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No-Arbitrage Option Pricing: New Evidence on the Validity of the Martingale Property

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by

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Abstract: The no-arbitrage approach to option pricing implies that risk-neutral prices follow a martingale. The validity of this property has been tested and rejected by Longstaff(1995). Since he tested the general framework, his results have far reaching and disturbing implications for contingent claims pricing. This paper proposes a new method to test the martingale property. This method is based on the Laguerre polynomial series. The tests use options and futures on the S&P500 index. The new methodology and data show that the martingale property cannot be rejected. This result implies that the general approach is still valid and the existence of frictions only adds noise. Testing more specific pricing models is relevant again.

Keywords: Option Pricing, Martingale Pricing, Semi-Nonparametric Method.

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1 Introduction

The derivation of risk-neutral valuation for contingent claims is one of the most important innovations in modern asset pricing theory. In risk-neutral valuation, the arbitrage-free price of a contingent claim can be obtained as an expectation of the discounted payoff of the contingent claim, modified by replacing the original price process with a risk-neutral process whose expected rate of return is the riskless interest rate. One implication of risk-neutral valuation is that the risk-neutral price process should be a martingale. That is, the expected asset price, discounted by the riskless interest rate under a risk-neutral probability density, should be equal to the current price of the underlying asset.

Recently, Longstaff (1995) questioned the empirical validity of risk-neutral valuation. He shows that the martingale property implied by risk-neutral valuation is strongly rejected based on tests using S&P 100 call options. His empirical results have far reaching implications for theoreticians, empiricists and practitioners alike.

His results have two aspects. First, his empirical results show that the restriction implied by the martingale property is not only rejected for a model with a specific distributional specification of risk-neutral density such as the Black-Scholes-Merton model, but also for a class of models with a general specification. Ideally, constructing a specification-free test would be desirable in testing a theory, because it provides results which are more robust and less subject to the classic joint hypotheses issue. Namely, if a test is based on specific assumption of the distribution of the underlying asset, the rejection of the martingale property may not imply rejection of the theory itself but imply that distributional assumption is wrong. Or, if the martingale property is not rejected, the risk-neutral valuation as well as the distributional specification would be jointly accepted. He deals with this issue by generalizing the distributional assumption of the risk-neutral density, and still rejects the martingale property.

Second, Longstaff's empirical results show that the deviations from the martingale property are related to market illiquidity as evidenced by the bid and ask spreads of options and their trading volumes. Since S&P 100 options are the most liquid equity options contracts in the world, frictions in the market, such as transaction costs and bid-ask spreads, are relatively less important than in other equity options contracts. Therefore, it is expected that market frictions are less likely to explain the deviations from risk-neutral valuation. If tests of basic arbitrage fail in this market, it seems more likely that such tests will fail in other less liquid markets.

This paper reexamines the validity of the martingale property based on a different methodology and a different set of data. The results that we obtain contradict the results obtained by Longstaff (1995).

In this paper, we propose a new method to test the martingale property without assuming a specific distribution. Recent related papers (e.g., Rubinstein (1994), Derman and Kani

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(1994), Jackwerth and Rubinstein (1996), Stutzer (1996)) have used option prices to derive risk neutral distributions. This paper also obtains the implied distribution from option prices but the main objective here is to test the validity of the martingale property. Specifically, we test the martingale property by using closed form solutions of option prices with a general probability density approximated by the Laguerre orthogonal polynomials. We show that the method used in Longstaff (1995) is biased against the martingale property. In particular, the bias in the tests increases as the true distribution departs further from the lognormal distribution assumed in the Black-Scholes-Merton model. The proposed method does not suffer from the above mentioned bias. The proposed method also has minimal approximation errors for a wider spectrum of possible distributions, and has substantially smaller errors than previous methods. Thus, the proposed method is less sensitive to approximation errors in the estimation of the general distribution.

In addition, we test the martingale restriction with European-style S&P 500 index (SPX) call and put options, whereas American-style S&P 100 (OEX) call options are used in Longstaff (1995). The empirical results in this paper are free from the problems of early exercise that exist for American index options. The valuation of OEX options is further complicated by the 'wildcard' feature due to cash-settlement for stock index options. SPX options, which are European and are accompanied by a futures contract used as a hedging vehicle, are a natural choice for our tests. Given that futures transactions entail low costs and impose no restrictions on short sales, the impact of market frictions on option values is minimized for S&P 500 index options. Furthermore, we extend the empirical tests to check the robustness of the results by using no-arbitrage relationship between option prices and futures prices.

Based on tests using the S&P 500 call and put options for the period, January 1993 to December 1994, the empirical results presented here contradict the results obtained by Longstaff (1995). Specifically, the empirical results show that: 1) The magnitude of deviations from the martingale restriction is much smaller than that in Longstaff (1995), and that the martingale restriction cannot be rejected; 2) The pricing errors from the proposed general distribution are less than the options' bid and ask spreads; 3) The deviation from the martingale property is not related to the bid and ask spreads of options; 4) The results are robust with respect to the pricing relationship between futures contracts and options.

The paper is organized as follows. Section 2 presents a method for estimating the implied density which is based on the Laguerre orthogonal polynomials series and discusses econometric methods for testing the martingale restriction. Section 3 describes the data. Section 4 presents the results of the tests and describes diagnostic tests. Section 5 summarizes the paper and discusses extensions for future research.

2 Semi-Nonparametric (SNP) Option Pricing: Theory and Econometric Tests

In this section, we propose a new semi-nonparametric estimation method for estimating the implied density, which is based on the Laguerre orthogonal polynomials series, and discuss econometric methods for testing the martingale property. The new proposed method has advantages over the Hermite polynomials adopted in Longstaff (1995). Especially, we show that the estimated moments based on the Hermite polynomials, used in Longstaff (1995) are biased against the martingale restriction. In particular, the bias in the tests increases as the moneyness bias of the Black-Scholes-Merton model increases. The proposed density estimator based on the Laguerre polynomials is free from the bias.

2.1 Theory

Consider the stochastic differential equation of the form

$$dS_t = \alpha(S, t)dt + \sigma(S_t, t)dB_t$$

where $\alpha : R \times [0, \infty) \to R$ and $\sigma : R \times [0, \infty) \to R$ are given functions which satisfy some regularity conditions. Let $C(S_t, t)$ denote the time t value of a derivative security which has the contractual payoff function $g(.)$, meaning that at the expiration date T, $C(S_T, T) = g(S_T)$. The existence of a trading strategy which replicates the derivative security implies that the arbitrage-free price of a derivative security can be obtained as an expectation of the discounted payoff of the derivative security, modified by replacing the original price process with a pseudo-price process (risk-neutral process), the expected rate of return of which is the riskless interest rate. Specifically, under risk-neutral valuation, instead of solving the fundamental partial differential equation, the arbitrage-free price of the derivative security $C(S_t, t)$ can be obtained by evaluating the expectation,

$$C(S_t, t) = e^{-r(T-t)} E_t[g(S_T^*)],$$

where S_t^* denotes the diffusion that obeys the stochastic differential equation [1]

$$dS_t^* = rS_t^* dt + \sigma(S_t^*, t)dB_t.$$

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[1]For example, the Black-Scholes option formula can be obtained by directly evaluating the integral

$$C(S_t, t) = e^{-r(T-t)} \int_0^\infty g(S_T^*) f(S_T^*) dS_T^*,$$

where $f(.)$ denotes the lognormal density given by

$$f(S_T^*) = \frac{1}{S_T^* \sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{\{\ln S_T^* - (\ln S_t + r\tau - \sigma^2\tau/2)\}^2}{2\sigma^2\tau}\right].$$

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One of the testable implications of risk-neutral valuation is that a pseudo-price process is a martingale; that is,

$$D_{t,T} E_t[S_T^* | S_t] = S_t,$$

where $D_{t,T}$ represents the time t price of a unit discount bond with maturity T. This is known as the *martingale property*.

In testing the martingale property we need to specify the risk-neutral probability density of the underlying asset. Since, however, the martingale property is true for any distributions, a proper test should be applied to a specification which is not so narrow that the test will be contingent on the particular specification (*e.g.*, assuming the lognormal distribution as in the Black-Scholes-Merton model). As previously mentioned, a general specification of the risk-neutral density is crucial in testing the martingale property, because the rejection of a model with a specific distributional assumption may not imply the violation of the martingale restriction if the distribution is misspecified. If the martingale restriction is not rejected, nested tests on the specific distributional assumption can be performed.

