $B(t,T)$ up to a numerical integration.

6.4 Pricing Defaultable Coupon Bonds

Up to this point we have been focusing on defaultable zero-coupon bonds. Intuitively, a coupon bond with rating i should have a price equal to the present value of its promised cash flows, discounted by the defaultable discount function $B^i(t,T)$. This is true under the assumption that the loss rates $l^i(Y_t, t)$ depend only on the rating class i and the economy-wide state vector, but not on characteristics of the cash flows of the bond being priced. In this case, given the Y_t, all defaultable securities with the same rating i lose the same fraction $l^i(Y_t, t)$ if default occurs at t.

When the loss rates depend on cash flow characteristics, then strictly speaking, there does not exist a universal defaultable discount function. Instead, each security (or subset of securities with particular cash flow patterns) will have its own set of defaultable discount functions, reflecting the unique impact of default on its pricing under non-default states.

For the recovery of market value model, the loss rate $l^i(Y_t, t)$ is applied uniformly to the construction of the $B^i(t,T)$ for discounting coupons and face value. On the other hand, for the recovery of face value models, in constructing the discount factors $B^i(t,T)$, it is typically assumed that $l^i(Y_t,t) = 1$ when discounting coupon payments (zero recovery of coupon payments) and $0 < l^i(Y_t,t) < 1$ when discounting the face value of a bond.

Finally, we note that, in the case of coupon bonds, the assumption in Jarrow, Lando, and Turnbull [1997] that bond holders recover a fraction of an otherwise equivalent treasury bond represents a third and distinct recovery convention. For creditors now recover a fraction of both face value and promised future coupons through their recovery of a coupon-paying treasury bond. Outside of the special case they examined, this recovery convention has not, to our knowledge, been widely studied.

6.5 Pricing Eurodollar Swaps

We can, and the literature often does, treat LIBOR-based plain-vanilla swaps as a special case of the pricing relations developed under the fraction-recovery-of-market-value assumption. We let $(B(t,T) : T \geq t)$ be the discount curve for the LIBOR rating class. Following Duffie and Singleton [1997], if (i) both counterparties have the same credit rating as LIBOR issuers and maintain this rating up to the time of any defaults ("refreshed" LIBOR quality issuers); (ii) upon default, the counterparty who is in the money recovers a fraction l_t of the marked-to-market value of the swap, where l_t does not depend on cash flow characteristics; and (iii) the floating index is a LIBOR rate with tenor matching the payment frequency, then the swap rate $s(t,T)$ on any reset date $t < T$ is equal to the par coupon yield with maturity T for the LIBOR rating class:

$$s(t,T) = \frac{1}{\delta} \frac{1 - B(t,T)}{\sum_{j=1}^{(T-t)/\delta} B(t, t+\delta j)}, \tag{37}$$

where δ is the length of each payment period, typically three months or half a year.

The presumption of "refreshed" LIBOR quality clearly makes (37) an approximation to the true pricing relation. In fact, the counterparties to a swap may have asymmetric credit qualities and, even if this is not true at the inception of a swap, the relative qualities of the counterparties may change over time. In these cases of two-sided credit risk, the effective default arrival intensity and recovery for pricing depends on current market price of the swap (that is, on which counterparty the swap is "in the money" to.) Duffie and Huang [1996] and Duffie and Singleton [1997] treat these issues in the case of a single ratings class. Huge and Lando [1999] discuss swap pricing with two-sided risk in a ratings-based model.

Gupta and Subrahmanyam [2000] document and explain the mispricing of Eurodollar swaps when they are priced off the Eurodollar futures strips. Since futures rates are higher than the forward rates due to marking to market of futures contracts, swap rates computed by treating futures rates as forward rates are higher than the true "fair market" swap rates. The difference is referred to as the "convexity bias".

7 Pricing of Fixed-Income Derivatives

This section overviews the pricing of fixed-income derivatives using *DTSM*s, forward-rate based models, and models that adopt specialized "pricing measures" to simplify the computation of derivatives prices.

7.1 Derivatives Pricing using *DTSM*s

As with term structure modeling, much of the academic literature on derivatives pricing using *DTSM*s has focused on affine and *QG* models. The tractability of affine models is captured by the "extended transform" result of Duffie, Pan, and Singleton [2000] which gives

$$\mathcal{G}(Y_t, t; T; \rho_0, \rho_1, v_0, v_1, u) = E^{\mathbb{Q}}\left[e^{-\int_t^T (\rho_0 + \rho_1' Y_s) ds} (v_0 + v_1' Y_T) e^{u' Y_T} \,\middle|\, Y_t \right], \quad (38)$$

for Y following an affine jump-diffusion, in closed-form. Specifically,

$$\mathcal{G}(Y_t, t; T; \rho_0, \rho_1, 1, 0, u) = e^{\alpha(\tau) + \beta(\tau)' Y_t}, \quad (39)$$

$$\mathcal{G}(Y_t, t; T; \rho_0, \rho_1, 0, v, u) = e^{\alpha(\tau) + \beta(\tau)' Y_t} [A(\tau) + B(\tau)' Y_t], \quad (40)$$

where $\alpha(\tau)$, $\beta(\tau)$, $A(\tau)$, and $B(\tau)$ are all explicitly known up to a set of *ODE*s (and, again, $\tau = (T - t)$). See Bakshi and Madan [2000] and Chacko and Das [2001] for related pricing results.

The Green's function can be obtained for affine jump-diffusion models by inverse Fourier transform of $\mathcal{G}$, based on the fact that

$$\mathcal{G}(Y_t, t; T; \rho_0, \rho_1, 1, 0, u) = \int e^{u' Y} G(Y_t, t; Y, T)\, dY.$$

It follows that European-style derivatives prices can be easily computed by integrating the product of the payoff function and the Green's function. This observation underlies the pricing formulas for various fixed-income derivatives discussed, for example, in Buttler and Waldvogel [1996], Das and Foresi [1996], Nunes, Clewlow, and Hodges [1999], Duffie, Pan, and Singleton [2000], Bakshi and Madan [2000], and Chacko and Das [2001].

From (38) we see that payoffs that are exponential affine in the state or the product of an affine and exponential affine function of Y are accommodated by these pricing models. This covers options on zero-coupon bonds, for example, because the prices of zero-coupon bonds are exponential affine functions of Y. However, these results do not cover the most common form of bond option, namely, options on coupon bonds. Jamshidian [1987] derived coupon bond option pricing formulas for the case of one-factor models in which zero-coupon bond prices are strictly monotonic functions of the (one-dimensional) state. Gaussian and square-root diffusion models are examined in Jamshidian [1989] and Longstaff [1993], respectively.

Taking a different approach, Wei [1997] showed that the price of a European option on a coupon bond is approximately proportional to the price of an option on a zero-coupon bond with maturity equal to the *stochastic duration* (Cox, Ingersoll, and Ross [1979]) of the coupon bond. Subsequently Munk [1999] extended Wei's Stochastic Duration approximation to the general case of multi-factor affine models. These approximations work very well for options that are either far in or far out of the money, while having relatively large approximation errors (though still absolutely small) for options that are near the money.

Approximate pricing formulas for coupon options that are computationally fast and very accurate over a wider range of moneyness, including nearly at the money options, were proposed by Collin-Dufresne and Goldstein [2001b] and Singleton and Umantsev [2002]. The former approach uses an Edgeworth expansion of the probability distribution of the future price of a coupon bond. The latter exploits the empirical observation that the optimal exercise boundary for coupon bond options in affine $DTSM$s can be accurately approximated by straight line segments.

Leippold and Wu [2001] derive the counterpart to the transform (39) for QG models which allows them to price derivatives with payoffs that are exponential quadratic functions of the state (which includes zero-coupon bonds as a special case). The approximate pricing of options on coupon bonds developed in Collin-Dufresne and Goldstein [2001b] and Singleton and Umantsev

[2002] for affine models may be adapted to QG models, though to our knowledge this adaptation has not been developed.

7.2 Derivatives Pricing using Forward Rate Models

A significant part of the literature on fixed-income pricing has focused on forward-rate models in which the terminal payoff $Z(T)$ is assumed to be completely determined by the discount function $(D(t,T) : T \geq t)$ (as in Ho and Lee [1986]), or equivalently, the forward curve $(f(t,T) : T \geq t)$ (as in Heath, Jarrow, and Morton [1992]) defined by

$$f(t,T) = -\frac{\partial \log D(t,T)}{\partial T}, \quad \text{for any } T \geq t. \tag{41}$$

The time t price of a fixed-income derivative with terminal payoff $Z(T) = Z(f(T,T+x) : x \geq 0)$ is then given by

$$Z(t) = E^{\mathbb{Q}} \left[e^{-\int_t^T f(u,u)du} Z(f(T,T+x) : x \geq 0) \middle| f(t,t+x) : x \geq 0 \right]. \tag{42}$$