In this paper, a general specification of the density is employed in the test to circumvent this problem by using the fact that any probability density can be well approximated by a finite series of orthogonal polynomial functions such as the Laguerre orthogonal polynomial series. Expansion of the distribution functions into a series of orthogonal functions is a well known mathematical technique in approximating statistical distributions. The coefficients of the orthogonal functions in the expansions are functions of the moments or cumulants of the distribution and can be determined whenever these moments or cumulants are known. Examples of such expansions are the Gram-Charlier series of type A, Edgeworth's form involving Hermite polynomials and the series involving Laguerre, Jacobi and other classical orthogonal polynomials. Jarrow and Rudd (1982) approximate the underlying distribution by a lognormal distribution with correcting term based on the Edgeworth series expansion. Rather than specifying the distributional assumption on the risk-neutral process, Longstaff (1995) uses the Hermite polynomial series for his general option pricing model, whereas Edgeworth's form involving Hermite polynomials are also used in the semi-nonparametric estimation approaches of Gallant and Nychka (1987) and Gallant and Tauchen (1989).[2]

We next develop our method and show why the general form of the distribution based on the Laguerre orthogonal polynomial series is preferable to the Hermite polynomials adopted in Longstaff (1995). Let the true risk-neutral probability density function $f(x)$ be expressed as

$$f(x) \simeq g(x) \left[1 + \sum_{j=1}^{m} a_j p_j(x_s) \right],$$

where $g(x)$ is a parametric key function with key parameters, $p_j(x_s)$ the j-th relevant polynomial and x_s is a standardized x value. For example, if the key function $g(x)$ is a normal

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[2]Orthogonal series density estimation is similar to the kernel density estimations employed in Aït-Sahalia (1995), Stanton (1995), and Boudoukh, Richardson, Stanton and Whitelaw (1995) in the sense that neither estimation methods requires a parametric specification of the density.

density, Hermite polynomials are preferred because the coefficients (a_j) can be determined by their orthogonality properties [see Kendall and Stuart (1987)]. If a gamma function is chosen as a key function, the Laguerre polynomials have this orthogonal property.

Specifically, we approximate the conditional density of S_T given S_t, $f_k(S_T|S_t)$, with the first k conditional moments of the distribution by the Laguerre orthogonal polynomial series as follows;

$$f_k(S_T|S_t) = \Psi_\delta(S_T|S_t) + \sum_{i=3}^{k} b_i \sum_{j=0}^{i} \binom{i}{j} (-1)^j \, \Psi_{\delta+j}(S_T|S_t),$$

where

$$\Psi_\delta(S_T) = \frac{1}{\theta^\delta \Gamma(\delta)} S_T^{\delta-1} e^{-S_T/\theta}, S_T \geq 0$$

and

$$
\begin{aligned}
b_i &= E\{L_i^{(\delta)}(S_T/\theta)\} \\
&= \frac{1}{i!} \sum_{j=0}^{i} \binom{i}{j} (-1)^j \left(\frac{\mu_j}{\theta^j}\right) \frac{\Gamma(\delta+i)}{\Gamma(\delta+j)},
\end{aligned}
$$

$i = 0, 1, 2, ..., k$ ($b_0 = 1, b_1 = 0$), $\mu_j = E(S_T^j|S_t)$, $\theta = var(S_T|S_t)/\mu_1$, $\delta = \mu_1/\theta$, and $I(.)$ is an incomplete gamma function defined as $I(u, a) = \int_0^u \frac{t^{a-1} e^{-t}}{\Gamma(a)} dt$.

Based on the general density with two-parameters gamma distribution as the key function, we derive the closed form solutions of option prices. The detailed forms of solutions for call and put option are provided in the appendix. The closed form solution of option values depends on the first four moments of the distribution as well as on the other parameters such as exercise price, interest rate and maturity.

2.1.1 The Unbiasedness of Moments

An interesting property of the proposed method is that the coefficients of the Laguerre polynomial series, b_i ($i = 0, 1, 2, ..., k$), are chosen such that, for $j = 1, ..., k$,

$$\int_0^\infty S_T^j f(S_T|S_t) dS_T \equiv \int_0^\infty S_T^j f_k(S_T|S_t) dS_T$$

where $f(S_T|S_t)$ is the true conditional density conditioned on S_t and $f_k(S_T|S_t)$ is the approximated conditional density conditioned on S_t. Thus, moments evaluated by the approximated distribution are identical to the corresponding true moments. Given that finance theory provides testable implications only for moments of asset returns as in Hansen and Richard (1987), the orthogonal series estimator provides the natural setting to test moment

restrictions implied by the theories. Specifically, in testing the martingale property, we test whether the first conditional moment of the distribution is

$$\mu_1 = S_t e^{(r-d)(T-t)},$$

where r is the interest rate and d is the dividend yield.

Note that the domain of the general density function based on the Laguerre polynomials is $[0, \infty)$, while the Hermite polynomials form an orthogonal system on the domain of $(-\infty, \infty)$. Thus, when the Hermite series is adopted, a logarithmic transformation is often made in order to ensure the non-negativity of price. However, a logarithmic transformation has non-trivial effect on the estimation of the moments. For example, in Longstaff (1995), the general implied distribution is the Hermite polynomial series given by the form

$$f_4(z) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2}) \left[1 + \frac{\mu_3}{6} \left(z^3 - z \right) + \frac{\mu_4 - 3}{24} (z^4 - 6z^2 + 3) \right],$$

where

$$z = \frac{\ln S_T - \mu}{\sigma},$$

and $\mu = E(\ln S_T)$, $\sigma = \sqrt{Var(\ln S_T)}$, $\mu_3 = E(z^3)$ and $\mu_4 = E(z^4)$. The mean of the transformed Hermite series density is

$$\widetilde{E}(S_T) = e^{\mu + \frac{\sigma^2}{2}} \left(1 + \frac{\mu_3}{6} \sigma^3 + \frac{\mu_4 - 3}{24} \sigma^4 \right),$$

which may not equal to the true mean. In fact, if z follows a normal distribution or the price follows a lognormal distribution, the mean of the distribution, approximated by the Hermite series, is the same as the true mean. This would not be true in other cases. To illustrate this, suppose that the true density of S_T follows a two parameter gamma distribution given by

$$f(S_T; \alpha, \beta) = \frac{(S_T)^{\alpha-1} e^{-\frac{S_T}{\beta}}}{\Gamma(\alpha) \beta^\alpha}$$

and $E(S_T) = \alpha\beta$. Then it can be easily seen that

$$\begin{aligned}
&\widetilde{E}(S_T) \\
&= \exp\left[\Psi(\alpha) + \ln(\beta) + \frac{1}{2}\Psi'(\alpha) \right] \left[1 + \frac{1}{6}\Psi''(\alpha) + \frac{1}{24}\left(\Psi'''(\alpha) - 3\Psi'(\alpha)^2 \right) \right] \\
&\neq \alpha\beta.
\end{aligned}$$

where $\Psi(.)$ is a digamma function and $\Psi^{(n)}(.)$ is its n-th derivative. In fact, as $\alpha \to \infty$, or as the log-gamma distribution converges to the normal distribution, $\widetilde{E}(S_T) = E(S_T)$. Thus, the estimated mean from the transformed Hermite polynomials becomes equal to the true mean only when the true distribution is lognormally distributed.

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2.1.2 Numerical Analysis of The Approximation Error

The choice of a key function is also important in obtaining a better approximation of the true distribution. Intuitively, if a key function is a good fit to the true density, few adjustments need to be made by adding more terms. On the other hand, if the key function is a poor fit, more terms are required for approximation. For example, the Gram-Charlier series of type A or the Edgeworth's form yields a good result when the distribution is fairly close to a normal, which is the weight function for these expansions. Since log-gamma distribution is more skewed and leptokurtic than normal, a gamma distribution as the key function can better resolve the moneyness bias reported by Rubinstein (1985) and Heston (1993)[3], and it is likely to reduce approximation errors in option prices without adding many additional terms.

One concern in using the semi-nonparametric method is that errors stemming from the approximation of the density may add too much noise to distinguish between theoretical models in empirical studies.[4] To assess the performance of the proposed method, we perform an experiment to check whether the proposed method recovers theoretical values of different models, namely, the square root model of Cox and Ross (1976) and the Black-Scholes-Merton model. In the experiment, we choose the values of both models such that the annualized conditional variances are the same for both model, that is,

$$\sigma^2_{SQR} = rS_0 \frac{\exp(r\tau)\left[\exp(\sigma^2_{BS}) - 1\right]}{[\exp(\mu\tau) - 1]} .$$

For example, when the annualized volatility (σ_{BS}) in the Black-Scholes-Merton model is 0.06, σ_{SQR} should be 1.2161, interest rate (r) is 5% and the current index level (S_0) is 400. When σ_{BS} is 0.16, σ_{SQR} is 3.2609. In the experiment, the mean absolute dollar differences of the Black-Scholes-Merton model are calculated based on 13 annualized volatilities (σ_{BS}) which are 4%, 5%, 6%, ..., and 16%. σ_{SQR}'s in the square root model are also chosen to match the annualized variance of the Black-Scholes-Merton. In the experiment, the current price (S_0) is set to 400 and the interest rate is equal to 5%.

Table 1 summarizes the mean absolute dollar differences between theoretical option values and the option values approximated by Laguerre polynomials for maturities of one month and three months. For a one month maturity, the mean absolute dollar difference between theoretical square root option value and the approximated option value is 0.05 cents. For

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[3]This feature is consistent with option pricing models including the constant elasticity of variance model, Geske's(1979) compound option model, Merton's(1976) and Bates's(1991) jump-diffusion with negative average jump size, the stochastic variance model with a negative correlation and Heston's(1993) log-gamma model.

[4]In Jarrow and Rudd (1982, p 364), for example, the mean absolute dollar error for the square root model is 3.8 cents which reduces the average difference between the square root option value and the Black-Scholes-Merton option value by 27% from 5.2 cents. Since their apporoximation method can explain the difference of theoretical prices between the constant elasticity of variance model and the Black-Scholes-Merton model by only 27%, it may be difficult to distinguish one from the other in the actual empirical studies.

the Black-Scholes-Merton model, the mean absolute dollar difference is also 0.05 cents, thus the proposed method also recovers the Black-Scholes-Merton model quite well. Note that the average absolute dollar difference between the square root model and the Black-Scholes-Merton model is 4.34 cents for a one month maturity, and errors due to approximation add almost no noise in distinguishing one model from the other. For a three month maturity, the mean absolute dollar difference between theoretical square root option value and the approximated option value is 0.37 cents. For the Black-Scholes-Merton model, the mean absolute dollar difference is also 0.34 cents. Given that the absolute difference between the square root model and the Black-Scholes-Merton model is 8.47 cents, the approximation still adds errors that are far less than 5% of the theoretical difference.

2.2 Econometric Estimation; A Test of the Martingale Property

In this section, we discuss econometric methods for estimating the implied parameters of a proposed option pricing formula with a set of option prices and testing the martingale property. The no arbitrage profit condition requires that the restriction is only on the mean of risk-neutral density, that is,

$$E(S_T) = S_t e^{(r-d)(T-t)} \,,$$

where $E(S_T)$ is the expected index value at the maturity and S_t is the current index value corresponding to option prices. In other words, the current index value should be the present value of the expected index value at the maturity of a given option.

In estimating the expected index value at the maturity of options, ordinary least square estimation is performed for the following system of non-linear regression:

$$V_{i,t} = f(\theta_t : K_i) + \varepsilon_{i,t} \quad (i = 1, \ldots, N, \, t = 1, \ldots, T) \,,$$

where $V_{i,t}$ is an actual option value with the exercise price K_i at time t, $f(\theta_t : K_i)$ is the theoretical option prices, and $\theta \in R^{M \times T}$ ($M \leq N$) is a set of parameters of the option pricing model $f(\theta_t : K_i)$ at time t.[5] When the general risk-neutral density based on Laguerre polynomials is employed, the theoretical option prices $f(\theta_t : K_i)$ in the estimation correspond to the explicit form provided in the previous section. Or, if the lognormal distribution is adopted for the risk-neutral density, the Black-Scholes formula is used as the theoretical option price.

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[5]If the same parameters appear in more than one of the regression functions, the system is subject to 'cross-equation restrictions'. This system of equations is termed nonlinear seemingly unrelated regression (NL-SUR) by Zellner (1962). In the presence of such restrictions, estimating all equations simultaneously rather than individually is more efficient even in the case that $Var(\varepsilon_t)$ is diagonal because $\varepsilon_{i,t}$ is correlated with $\varepsilon_{j,t}$ for $i \neq j$. However, the implementation of the NL-SUR estimation is virtually impossible due to intensive computational burden of our estimation problem.

To test the martingale property, we form an F-test based on the vector of residuals from the nonlinear regressions. Specifically, we use the nonlinear regression both for the unrestricted model and for the restricted model in which the conditional first moment is restricted by the martingale property. As discussed, the martingale property implies that the expected price of the underlying asset discounted by the riskless interest rate under the risk-neutral probability density is equal to the current underlying asset price. In the restricted model, the expected price of the underlying asset discounted by the riskless interest rate is equal to the current underlying price. For example, in the case of the general distribution, only three parameters are estimated for the restricted model, whereas four parameters including the implied mean of the distribution are estimated in the unrestricted model. Based on the residuals both from the unrestricted model, ε_t^U, and the restricted model, ε_t^R,

$$F = \frac{\left[\sum_{t=1}^T \varepsilon_t^{R\prime} \varepsilon_t^R - \sum_{t=1}^T \varepsilon_t^{U\prime} \varepsilon_t^U \right] / TJ}{\left[\sum_{t=1}^T \varepsilon_t^{U\prime} \varepsilon_t^U \right] / (N - M) T},$$

where J is the number of restrictions at each cross-section and F is asymptotically distributed with an $F[JT, (N - M)T]$ distribution. $J = 1$ applies to the case of the martingale property. Note that this statistic is asymptotically valid because neither the numerator nor the denominator exactly has a chi-squared distribution in the nonlinear setting. This F-statistic can be used to test the specification of a specific model as well as of a class of distributions.

As in Longstaff (1995), the martingale restriction is also tested based on the percentage difference between the actual index value and the estimated current index value from option prices, that is;

$$\frac{\widehat{E(S_T)} e^{-(r-d)(T-t)} - S_t}{S_t} \times 100.$$

If the martingale property holds, the percentage difference between the implied spot prices and the actual prices is expected to be zero. Specifically, t-tests for the mean difference are performed to test the null hypothesis that the actual index value is equal to the estimated (implied) current index value from option prices.

The test based on the F-statistic in this paper is essentially the likelihood ratio test, while the t-statistic is based on the Wald type test. Tests based on these two statistics are equivalent in a linear regression, but differ in a nonlinear regression. A disadvantage of the F-statistic is the requirement to estimate both the unrestricted model and the restricted model. In contrast with the F-statistic, the t-statistic requires estimating only the unrestricted model. However, the Wald test in nonlinear regression suffers from important drawbacks. Simulation results reported in Gallant (1975) show that the asymptotic distribution of the Wald test is a poorer approximation to its small sample distribution than the likelihood ratio test is, and that the likelihood ratio test has more power than the Wald test. Another important drawback of the t-statistic is that it is *not* invariant to reparameterization of the model, while the likelihood ratio test is invariant.[6]

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[6]Detailed analysis and Monte Carlo evidence on this issue are discussed in Gallant (1987), and Gregory

Leaving the finite sample properties of test statistics employed in this paper for future exercise, the $F-$statistic here is preferred because it also provides probability statements of differences in pricing errors. Intuitively, the $F-$statistic tests whether pricing errors reduced by relaxing martingale property are large enough to reject the martingale property, while the $t-$ statistic provides no information on this issue. Since parameters in nonlinear regression tend to be correlated, it is important to consider a possibility that, while the first moment is restricted, higher moments can be fitted such that pricing errors may not be large enough to reject the martingale property.

3   The Data

In this paper we use call and put option prices for S&P 500 index options traded at the Chicago Board Options Exchange (CBOE). We also use the S&P 500 index futures data. There are two main reasons why we prefer using S&P 500 index options rather than S&P 100 index options which are used in Longstaff (1995). First, the S&P 500 index options are European-style options while the S&P 100 index options are American-style options. Brenner, Courtadon and Subrahmanyam (1989) argue that American-style options could deviate substantially from European-style options. First, assuming a constant dividend yield and physical delivery, they show that the values of Americanx calls are almost identical to European calls, while the values of index puts especially for in-the-money options may significantly exceed their European counterparts. To avoid the measurement errors associated with the American feature, Longstaff (1995) uses data for call options only. The use of call options does not, however, deal with the more important issue with American index options, the 'wildcard' option, arising from the cash-settlement arrangements (See also Brenner (1990)). This feature which distinguishes stock options from index options is the main reason for the deviations of American options, calls and puts, from European options. Ignoring the American feature may potentially lead to the rejection of the martingale property. European options used in this paper are free from these measurement errors.

Another important advantage of S&P 500 options is that we can use the actively trading S&P 500 index futures contract. Riskless hedge arguments require that the underlying asset trade continuously and that transaction costs are negligible. Hedging index options with stock index futures contracts, which is a standard market practice, enables us to use the futures data to test the martingale property with data which may be even less subject to measurement errors. Though in recent years program trading, buying and selling simultaneously hundreds of stocks, has become almost instantaneous and relatively inexpensive, futures trading is still faster and somewhat less costly.[7] On the other hand, the market for

and Veall (1985). Marcel, Dagenais and Dufour (1991) also show that a simple rescaling of parameters in the Wald test can lead to vastly different results in nonlinear setting even when the equivalent null hypotheses are tested.

[7]Recently, the American Stock Exchange introduced a security called SPIDER which replicates the S&P 500 index and trades as a single security.

S&P stocks is still deeper than the futures market.[8]

The option prices in the sample are obtained from the Interactive Data Corporation (IDC) database for the period; January, 1993 to December, 1994. Specifically, the bid and ask quotes of options at 3:15 p.m. on Thursday are collected. The mid-points of the bid-ask quotes are used for empirical analysis. As argued by Longstaff (1995), inferences based on transaction data could be affected by whether the actual transaction data was at the bid or the ask especially for options away from the money. Moreover, the bid and ask prices are set at the end of trading based on the available information, mainly the stock prices at the close while transaction prices may be minutes or hours old. Chan, Chung and Johnson (1993) show that price movements in the stock option market and the underlying stock market are virtually simultaneous if option prices are based on bid/ask midpoints rather than trade prices. Given that the stock market closes at 3:00 p.m., bid-ask quotes from the options' market close may be somewhat nonsynchronous. To assess the effect of staleness in the stock market index, the sensitivity of results is tested by using the settlement prices of the futures contracts at 3:15 p.m.