For this model to be free of arbitrage opportunities, Heath, Jarrow, and Morton [1992] show that the risk-neutral dynamics of the forward curve must be given by

$$df(t,T) = \left[\sigma(t,T) \int_t^T \sigma(t,u)\,du \right] dt + \sigma(t,T)\,dW^{\mathbb{Q}}(t), \text{ for any } T \geq t, \tag{43}$$

and for a suitably chosen volatility function $\sigma(t,T)$. This forward-rate representation of prices is particularly convenient in practice, because the forward curve can be taken as an input for pricing derivatives and, once the functions $\sigma(t,T)$, for all $T \geq t$, are specified, then so are the processes $f(t,T)$ under $\mathbb{Q}$. This approach, as typically used in practice, allows the implied r_t and Λ_t to follow general Ito processes (up to mild regularity conditions); there is no presumption that the underlying state is Markov in this forward-rate formulation. Additionally, taking $(f(t,T) : T \geq t)$ as an input for pricing means that a forward-rate based model can be completely agnostic about the behavior of yields under the actual data generating process.

Building off of the original insights of Heath, Jarrow, and Morton, a variety of different forward-rate based models have been developed and used in practice. The finite dimensionality of $W^{\mathbb{Q}}$ was relaxed by Musiela [1993], who models the forward curve as a solution to an infinite-dimensional stochastic partial differential equation (*SPDE*) (see Da Prato [1992] and Pardoux [1993] for some mathematical characterizations of the *SPDE*). Specific formulations of infinite-dimensional *SPDE*s have been developed under the labels of "Brownian sheets" (Kennedy [1994]), "random fields" (Goldstein [2000]), and "stochastic string shocks" (Santa-Clara and Sornette [2001]). The high dimensionality of these models gives a better fit to the correlation structure, particularly at high frequencies. Since solutions to *SPDE*s can be expanded in terms of a countable basis (cylindrical Brownian motions – see, e.g., Da Prato [1992] and Cont [1999]), the *SPDE* models can also be viewed as infinite-dimensional factor models. Though these formulations are mathematically rich, in practice, they often add little generality beyond finite-state forward-rate models, because practical considerations often lead modelers to work with a finite-dimensional $W^{\mathbb{Q}}$.

Key to all of these formulations is the specification of the volatility function, since this determines the drift of the relevant forward rates under $\mathbb{Q}$ (as in Heath, Jarrow, and Morton [1992]). Amin and Morton [1994] examine a class of one-factor models with the volatility function given by

$$\sigma(t,T) = [\sigma_0 + \sigma_1(T-t)]\, e^{-\lambda(T-t)} f(t,T)^{\gamma}. \tag{44}$$

This specification nests many widely used volatility functions, including the continuous-time version of Ho and Lee [1986] ($\sigma(t,T) = \sigma_0$), the lognormal model ($\sigma(t,T) = \sigma_0 f(t,T)$), and the Gaussian model with time-dependent parameters as in Hull and White [1993]. When $\gamma \neq 0$, (44) is a special case of the "separable specification" $\sigma(t,T) = \xi(t,T)\eta(t)$ with $\xi(t,T)$ a deterministic function of time and $\eta(t)$ a possibly stochastic function of Y. The state vector may include the current spot rate $r(t)$ (see, e.g., Jeffrey [1995]), a set of forward rates with fixed time-to-maturity, or an autonomous Markovian vector of latent state variables (Cheyette [1994], Brace and Musiela [1994], and Andreasen, Collin-Dufresne, and Shi [1997]). In practice, the specification of $\eta(Y,t)$ has been kept simple to preserve computational tractability, often simpler than the specifications of stochastic volatility in yield-based models. On the other hand, Y often has a large dimension (many forward rates are used) and $\xi(t,T)$ is given a flexible functional form. Thus, there is the risk with forward-rate models of mis-specifying the dynamics through restrictive specifications of η, while "over-fitting" to current market information through the specification of $\xi(t,T)$.

More discipline, as well as added computational tractability, is obtained by imposing a Markovian structure on the forward rate processes. Two logically distinct approaches to deriving Markov *HJM* models have been explored in the literature. Ritchken and Sankarasubramanian [1995], Bhar and Chiarella [1997], and Inui and Kijima [1998] ask under what conditions, taking as given the current forward rate curve, the evolution of future forward rates can be described by a Markov process in an *HJM* model. These papers show that an N-factor *HJM* model can be represented, under certain restrictions, as a Markov system in $2N$ state variables. While these results lead to simplifications in the computation of the prices of fixed-income derivatives, they do not build a natural bridge to Markov, spot-rate based *DTSM*s. The distributions of both spot and forward rates depend on the date and shape of the initial forward rate curve.

Carverhill [1994], Jeffrey [1995], and Bjork and Svensson [2001] explore conditions under which an N-factor *HJM* model implies an N-factor Markov representation of the short rate r. In the case of $N = 1$, the question can be posed as: Under what conditions does a one-factor *HJM* model – that by construction matches the current forward curve – imply a diffusion model for r with drift and volatility functions that depend only on r and t? Under the assumption that the instantaneous variance of the T-period forward rate is a function only of (r, t, T), $\sigma_f^2(r, t, T)$, Jeffrey proved the remarkable result that $\sigma_f^2(r, t, T)$ must be an affine function of r (with time-dependent coefficients) in order for r to follow a Markov process. Put differently, his result essentially says that the only family of "internally consistent" one-factor *HJM* models (see also Bjork and Christensen [1999]) that match the current forward curve and imply a Markov model for r is the family of affine *DTSM*s with time-dependent coefficients. Bjork and Svensson discuss the multi-factor counterpart to Jeffrey's result.

7.3 Defaultable Forward Rate Models with Rating Migrations

No-arbitrage restrictions on the risk-neutral drifts of defaultable forward rates have been derived in rating migration models (see, e.g., Schonbucher [1998], Duffie [1998], Bielecki and Rutkowski [2000], and Acharya, Das, and Sundaram [2002]). The resulting risk-neutral specifications of the defaultable forward curves can be used to construct arbitrage-free pricing models for credit derivatives, in very much the same way as equation (43) can be used to construct an arbitrage-free pricing model for default-free interest rate derivatives. In general, the no-arbitrage restrictions depend on the recovery scheme for the underlying default pricing model. We illustrate this by giving a heuristic derivation of the no-arbitrage restrictions under two widely used recovery schemes.

For a partition $t = T_0 < T_1 < \dots T_{N-1} < T_N \equiv T$ of the the time interval $[t, T]$, let $\{g^k_{t,i} : 0 \le i \le N-1\}$ be a consecutive sequence of forward rates for rating class k with settlement dates T_i:

$$g^k_{t,i} = \frac{1}{\delta_i}\left[\frac{B^k(t,T_i)}{B^k(t,T_i+\delta_i)} - 1\right],$$

where $\delta_i \equiv T_{i+1} - T_i$ is the tenor of the underlying zero-coupon bond for the i^{th} forward contract. Inverting, the discount function for rating class k with maturity T_n, $1 \le n \le N$, is given by

$$B^k(t,T_n) = \exp\left(-\sum_{i=0}^{n-1}\delta_i g^k_{t,i}\right). \tag{45}$$

Assuming that the defaultable forward rates in rating class k have the following risk-neutral dynamics:

$$dg^k_{t,i} = \mu^k_{t,i}\,dt + \sigma^k_{t,i}\,dW^{\mathbb{Q}}_t,\ \ 0 \le i \le N-1,$$

we now proceed to derive no-arbitrage restrictions on $\mu^k_{t,i}$, $0 \le k \le S-1$, $0 \le i \le N-1$, under different recovery schemes.

Fractional Recovery of Market Value

In this case, the loss rate l^k_t does not depend on $B^k(t,T_n)$, for all k. Substituting the discount functions (45) into equation (26) yields, for $1 \le \forall n \le N$ and $0 \le \forall k \le S-1$,

$$-\sum_{i=0}^{n-1}\delta_i\mu^k_{t,i} + \frac{1}{2}\sum_{i=0}^{n-1}\sum_{i'=0}^{n-1}\delta_i\delta_{i'}\,\sigma^k_{t,i}\cdot\sigma^k_{t,i'} + \sum_{k'=0}^{S-1}\hat{R}^{kk'}_t\frac{B^{k'}(t,T_n)}{B^k(t,T_n)} - r = 0. \tag{46}$$

For a given k, differencing (46) with respect to index n yields

$$\mu^k_{t,n} \;=\; \sigma^k_{t,n}\cdot\sum_{i=0}^{n-1}\delta_i\sigma^k_{t,i} + \frac{\delta_n}{2}\sigma^k_{t,n}\cdot\sigma^k_{t,n} + \sum_{k'=0}^{S-1}\hat{R}^{kk'}_t\,\frac{e^{\delta_n s^{kk'}_{t,n}}-1}{\delta_n}\,e^{\sum_{i=0}^{n-1}\delta_i s^{kk'}_{t,n}},$$

where $s^{kk'}_{t,i} \equiv g^k_{t,i} - g^{k'}_{t,i}$ is the spread between two forward rates with the same settlement date T_i but different rating classes k and k'. Taking the limit as $N \to \infty$ and $\sup_{i=0}^{N-1} \delta_i \to 0$, in such a way that $\sum_{i=0}^{N-1} \delta_i = T$, we obtain