The number of option prices available each day, in the sample period, ranges from 9 to 46 depending on the option's maturity. Thanks to the abundance of data, option bid and ask prices are collected based on two criteria, maturity and moneyness. Given that longer maturity options are much less liquid, for each maturity of options contract, options prices on every Thursday for the period from ten weeks to one week prior to maturity are collected. By limiting the options' maturities to less than ten weeks, the impact of stochastic interest rates is also minimized. Both the bid and ask quotes must be available, and ask prices of options must be larger than ten cents. Seven call prices and seven put prices which are closest to being at-the-money are then selected.[9] The selected options prices are also checked for Merton's (1973) distributional options bounds, but no violation is found.

To make dividend adjustments, the dividend yield series are obtained from DATAS-TREAM. For options less than one month to maturity, one-month Treasury Bill rates are used, while for options longer than one month to maturity, linear interpolation between the one-month Treasury Bill rate and the three-month Treasury Bill rate, as obtained from DATASTREAM, is used.

Table 2 presents descriptive statistics of the sample used in this paper. There are 1,680 call prices with an average option value of seven dollars. The moneyness of these call options ranges from about 0.95 to 1.05, and the average bid and ask spread is about 35 cents with a range from 1/16 to one dollar. An equal number of put prices is also included. The average bid and ask spread for these put options is a slightly larger. 37 cents. The moneyness of put options also ranges from 0.95 to 1.05. Table 2 also includes the summary statistics for observations of the corresponding S&P 500 index values and the discount rates. The average

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[8]Large trades, baskets of 20 millions or more, will have very little price 'impact' on the NYSE while the equivalent, 50 contracts or more, traded on the CME will have a larger effect on price.

[9]$S_t/Ke^{-r\tau}$ is the definition of moneyness.

interest rate for the sample period is 3.44% ranging from 2.63% to 5.76%. The average annualized dividend yields for the S&P 500 index is 2.8%, ranging from 2.65% to 3.02%.

4 Empirical Tests

4.1 Tests of the Martingale Property

We test the martingale property, using three distributional specifications: the lognormal density, the general density based on Hermite polynomials and the general density based on Laguerre polynomials. The density based on Hermite polynomials, as used by Longstaff (1995), is implemented in order to assess the potential bias induced by the logarithmic transformation. A nonlinear regression is performed with seven call option prices to estimate the implied parameters. For the general option pricing model, the first four implied conditional moments are estimated by the nonlinear regression. In the case of a lognormal distribution, two parameters are sufficient to describe the entire distribution. Once the implied parameters are estimated, the entire density and the implied mean of the distribution at the maturity can be easily recovered.

Table 3 presents summary statistics for the percentage pricing difference for three distributional specifications. In the case of the lognormal specification, as in Longstaff (1995), the martingale property is significantly rejected. The average percentage difference between the implied index value and the actual index value is 0.542%. The median percentage difference is 0.5067%. The null hypothesis of the martingale restriction is significantly rejected with t-statistic of 27.5384. Given that the average S&P index value for the sample period is 454.6028, the estimated mean value has a deviation of 2.46 from the index value.

Longstaff (1995), using the S&P 100 call option prices, finds an average percentage difference for the lognormal specification of 0.465% with a t-statistic of 31.52 and a median percentage difference of 0.410% for all observations in his sample. When 6-7 call options are used in his empirical test, the average percentage difference is 0.628% with a median of 0.530%. Although the S&P 500 index option is generally less liquid than the S&P 100 option[10], the pattern of percentage difference in the S&P 500 call options is consistent with that in the S&P 100 options when the lognormal distributions is used.

Given the very specific lognormal assumption it is not surprising that our tests as well as Longstaff's reject the martingale property. However, the empirical results of the martingale property test based on the proposed general density are quite different from those based on the lognormal specification. Table 3 also reports summary statistics for the percentage pricing differences between the actual and the implied index estimated from the general

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[10]For example, the average bid and ask spreads of the S&P 100 index options reported in his study is 0.153 which is less than half of the S&P 500 call options spread used in this paper.

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density based on the Laguerre polynomials. The average percentage difference is 0.1123% with a t-statistic of 7.15. Given that the average S&P index value for the sample period is 454.6028, the mean of deviation, estimated from the general distribution, is just about 0.5105. The median percentage difference is only 0.0108%. The median deviation is about 0.0491.

The empirical results of Longstaff (1995), based on the Hermite polynomial series estimator, are significantly different from the results based on the Laguerre polynomials. Based on the distribution approximated by Hermite polynomials, Longstaff (1995) obtains a mean percentage difference of 0.400% with a t-statistic of 31.08, and a median of 0.311%. The percentage difference for the lognormal specification based on three to four call options is 0.509%, so that about 10% of the mean percentage difference is reduced due to the application of the general distribution. However, in our tests based on the Laguerre density estimator, the reduction of the mean difference is 0.43%, which is about 80% reduction from the mean percentage difference of the lognormal specification.

To ascertain whether these different empirical results are due to a different data set, the Hermite polynomial series used by Longstaff is also applied to our sample. Table 3 also reports summary statistics of percentage pricing differences between the actual and the implied indices estimated from the Hermite polynomial series. The average percentage difference based on the Hermite polynomial series is 0.2564% with a t-statistic of 12.11, which is about twice that based on the Laguerre series. The median percentage difference for call options estimated by the Hermite polynomial series is 0.1941%. These results imply that the bias in the implied mean estimated from the transformed Hermite polynomials may substantially affect tests of the martingale property as discussed in the previous section.

Empirical results of the martingale property tests based on the F-statistic are quite different from those based on the t-statistic. Specifically, this paper empirically analyzes 1,680 (= 240 × 7) call options data with a total of 1,120 (= 240 × 4) parameters for the unrestricted general model, and a total of 480 (= 240 × 2) for the lognormal specification. Thus F-statistic is asymptotically distributed with an $F[240, (7-M) \times 240]$ distribution with $M = 2$ for the lognormal specification and $M = 4$ for the proposed general distribution.

In the case of the lognormal specification, the value of the $F-$statistic for call options is 49.7506 which clearly rejects the null hypothesis of the martingale property.[11] However, in the case of the proposed general model, the value of the $F-$statistic for call options becomes 0.3374.[12]

These empirical results suggest that Longstaff's (1995) strong rejection of the martingale property is questionable. Even though the martingale property is rejected here based on the $t-$statistic, the magnitudes of the deviations from the martingale property are much smaller than those in Longstaff (1995). Based on the $F-$statistic, the martingale property is rejected

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[11]The critical values at 5% and 1% significance levels are respectively 1.1729 and 1.2522.

[12]The critical values at 5% and 1% significance levels are respectively 1.1844 and 1.2698.

for the lognormal specification but not rejected for the general specification. Relaxing the restriction does not reduce much the pricing errors.

4.2 Tests with Put Options

The pattern of empirical results for put options is quite similar to that for call options. In the case of the lognormal specification, the average percentage difference between the implied index value and the actual index value is 0.610%, while the median is 0.5067%. The martingale property is significantly rejected with a $t-$statistic of 17.0291. Overall, the empirical results from the lognormal specification strongly reject the martingale restriction.

In the case of the general distribution based on the Laguerre series, the average percentage difference is -0.0485% with a t-statistic of -4.85, which implies that the deviation from the index value is about -0.2205. The median percentage difference for put options estimated by the proposed general distribution is -0.0120%, which is almost the same magnitude as the median percentage difference for call options, but with a different sign. The negative percentage difference implies that the price of put options is on average higher than that implied by the current index level.

Here too the F-statistics tells a different story.[13] In the case of the lognormal specification, the value of the $F-$statistic for put options is 11.4654, which clearly rejects the null hypothesis of the martingale property . However, in the case of the proposed general model, the value of the $F-$statistic for put options is 0.6832, which fails to reject the martingale property.

In sum, we cannot reject the martingale property based on the F-statistic using call options and put options.

4.3 Implications for Pricing Errors

Table 4 reports summary statistics of the absolute pricing errors for S&P 500 call options. The pricing errors are obtained from the actual option prices and the fitted option prices of different models. A total of 1,680 pricing errors from call options are used for subsequent analysis.

For the lognormal specification, the mean of the absolute pricing errors in the restricted model is 53 cents, and the median of the absolute pricing errors is 46 cents. As also noted by Longstaff (1995), the pricing errors from the unrestricted lognormal specification are dramatically less than those from the restricted model. For call options, the mean of absolute

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[13]Though the two samples are not entirely independent due to put-call parity , the facts are that, unlike options on individual stocks, put-call parity holds only within certain wide bands due to transaction costs.

pricing errors from the unrestricted model is reduced to 14 cents from 53 cents, a reduction of 73%. The median from the unrestricted lognormal specification is reduced to 10 cents from 46 cents, a reduction of 77%.

For the proposed Laguerre series density, the mean of the absolute pricing error is substantially reduced even without relaxing the martingale restriction. Based on this model, the mean of absolute pricing errors for call options is ten cents and the median is seven cents. These represent reductions of 81% and 86%, respectively, from the restricted lognormal specification. The mean of absolute pricing errors from the proposed model without relaxing the martingale property is less than that from the unrestricted lognormal specification which relaxes the martingale property. When the martingale restriction is relaxed for the proposed model, the pricing error declines a marginal one cent. Thus, the failure to reject the martingale property based on the F−statistic is due to the fact that the proposed density estimator can match option prices well without relaxing the martingale restriction. Given that the bid and ask spread for S&P 500 options is usually larger than $\frac{1}{8}$, the proposed model has a median of pricing errors that is less than the usual bid-ask spread of options.

In Figure 3 and Figure 4, the pricing errors from each model are plotted against the moneyness of options. Clearly, the main source of mispricing in the lognormal specification is the moneyness bias or 'smirk' effect. By relaxing the martingale property of the lognormal specification, the mean of distribution shifts such that the moneyness bias is reduced. However, for put options, the shifted distribution does not make the 'smile' effect disappear. The proposed density estimator reduces the moneyness bias by shifting the skewness and kurtosis of the distribution, instead of shifting the mean as in the lognormal specification.

Table 5, which reports the regression results from regressing pricing errors on the moneyness of options, time to maturity and the bid and ask spread, shows that the successful reduction of pricing errors is mainly due to the elimination of the moneyness bias. Note that the R^2 of the regression for for the lognormal specification is 71.7%.

4.4 The Martingale Property with Futures Prices

One advantage of analyzing the S&P 500 index options market is that the options can be hedged by using S&P 500 index futures. Under the assumption of stochastic interest rates, the forward price for a non-dividend paying asset is

$$F_t = \frac{E_t^Q \left[\exp\left(-\int_t^T r(s)\,ds\right) S_T \right]}{E_t^Q \left[\exp\left(-\int_t^T r(s)\,ds\right) \right]} \cdot$$

If the short-interest rate (r) and the underlying asset (S) are statistically independent, or the short-interest rate (r) is deterministic, the forward price is equal to the futures price,

which simplifies to[14]

$$F_t = E_t^Q [S_T] \, .$$

In the case of a dividend paying asset, if the short-rate process and the dividend yield are constant, the futures price simplifies to the well-known cost-of-carry formula

$$F_t = S_t e^{(r-d)(T-t)} \, ,$$

which is the martingale property imposed on the conditional mean of the implied distribution at maturity. Instead of analyzing the percentage difference between the actual index and the implied index, the percentage difference between the actual futures price and that implied from option prices is tested, using futures and options with the same maturity date

$$\left(\frac{\widehat{E_t [S_T]} - F_t}{F_t} \right) \times 100 \, .$$

If the martingale property holds, the percentage pricing difference between the actual futures price and the conditional mean implied from the option prices should be the same.

As argued before, futures prices have an advantage being more synchronous with the options prices, both closing at 3:15 p.m. while the stock market closes at 3:00 p.m.. Another advantage of using futures prices is that we do not need to use interest rates data to test the martingale property.

In testing the martingale property, options which expire on the same date as the futures contracts, that is 80 samples out of the total of 240 observations, are used. Before we use the implied index of option prices, we test the martingale property by using the cost-of-carry relationship between actual spot prices and futures prices. The mean of percentage pricing difference from the cost-of-carry model is 0.0148% with a t-statistic of 1.07 when the actual spot price with the cost-of-carry model is used to price futures contracts.

The mean percentage pricing differences between the futures price and the implied mean from the lognormal specification are 0.5733% for call options and 0.7168% for put options. Their t-statistics are 16.23 for call options and 9.57, respectively, which rejects the martingale property. The rejection pattern is very similar to the test results based on the implied index and the actual index. For the proposed general density model, the mean of the percentage pricing differences for call options is 0.1178% with a t-statistic of 5.05, while, for put options, it is -0.0172% with a t-statistic of -0.73. These results imply that the nonsynchronous trading problem is of marginal importance.

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[14]The forward and futures equivalence may not hold due to the marking to market procedure if there is a correlation between interest rates and the stock index. It is generally accepted that such a correlation is rather small for stock indices.

4.5 Options' Liquidity and Deviations from the Martingale property

Longstaff (1995) argues that the differences between the actual index and the implied index from the unrestricted lognormal specification are strongly related to the proxies for market liquidity such as the bid and ask spread of options, open interest, and the total trading volume for call options. His empirical results show that the average bid and ask spread of the options used in estimating the implied index value is positively related to these differences. His results indicate that the martingale property could be violated due to market friction. Therefore, we analyze effect of potential frictions on the tests of the martingale property and use regression analysis to test what factors may have affected the deviations from the martingale property.

Bid and ask spreads may be related to the deviations from the martingale property because of the convexity effect. In implementing the estimation, the mid-points of the options' bid and ask quotes are used. Due to the convexity of options with respect to the underlying price, the implied index taken from the mid-point of the option's bid and ask price is not the same as the mid-point between the implied index from the bid price and the implied index from the ask price. If the true index level is distributed symmetrically between the implied index from the bid price and the implied index from the ask price, the convexity effect can bias upward the estimated difference between the actual and the implied index for the call option and can bias downward the estimated difference for the put option.

Another important factor in explaining deviations from the martingale property is the moneyness of options. First, as reported in many papers, the standard Black-Scholes-Merton model underprices in-the-money options. Thus, the violation of the martingale property may result from the model's inability to account for the moneyness bias of options. Alternatively the violation may be due to a sampling problem. If more out-of-the-money options are used in the estimation, the mean of the density could be biased because out-of-the-money options tend to be less liquid. On the other hand, options with more strike prices provide more information about the entire density. Thus, the standard deviation of the options moneyness' is used as a proxy for the dispersion of moneyness.

Finally, the interest rate used in the estimation is included in the regression analysis. In the estimation, interpolated interest rates based on one month and three month interest rates are used. Though option values are less sensitive to interest rates for short-term options, the interpolation could induce errors in measuring true interest rates.

Table 6 reports estimation results from regressing percentage differences of the lognormal specification on the selected variables. As in Longstaff (1995), the average bid and ask spread of options used in the estimation is significantly related to deviations from the martingale property in the lognormal specification. The percentage difference between the actual and the implied indices increases as the average of the bid and ask spread increases. The estimated difference increases as more in-the-money options are used in the estimation. The dispersion

of options' moneyness significantly relates to the estimated difference. Thus, the deviations from the martingale property contracts as moneyness becomes less concentrated.

Regression results from the proposed general distribution are very different from those of the lognormal specification. The estimated percentage difference is no longer significantly related to the bid and ask spread and the average moneyness of options used in the estimation. These regression results indicate that the estimated pricing difference is *not* related to market frictions, such as the bid and ask spreads as suggested by Longstaff (1995). The estimated deviations from the martingale property is not significantly different from zero based on the F-statistics and its magnitude is very small relative to those from the lognormal specification. Moreover, since S&P 500 options are generally less liquid than the S&P 100 options, rejection of the martingale property is less likely to be caused by the market frictions or liquidity, if it is to be rejected at all.

5 Conclusion

The derivation of risk neutral valuation for contingent claims is one of the most important innovations in modern asset pricing theory. However, it is empirically challenged by Longstaff (1995), which shows that the martingale property, implied by the risk-neutral valuation, is strongly rejected using S&P 100 call options data. Since rejection of the martingale restriction is of critical importance to both theoretical and practical aspects of option pricing, the martingale property is empirically re-examined with a new method and different sets of data. A new method using a general distribution based on Laguerre polynomials provides a general framework for computing option prices and for testing moment restrictions. The proposed density estimator has substantially smaller approximation errors for a wider spectrum of possible distributions, and, more importantly, it is free from the bias present in the Hermite polynomials density used in Longstaff (1995). The new method is applied to S&P 500 options that are free from the early exercise premium and 'wildcard' feature.

Empirical results, based on S&P 500 call and put options for the period from January, 1993 to December, 1994 show that: 1) the martingale property cannot be rejected for either call or put options using the F-tests; 2) most pricing errors from the proposed model are less than the options' bid and ask spreads even without relaxing the martingale restriction; 3) the deviation from the martingale property is not related to the bid and ask spread of options; and 4) results are robust when the pricing relation between futures contracts and options are used.

Empirically, the technique developed in this paper may resolve important issues in pricing derivative securities. First, by using the implied distribution from options prices, other derivative securities on the same underlying asset can be priced. Valuation of S&P 500 index futures and options on the S&P 500 index futures is but one example. Valuation of exotic options such as knock-out options in the currency market could be another interesting

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application.