$$\mu^k(t,T) = \sigma^k(t,T) \cdot \int_t^T \sigma^k(t,u)\,du + \sum_{k'=0}^{S-1} \hat{R}^{kk'}_t s^{kk'}(t,T)\, e^{\int_t^T s^{kk'}(t,u)\,du}, \quad (47)$$

where $\mu^k(t,T) = \lim_{\delta_N \to 0} \mu^k_{t,N}$ and $\sigma^k(t,T) = \lim_{\delta_N \to 0} \sigma^k_{t,N}$ are the risk-neutral drift and diffusion of the instantaneous forward rate $g^k(t,T) = \lim_{\delta_N \to 0} g^k_{t,N}$, and $s^{kk'}(t,T) = g^k(t,T) - g^{k'}(t,T)$ is the spread between two forward curves with different ratings k and k'. Equation (47) generalizes Duffie and Singleton [1999] for the case of $S = 1$ (no rating migrations).

Under the same recovery scheme, Acharya, Das, and Sundaram [2002] derive no-arbitrage restrictions on the risk-neutral drifts of inter-rating spreads $s^{kk'}(t,T)$ on a lattice. Due to their discrete-time and discrete state-space setup, these risk-neutral drifts must be determined numerically by solving a system of equations. The continuous-time and continuous state-space limit of their result, when expressed in terms of risk-neutral drifts of defaultable forward rates, converges to equation (47).

Schonbucher [1998] derives equation (47) under slightly more general assumptions about default events and recovery. In his setup default does not lead to a liquidation, but rather a reorganization of the issuer. Defaulted bonds lose a fraction of their face value and continue to trade, and the fractional loss is a random variable drawn from an exogenous distribution.

Fractional Recovery of Face Value, Payable at Maturity

In this case, the defaultable discount functions (45) must satisfy the PDEs (32), which implies that, for $0 \leq \forall k \leq S-1$ and $1 \leq \forall n \leq N$,

$$-\sum_{i=0}^{n-1} \delta_i \mu^k_{t,i} + \frac{1}{2} \sum_{i=0}^{n-1} \sum_{i'}^{n-1} \delta_i \delta_{i'}\, \sigma^k_{t,i} \cdot \sigma^k_{t,i'} + \sum_{k'=0}^{S-1} R^{kk'}_t \frac{B^{k'}(t,T_n)}{B^k(t,T_n)} - r + \omega^k_t R^{kS}_t \frac{D(t,T_n)}{B^k(t,T_n)} = 0. \quad (48)$$

Differencing with respect to the index n, dividing both sides by δ_n, and taking the limit as $N \to \infty$ and $\sup_{i=0}^{N-1} \delta_i \to 0$ in such a way that $\sum_{i=0}^{N-1} \delta_i = T$, we obtain

$$\begin{aligned}\mu^k(t,T) = \sigma^k(t,T) \cdot \int_t^T \sigma^k(t,u)\,du + \sum_{k'=0}^{S-1} R_t^{kk'} s^{kk'}(t,T) e^{\int_t^T s^{kk'}(t,u)\,du} \\ + \omega_t^k R_t^{kS} s^k(t,T) e^{\int_t^T s^k(t,u)\,du},\ 0 \le k \le S-1,\end{aligned} \tag{49}$$

where $\mu^k(t,T)$ and $\sigma^k(t,T)$ are the instantaneous drift and diffusion of the instantaneous forward rate $g^k(t,T)$, and $s^k(t,T) \equiv g^k(t,T) - f(t,T)$ is the forward credit spread of rating class k relative to the default-free forward curve. Equation (49) was first derived by Bielecki and Rutkowski [2000] as one of the "consistency conditions" for an arbitrage-free pricing model with rating migrations under the current recovery scheme.

Fractional Recovery of Face Value, Payable at Default

In this case, the defaultable discount functions must satisfy the PDEs (36). It is straight forward to show that the no-arbitrage restriction now takes the following form:

$$\begin{aligned}\mu^k(t,T) = \sigma^k(t,T) \cdot \int_t^T \sigma^k(t,u)\,du + \sum_{k'=0}^{S-1} R_t^{kk'} s^{kk'}(t,T) e^{\int_t^T s^{kk'}(t,u)\,du} \\ + \omega_t^k R_t^{kS} g^k(t,T) e^{\int_t^T g^k(t,u)\,du},\ 0 \le k \le S-1,\end{aligned} \tag{50}$$

which generalizes Duffie [1998] for the case of $S = 1$.

Under all of the recovery schemes discussed above, the risk-neutral drift of the defaultable forward curve for a given rating class depends on the diffusion and the initial forward curves for all rating classes. The defaultable forward curves are coupled, because the defaultable discount functions are strongly coupled through rating migrations. When $S = 1$, the no-arbitrage restriction under fractional recovery of market-value has the same form as in the default-free case. This is not the case under fractional recovery of face-value.