6 Appendix: Option formula based on the Laguerre series

Many techniques have been presented over the years to solve the so-called finite problem of moments, that is, the problem of determining or approximating a probability distribution from a finite number of its moments. When a probability distribution has a nonzero value for its p.d.f only in the range $0 < x < \infty$, then a series developed from the gamma distribution is used for approximating the p.d.f. from the moments of the distribution. Using a derivation similar to that used for the Hermite polynomial series, one can obtain the following general form of the series expansion of the p.d.f. $f(x)$

$$f(x) = \sum_{j=0}^{\infty} \alpha_j L_j^{(m)}(x) \Phi_m(x), \ 0 < x < \infty,$$

where

$$L_r^{(m)}(x) = \frac{1}{r!} \sum_{j=0}^{n} \binom{r}{j} (-1)^j \frac{\Gamma(m+r)}{\Gamma(m+j)} x^j,$$

$$\Phi_m(x) = \frac{x^{m-1} \exp(-x)}{\Gamma(m)},$$

and

$$\alpha_r = \frac{E\left[L_r^{(m)}(x)\right]}{\binom{m+r-1}{r}}.$$

Let $f(x)$ be the approximated by $f_k(x)$, a finite series involving Laguerre polynomials

$$f_k(x) = \sum_{j=0}^{k} \alpha_j L_j^{(m)}(x) \Psi_m(x).$$

To be explicit,

$$L_0^{(m)}(x) = 1,$$

$$L_1^{(m)}(x) = m - x,$$

$$L_2^{(m)}(x) = \frac{1}{2!} m(m+1) - (m+1)x + \frac{x^2}{2!},$$

$$L_3^{(m)}(x) = \frac{1}{3!} m(m+1)(m+2) - \frac{1}{2!}(m+1)(m+2)x + \frac{1}{2!}(m+2)x^2 - \frac{1}{3!}x^3,$$

$$L_4^{(m)}(x) = \frac{1}{4!}m(m+1)(m+2)(m+3) - \frac{1}{3!}(m+1)(m+2)(m+3)x +$$
$$\frac{1}{2!}(m+2)(m+3)x^3 - \frac{1}{4!}x^4.$$

For computational simplicity, the equation is written as

$$f_k(x) = \Phi_m(x) + \sum_{i=3}^{k} \beta_i \sum_{j=0}^{r} \binom{i}{j} (-1)^j \Phi_{m+j}(x),$$

where

$$\beta_r = \binom{m+r-1}{r} \alpha_r.$$

Let $E(x) = \mu$, $\theta = var(x)/\mu$, and $\delta = \mu/\theta$. By taking the transformation of x/θ,

$$f_k(x) = \Psi_\delta(x) + \sum_{i=3}^{k} b_i \sum_{j=0}^{i} \binom{i}{j} (-1)^j \Psi_{\delta+j}(x).$$

where

$$\Psi_\delta(x) = \frac{x^{\delta-1}\exp(-x/\theta)}{\theta^\delta \Gamma(\delta)}.$$

and

$$b_i = E\{L_i^{(\delta)}(x/\theta)\}$$
$$= \frac{1}{i!}\sum_{j=0}^{i} \binom{i}{j} (-1)^j E\left\{\left(\frac{x}{\theta}\right)^j\right\} \frac{\Gamma(\delta+i)}{\Gamma(\delta+j)}.$$

Using the general density based on the Laguerre polynomial series, we can derive the explicit form of the expected value of options' payoff function. Consider the evaluation of $E[\max(x - k), 0]$, which is

$$\int_K^\infty (x - K)f_k(x)dx.$$

By making use of the fact that

$$\int_0^K \frac{x^{\delta+j-1}\exp(-x/\theta)}{\theta^{\delta+j}\Gamma(\delta+j)}dx = I(K/\theta, \delta+j),$$

where $I(.,.)$ is an incomplete gamma function

$$I(u, a) = \int_0^u \frac{t^{a-1}e^{-t}}{\Gamma(a)}dt$$

and the fact that

$$\int_0^K x\frac{x^{\delta+j-1}\exp(-x/\theta)}{\theta^{\delta+j}\Gamma(\delta+j)}dx = (\delta+j)I(K/\theta, \delta+j+1),$$

it can be seen that

$$\int_0^K x f_k(x) dx$$

$$= \theta \delta I(K/\theta, \delta + 1) + \sum_{i=3}^k b_i \sum_{j=0}^i \binom{i}{j} (-1)^j (\delta + j) \theta I(K/\theta, \delta + 1 + j)$$

and

$$\int_0^K f_k(x) dx$$

$$= I(K/\theta, \delta) + \sum_{i=3}^k b_i \sum_{j=0}^i \binom{i}{j} (-1)^j I(K/\theta, \delta + j).$$

Thus, by making the use of the analytical evaluation of $E\left[\max(x - k), 0\right]$, we can easily see that the resulting call option value is

$$C(S_t, K, r, T : \mu_1, \mu_2, \mu_3, \mu_4) =$$

$$e^{-r(T-t)} \left[(\mu_1 - K) - \left\{ \begin{array}{c} \theta \delta I(K/\theta, \delta + 1) + \\ \sum_{i=3}^k b_i \sum_{j=0}^i \binom{i}{j} (-1)^j (\delta + j) \theta I(K/\theta, \delta + 1 + j) \end{array} \right\} \\ -K \left\{ I(K/\theta, \delta) + \sum_{i=3}^k b_i \sum_{j=0}^i \binom{i}{j} (-1)^j I(K/\theta, \delta + j) \right\} \right]$$

where

$$\Psi_\delta(S_T) = \frac{1}{\theta^\delta \Gamma(\delta)} S_T^{\delta-1} e^{-S_T/\theta}, S_T \geq 0$$

and

$$b_i = E\{L_i^{(\delta)}(S_T/\theta)\}$$

$$= \frac{1}{i!} \sum_{j=0}^i \binom{i}{j} (-1)^j \left(\frac{\mu_j}{\theta^j}\right) \frac{\Gamma(\delta + i)}{\Gamma(\delta + j)},$$

$i = 0, 1, 2, ...(b_0 = 1, b_1 = 0)$, $\mu_j = E(S_T^j | S_t)$, $\theta = Var(S_T | S_t)/\mu_1$, $\delta = \mu_1/\theta$, and $I(.)$ is an incomplete gamma function defined as $I(u, a) = \int_0^u \frac{t^{a-1} e^{-t}}{\Gamma(a)} dt$.

The put option value is

$$P(S_t, K, r, T : \mu_1, \mu_2, \mu_3, \mu_4) =$$

$$e^{-r(T-t)} \left[K \left\{ I(K/\theta, \delta) + \sum_{i=3}^k b_i \sum_{j=0}^i \binom{i}{j} (-1)^j I(K/\theta, \delta + j) \right\} \\ - \left\{ \begin{array}{c} \theta \delta I(K/\theta, \delta + 1) + \\ \sum_{i=3}^k b_i \sum_{j=0}^i \binom{i}{j} (-1)^j (\delta + j) \theta I(K/\theta, \delta + 1 + j) \end{array} \right\} \right]$$

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Table 1

Mean Absolute Dollar Difference Between Theoretical Option Values and Approximated Option Values by Laguerre Polynomials

Panel A. Maturity : One Month

S/Ke^{-rt}	$|CEV_T - CEV_L|$	$|BS_T - BS_L|$	$|CEV_T - BS_T|$	BS
0.950	0.000606	0.000244	0.004659	0.3677
0.975	0.000130	0.000650	0.029376	1.4133
1.000	0.000611	0.000611	0.069649	4.6063
1.025	0.000704	0.000280	0.069757	11.1780
1.050	0.000433	0.000705	0.043473	19.4449
Avg	0.000496	0.000498	0.043383	

Panel C. Maturity : Three Month

S/Ke^{-rt}	$|CEV_T - CEV_L|$	$|BS_T - BS_L|$	$|CEV_T - BS_T|$	BS
0.950	0.002406	0.004544	0.007687	1.9994
0.975	0.001133	0.004525	0.047829	4.1255
1.000	0.004247	0.002467	0.103868	7.9776
1.025	0.005726	0.001180	0.134443	13.8498
1.050	0.005414	0.004218	0.129753	21.0930
Avg	0.003785	0.003386	0.084716	

For each moneyness and maturity, the mean absolute dollar differences of the Black-Scholes-Merton (BSM) model are calculated based on 13 annualized volatilities which are 4%, 5%, 6%, ..., and 16%. Parameters of the Constant Elasticity of Variance (CEV) model are chosen to match the annualized variance of the CEV model to that of BSM with the elasticity of 1 (Square-Root Model). The current price (S_0) is set to 400, and the interest rate is set to 5%. Each column presents the mean absolute dollar difference based on 13 different volatilities. CEV_T represents the theoretical CEV option value and CEV_L represents the approximated option value using Laguerre polynomials. BSM_T also represents the theoretical BSM option value and BSM_L represents the approximated option value using Laguerre polynomials.

Table 2

Descriptive Statistics for the S&P 500 Index Options

Panel A. Spot Market Data

	Mean	Std. Dev.	Min	Max	N
Index Value	454.6028	11.8922	418.3400	480.7000	240
Interest Rate	3.4435	0.6935	2.6300	5.7600	240
Dividend Yield	2.8009	0.0709	2.6500	3.0200	240

Panel B. Call Options

	Mean	Std. Dev.	Min	Max	N
Avg. Option Value	6.9371	5.3357	0.0938	21.6250	1680
Moneyness	1.0003	0.0213	0.9594	1.0516	1680
Time to Maturity	0.1125	0.0559	0.0250	0.2000	1680
Bid-Ask Spread	0.3526	0.2192	0.0625	1.0000	1680

Panel C. Put Options

	Mean	Std. Dev.	Min	Max	N
Avg. Option Value	7.9602	5.4142	0.0938	23.0000	1680
Moneyness	0.9999	0.0210	0.9578	1.0516	1680
Time to Maturity	0.1125	0.0559	0.0250	0.2000	1680
Bid-Ask Spread	0.3776	0.2144	0.0625	1.0000	1680

The sample consists of 1680 call options and 1680 put options with maturity dates between January 1993, and December, 1994. Average option prices are mid-points between bid and ask quotes at the close. Prices are collected every Thursday from 1 to 10 weeks prior to maturity. Moneyness is defined as the ratio of spot value to the present value of exercise price (S/Ke^{-rt}). For options which are less than 30 days to maturity, one month interest rates are used. Otherwise, interest rates are the interpolated value between one month and three month interest rates weighted by the options' maturities. N denotes the number of observations.

Table 3

Summary Statistics for the Estimates of the Percentage Difference between the Implied Index Value Implied by the S&P 500 Index Option and the Actual Index value

Panel A. Percentage Differences of Call Options

	Mean	Median	Std. Dev.	t-stat.	Min	Max	N
Lognormal	0.5420	0.5067	0.3049	27.5384	-0.1036	1.4967	240
Hermite	0.2564	0.1941	0.3280	12.1094	-0.9825	1.6996	240
Laguerre	0.1113	0.0108	0.2433	7.1541	-0.1175	1.4891	240

Panel B. Percentage Differences of Put Options

	Mean	Median	Std. Dev.	t-stat.	Min	Max	N
Lognormal	0.6100	0.4734	0.5549	17.0291	-0.1362	4.2989	240
Laguerre	-0.0485	-0.0120	0.1549	-4.8515	-0.8509	0.4635	240

Panel C. F-Tests

	Lognormal	Laguerre
Call Option	49.7506	0.3374
Put Option	11.4654	0.6832

The percentage difference between the implied index value and the actual index value is defined as

$$\frac{E(S_T)e^{-(r-d)\tau} - S_0}{S_0} \times 100$$

where r is the interest rate, τ is the time to maturity, $E(S_T)$ is the estimated mean of the implied distribution at the maturity, and S_0 is the current index value. The estimated mean is from the nonlinear regression based on the sevenoptions nearest to at-the-money. The null hypothesis is that the difference between the implied index value and the actual value is zero. N denotes the number of observations for the test. F-tests are based on the vector of residuals from the nonlinear regression both for the unrestricted model and the restricted model. The critical values for the lognormal model at 5% and 1% significance levels are respectively 1.1729 and 1.2522. The critical values for the Laguerre polynomial model at 5% and 1% significance levels are respectively 1.1844 and 1.2698.

Table 4

Summary Statistics for the Absolute Pricing Error between the Fitted and Actual Option Prices

Panel A. Call Option

Model	Mean	Median	Std. Dev.	Min	Max	N
Restricted BSM	0.5294	0.4635	0.3719	0.0002	2.1636	1680
Unrestricted BSM	0.1389	0.1040	0.1376	0.0001	2.1850	1680
Restricted Laguerre	0.1011	0.0672	0.1246	0.0000	1.9480	1680
Unrestricted Laguerre	0.0930	0.0589	0.1204	0.0000	1.9319	1680

Panel B. Put Option

Model	Mean	Median	Std. Dev.	Min	Max	N
Restricted BSM	0.5353	0.4635	0.3850	0.0004	3.5376	1680
Unrestricted BSM	0.2730	0.2363	0.2258	0.0001	3.4789	1680
Restricted Laguerre	0.1933	0.1134	0.3064	0.0001	3.0283	1680
Unrestricted Laguerre	0.1390	0.0892	0.1837	0.0001	3.0289	1680

The restricted lognormal model assumes a lognormal distribution for the risk-neutral density, while the unrestricted lognormal model relaxes the martingale property. Parameters are estimated with the fitted index value and volatility together. Absolute pricing errors from the restricted Laguerre model are obtained from the general distribution, while the first moment is restricted by the martingale property. Absolute pricing errors from the unrestricted Laguerre model is also obtained from the general distribution with the first four moments. All four moments are fitted with option prices. The absolute pricing errors are residuals obtained from the nonlinear regression based on the seven options which are nearest to at-the-money.

Table 5

Regression Results of the Pricing Error

	Int	M	τ	B/A	R^2
Restricted Lognormal	-24.0402 (-38.9052)	23.9909 (38.1794)	-0.3289 (-2.0177)	0.2194 (3.5093)	0.717
Unrestricted Lognormal	-1.3611 (-3.9143)	1.3604 (3.8457)	0.0403 (0.4391)	-0.0489 (-1.3893)	0.013
Restricted Laguerre Model	0.9122 (3.1875)	-0.9221 (-3.1671)	-0.1209 (-1.6011)	0.0949 (3.2763)	0.007
Unrestricted Laguerre Model	0.7868 (2.7825)	-0.7682 (-2.7676)	-0.1225 (-1.7063)	0.0770 (2.7969)	0.005

Each cell presents the estimated coefficient followed by t-statistics from ordinary least square regression of the pricing errors between the fitted and actual option values on independent variables. Int denotes the intercept of the regression, M denotes the moneyness, τ denotes the time to maturity, and B/A is the bid and ask spread. Each regressions includes 1,680 observations.

Table 6

Regression Results of the Difference Between the Actual Index and the Implied Index

Panel A. Lognormal

Int	M	$Std.M$	B/A	τ	r	R^2
-0.1742	-16.1694		0.9618	3.3009	0.0030	0.704
(-2.3166)	(-5.3103)		(5.8784)	(12.9598)	(0.1944)	
-0.0761		-3.4648	0.6255	3.8421	0.0123	0.670
(-0.7551)		(-1.0573)	(3.8903)	(15.4609)	(0.7226)	
-0.1654	-16.0877	-0.4535	0.9585	3.3058	0.0036	0.704
(-1.7016)	(-5.1828)	(-0.1433)	(5.7883)	(12.8352)	(0.2224)	

Panel B. Laguerre

Int	M	$Std.M$	B/A	τ	r	R^2
0.2127	-4.0944		0.1539	0.7332	-0.0685	0.093
(2.0253)	(-0.9626)		(0.6733)	(2.0609)	(-3.1135)	
0.4867		-13.9541	0.0200	0.9367	-0.051	0.130
(3.7251)		(-3.2851)	(0.0961)	(2.9079)	(-2.3144)	
0.4476	-1.6357	-13.6479	0.0538	0.8820	-0.0519	0.130
(3.5912)	(-0.3851)	(-3.1527)	(0.2378)	(2.5034)	(-2.3377)	

Each cell presents the estimated coefficient followed by the corresponding t-statistic in brackets from ordinary least square regression of the pricing difference on independent variables. Int is the intercept of the regression, while M is the average moneyness used in the estimation of each difference between the actual index and the implied index. Std. M, the standard deviation, measures the dispersion of moneyness of the options. B/A is the average bid and ask spread, τ denotes the time to maturity, and r is the interest rate. Each of regressions includes 240 observations.

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Figure 1: **Example of Implied Distribution: Call Options.** Figure 1 presents the implied distribution estimated from call options. The upper panel shows the estimated volatilities of seven S&P 500 call options based on Black-Scholes model on August 18, 1994. Time to maturity of options is one month, and the index level is 463.17. The moneyness of option is defined as spot index divided by the present value of the exercise price. The lower panel shows the estimated implied distributions using four specifications: i) lognormal distribution with the martingale restriction (BSM-Rstr), ii) Laguerre series distribution with the restriction (Lag-Rstr), iii) lognormal distribution without the restriction (BSM-Unrstr), and iv) Laguerre series distribution without the restriction (Lag-Unrstr).