7.4 The LIBOR Market Model

An important recent development in the HJM modeling approach, based on the work of Sandmann, Sondermann, and Miltersen [1995], Miltersen, Sandmann, and Sondermann [1997], Brace, Gatarek, and Musiela [1997], Musiela and Rutkowski [1997a], and Jamshidian [1997], is the construction of arbitrage-free models for forward LIBOR rates at an observed discrete tenor structure. Besides the practical benefit of working with observable forward rates (in contrast to the unobservable instantaneous forward rates), this shift overcomes a significant conceptual limitation of continuous-rate formulations. Namely, as shown by Morton [1988] and Sandmann and Sondermann [1997], a lognormal volatility structure for $f(t,T)$ is inadmissible, because it may imply zero prices for positive-payoff claims and, hence, arbitrage opportunities. With the use of discrete-tenor forwards, the lognormal assumption becomes admissible. The resulting *LIBOR market model* (LMM) is consistent with the industry-standard Black model for pricing interest rate caps.

In addition to taking full account of the observed discrete-tenor structure, the LMM framework also facilitates tailoring the choice of *pricing measures* to the specific derivative products. In the absence of arbitrage opportunities, Harrison and Kreps [1979] and Harrison and Pliska [1981] demonstrated that, for each traded security with price P_t, there exists a measure $\mathbb{M}(P)$ under which the price of any other traded security with payoffs denominated in units of the numeraire security is a Martingale. The probability measure $\mathbb{M}(P)$ is referred to as the *pricing measure* induced by the price P of the numeraire security. The risk-neutral measure, underlying our preceding discussions of both *DTSM*s and HJM models, is one example of a pricing measure. The LIBOR market model is based on either one of the following two pricing measures: the *terminal (forward) measure* proposed by Musiela and Rutkowski [1997a] and the *spot LIBOR measure* proposed by Jamshidian [1997].

To fix the notation for the tenor structure, let us suppose that, at time $t = 0$, there are N consecutive LIBOR forward contracts, with delivery dates T_n, $n = 1, 2, \ldots, N$. The underlying of the n^{th} forward contract is a Eurodollar deposit with tenor δ_n. Clearly, $\delta_n = T_{n+1} - T_n$, $n = 1, 2, \ldots, N$ (with $T_{N+1} \equiv T_N + \delta_N$). For $0 < \forall t \leq T_N$, let us denote the next delivery date $n(t) = \inf_{n \leq N} \{n : T_n \geq t\}$.

Let $B(t,T)$ be the LIBOR discount factor at time t with maturity date T. Then the time-t forward LIBOR rate with reset date $t < T_n \leq T_N$ is given by

$$L_n(t) = \frac{1}{\delta_n}\left[\frac{B(t,T_n)}{B(t,T_{n+1})} - 1\right].$$

A caplet is a security with payoff $\delta_n [L_n(T_n) - k]^+$, determined at the reset date T_n and paid at the settlement date T_{n+1} (payment in arrears), where

$L_n(T_n)$ is the spot LIBOR rate at T_n and k is the strike rate. Letting $C_n(t)$ denote the price of the caplet, Brace, Gatarek, and Musiela [1997] show that, in the absence of arbitrage, both $\frac{B(t,T_n)}{B(t,T_{n+1})}$ (and hence $L_n(t)$) and $\frac{C_n(t)}{B(t,T_{n+1})}$ are Martingales under the forward measure, $\mathbb{P}^{n+1} \equiv \mathbb{M}(B(t,T_{n+1}))$, induced by the LIBOR discount factor $B(t,T_{n+1})$. Furthermore, under the assumption that $L_n(t)$ is log-normally distributed,[11] the Black model for caplet pricing obtains:

$$C_n(t) \;=\; \delta_n B(t,T_{n+1})\left[L_n(t)\,N(d_1) - k\,N(d_2)\right], \tag{51}$$

$$d_1 \;\equiv\; \frac{\log\frac{L_n(t)}{k} + \frac{v_n}{2}}{\sqrt{v_n}}, \quad d_2 \equiv \frac{\log\frac{L_n(t)}{k} - \frac{v_n}{2}}{\sqrt{v_n}}, \tag{52}$$

where $N(\cdot)$ is the cumulative normal distribution function and v_n is the cumulative volatility of the forward LIBOR rate from the trade date to the delivery date: $v_n \equiv \int_t^{T_n} \sigma_n(u)'\sigma_n(u)\,du$. The price of a cap is simply the sum of all un-settled caplet prices (including the value of the caplet paid at settlement date $T_{n(t)}$ which is known at t).