Implied Volatilities of BS Model
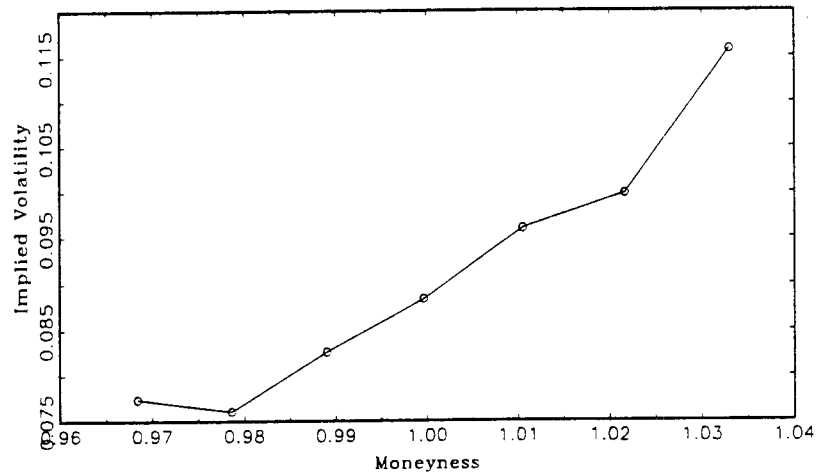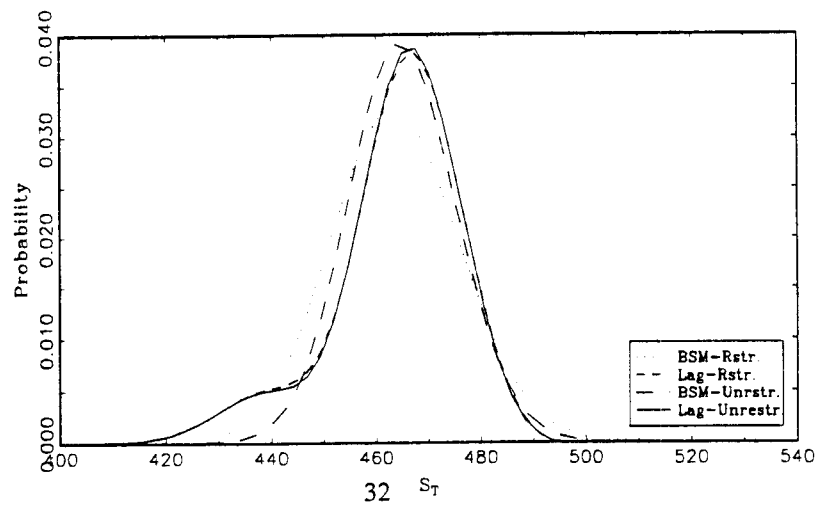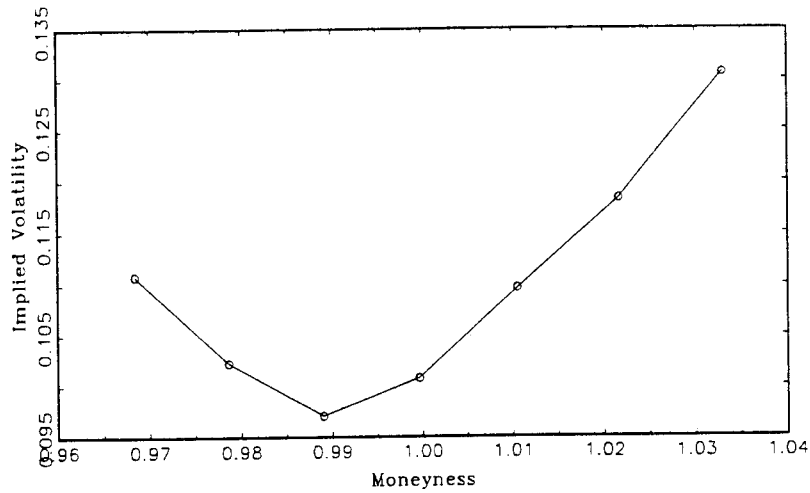

Implied Distribution At Maturity

Figure 2: **Example of Implied Distribution: Put Options.** Figure 2 presents the implied distribution estimated from put options. The upper panel shows the estimated volatilities of seven S&P 500 put options based on Black-Scholes model on August 18, 1994. Time to maturity of options is one month, and the index level is 463.17. The moneyness of option is defined as spot index divided by the present value of the exercise price. The lower panel shows the estimated implied distributions using four specifications: i) lognormal distribution with the martingale restriction (BSM-Rstr), ii) Laguerre series distribution with the restriction (Lag-Rstr), iii) lognormal distribution without the restriction (BSM-Unrstr), and iv) Laguerre series distribution without the restriction (Lag-Unrstr).

Implied Volatilities of BS Model
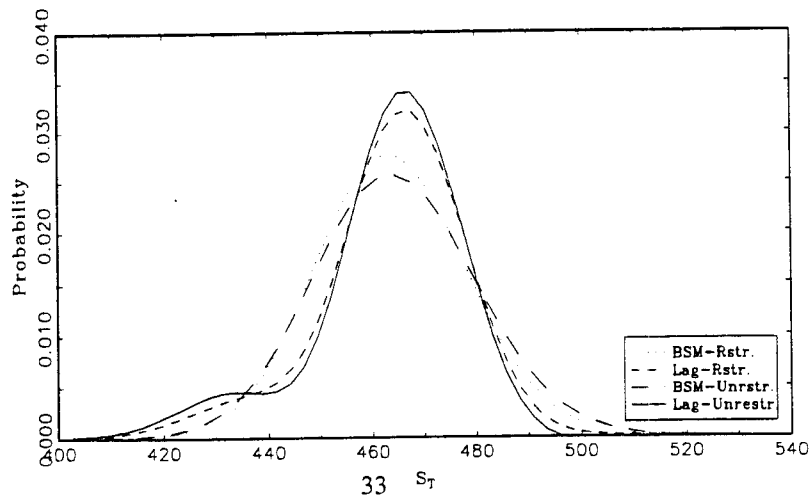

Implied Distribution At Maturity

Figure 3: **Pricing Errors; Call Options.** Figure 3 presents the pricing errors of call options using four specifications: i) lognormal distribution with the martingale restriction, ii) Laguerre series distribution with the restriction, iii) lognormal distribution without the restriction , and iv) Laguerre series distribution without the restriction. X-axis represents the moneyness of optins defined as the spot index divided by the present value of the exercise price. The sample period is January 1993 - December 1994, and the total number options is 1,680.
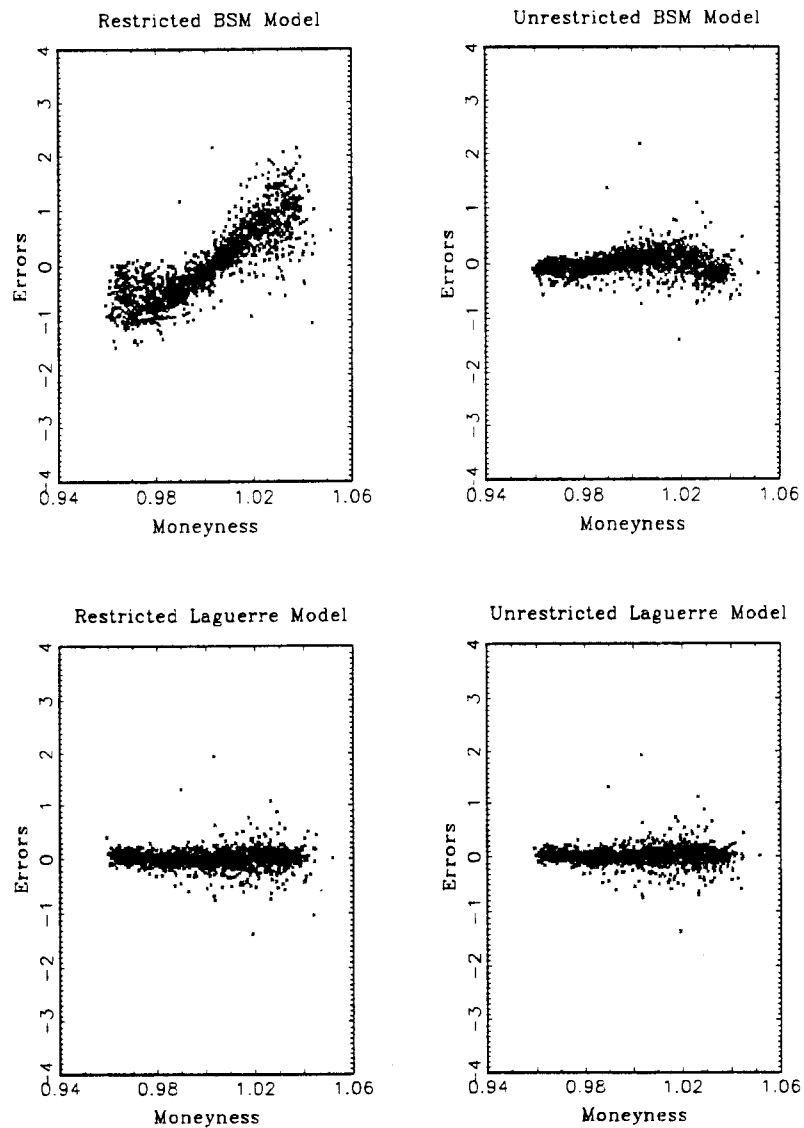
Figure 4: **Pricing Errors; Put Options.** Figure 4 presents the pricing errors of put options using four specifications: i) lognormal distribution with the martingale restriction, ii) Laguerre series distribution with the restriction, iii) lognormal distribution without the restriction , and iv) Laguerre series distribution without the restriction. X-axis represents the moneyness of optins defined as the spot index divided by the present value of the exercise price. The sample period is January 1993 - December 1994, and the total number options is 1,680.