The Black-Scholes type pricing formula (51)–(52) for caps is commonly referred to as the *cap market model*. The simplicity of the cap market model derives from the facts that (a) each caplet with reset date T_n and payment date T_{n+1} is priced under its own forward measure $\mathbb{P}^{n+1}$; (b) we can be completely agnostic about the exact nature of the forward measures and their relationship with each other; and (c) we can be completely agnostic about the factor structure: the caplet price C_n does not depend on how the total cumulative volatility v_n is distributed across different shocks W^n.

The simplicity of the cap market model does not immediately extend to the pricing of securities whose payoffs depend on two or more spot LIBOR rates with different maturities, or equivalently two or more forward LIBOR rates with different reset dates. A typical example is a European swaption with expiration date $n \geq n(t)$, final settlement date T_{N+1}, and strike k. Let

$$S_{n,N}(t) = \frac{B(t,T_n) - B(t,T_{N+1})}{\sum_{j=n}^{N} \delta_j B(t,T_{j+1})}$$

[11] That is,

$$\frac{dL_n(t)}{L_n(t)} = \sigma_n(t)'\,dW^n(t),$$

where W^n is a vector of standard and independent Brownian motions under $\mathbb{P}^n$, and $\sigma_n(t)$ is a deterministic vector commensurate with W^n.

be the forward swap rate, with delivery date T_n and final settlement date T_{N+1}, the payoff of the payer swaption at T_n is a stream of cash flows paid at T_{j+1} and in the amount $\delta_j[S_{n,N}(T_n) - k]^+$, $n \leq j \leq N$, where the spot swap rates $S_n(T_n)$ are completely determined by the forward LIBOR rates $L_j(T_n)$, $n \leq j \leq N$. The market value of these payments, as of T_n, is given by $\sum_{j=n}^{N} \delta_j B(T_n, T_{j+1}) \, [S_n(T_n) - k]^+ = \left[1 - B(T_n, T_{N+1}) - k \sum_{j=n}^{N} \delta_j B(T_n, T_{j+1})\right]^+$. In order to price instruments of this kind, we need the joint distribution of the forward LIBOR rates $\{L_j(t) : n \leq j \leq N, 0 \leq t \leq T_n\}$, under a *single* measure. The *LIBOR market model* arises precisely in order to meet this requirement.

Musiela and Rutkowski [1997a] show that under the *terminal measure* $\mathbb{P}^* \equiv \mathbb{P}^{N+1}$, i.e., the probability measure induced by the LIBOR discount factor $B(t, T_{N+1})$, the forward LIBOR rates can be modeled as a joint solution to the following stochastic differential equations (SDEs): for $n(t) \leq \forall n \leq N$,

$$\frac{dL_n(t)}{L_n(t)} = \sigma_n(t)' \left[- \sum_{j=n+1}^{N} \frac{\delta_j L_j(t)}{1 + \delta_j L_j(t)} \, \sigma_j(t) \, dt + dW^*(t) \right], \qquad (53)$$

where W^* is a vector of standard and independent Brownian motions under $\mathbb{P}^*$. These SDEs have a recursive structure that can be exploited in simulating the LIBOR forward rates: first, the drift of $L_N(t)$ is identically zero, because it is a Martingale under $\mathbb{P}^*$; second, for $n < N$, the drift of $L_n(t)$ is determined by $L_j(t)$, $n \leq j \leq N$.

Jamshidian [1997] proposes an alternative construction of the LIBOR market model based on the so-called the *spot LIBOR measure*, $\mathbb{P}^B$, induced by the price of a "rolling zero-coupon bond" or "rolling C.D." (rather than a continuously compounded bank deposit account which induces the risk-neutral measure):

$$B(t) \equiv \frac{B(t, T_{n(t)})}{B(0, T_1)} \prod_{j=1}^{n(t)-1} \left[1 + \delta_j L_j(T_j)\right].$$

He shows that, under this measure, the set of LIBOR forward rates can be modeled as a joint solution to the following set of SDEs: for $n(t) \leq \forall n \leq N$,

$$\frac{dL_n(t)}{L_{n(t)}(t)} = \sigma_n(L_{n(t)}(t), t)' \left[\sum_{j=n(t)}^{n} \frac{\delta_j L_j(t)}{1 + \delta_j L_j(t)} \sigma_i(L_i(t), t) dt + dW^B(t) \right], \quad (54)$$

where W^B is a vector of standard and independent Brownian motions under $\mathbb{P}^B$ and the possible state-dependence of the volatility function is also made explicit. These SDEs also have a recursive structure: starting at $n = n(t)$, $L_{n(t)}(t)$ solves an autonomous SDE; for $n > n(t)$, the drift of $L_n(t)$ is determined by $L_j(t)$, $n(t) \leq j \leq n$.

The price of a security with payoff $g(\{L_j(T_n) : n \leq j \leq N\})$ under the LIBOR market model is given by

$$\begin{aligned} P_t &= B(t, T_{N+1})\, E_t^* \left[g\left(\{L_j(T_n) : n \leq j \leq N\}\right)\right] \\ &= B(t, T_{n(t)})\, E_t^B \left[\frac{g\left(\{L_j(T_n) : n \leq j \leq N\}\right)}{\prod_{j=n(t)}^{n-1}(1 + \delta_j L_j(T_j))} \right], \end{aligned} \tag{55}$$

where $E_t^*[\cdot]$ denotes the conditional expectation operator under the terminal measure $\mathbb{P}^*$ and $E_t^B[\cdot]$ denotes the conditional expectation operator under the spot LIBOR measure $\mathbb{P}^B$. The Black model for caplet pricing or the cap market model is recovered under the assumption that the proportional volatility functions $\sigma_j(t)$ are deterministic.[12]

7.5 The Swaption Market Model

According to equation (55), the price of a payer swaption with expiration date T_n and final maturity date T_{N+1} is given by

$$\begin{aligned} P_{n,N}(t) &= B(t, T_{N+1})\, E_t^* \left[\left(1 - B(T_n, T_{N+1}) - k \sum_{j=n}^{N} \delta_j B(T_n, T_{j+1}) \right)^+ \right] \\ &= B(t, T_{n(t)})\, E_t^B \left[\frac{\left(1 - B(T_n, T_{N+1}) - k \sum_{j=n}^{N} \delta_j B(T_n, T_{j+1})\right)^+}{\prod_{j=n(t)}^{n-1}(1 + \delta_j L_j(T_j))} \right]. \end{aligned}$$

Under the assumption of deterministic proportional volatility for forward LIBOR rates, the above expression can not be evaluated analytically.

[12] The pricing equation (55) holds even when the proportional volatility of the forward LIBOR rates are stochastic. Narrowly defined, the LIBOR market model refers to the pricing model based on the assumption that the proportional volatilities of the forward LIBOR rates are deterministic. Broadly defined, the LIBOR market model refers to the pricing model based on any specification of state-dependent proportional volatilities (as long as appropriate Lipschitz and growth conditions are satisfied).

In order to calibrate theoretical swaption prices directly to market quoted Black volatilities for swaptions, a more tractable model for pricing European swaptions is desirable. Jamshidian [1997] shows that such a model can be obtained by assuming that the proportional volatilities of forward swap rates, rather than those of forward LIBOR rates, are deterministic. The resulting model is referred to as the *swaption market model.*

The swap market model is based on the *forward swap measure*, $\mathbb{P}^{n,N}$, induced by the price of a set of fixed cash flows paid at T_{j+1}, $n \leq j \leq N$, namely,

$$B_{n,N}(t) \equiv \sum_{j=n}^{N} \delta_j B(t, T_{j+1}).\ t \leq T_{n+1}.$$

Under $\mathbb{P}^{n,N}$, the forward swap rate $S_{n,N}(t)$ is a Martingale:

$$\frac{dS_{n,N}(t)}{S_{n,N}(t)} = \sigma_{n,N}(t)' dW^{n,N},$$

where $W^{n,N}$ is a vector of standard and independent Brownian motions under $\mathbb{P}^{n,N}$. Thus, the price of a European payer swaption with expiration date T_n and final settlement date T_{N+1} is given by

$$P_{n,N}(t) = B_{n,N}(t)\, E_t^{n,N} \left[(S_{n,N}(T_n) - k)^+ \right],\ t \leq T_n. \tag{56}$$

Under the assumption that the proportional volatility of the forward swap rate is deterministic, the swaption is priced by a Black-Scholes type formula:

$$\begin{aligned} P_{n,N}(t) &= B_{n,N}(t) \left[S_{n,N} N(d_1) - k N(d_2) \right], \\ d_1 &\equiv \frac{\log \frac{S_{n,N}}{k} + \frac{v_{n,N}}{2}}{\sqrt{v_{n,N}}},\ d_2 \equiv \frac{\log \frac{S_{n,N}}{k} - \frac{v_{n,N}}{2}}{\sqrt{v_{n,N}}}, \end{aligned}$$

where $v_{n,N} \equiv \int_t^{T_n} \sigma_{n,N}(u)' \sigma_{n,N}(u)\, du$ is the cumulative volatility of the forward swap rate from the trade date to the expiration date of the swaption.

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