NEW YORK UNIVERSITY
STERN SCHOOL OF BUSINESS
FINANCE DEPARTMENT

Working Paper Series, 1996

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FIN-96-12

forthcoming, *REVIEW OF ECONOMICS AND STATISTICS*

The Central Tendency: A Second Factor in Bond Yields

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May 1994. This revision, August 1996.

Abstract: We assume that the instantaneous riskless rate reverts towards a central tendency which, in turn, is changing stochastically over time. As a result, current short-term rates are not sufficient to predict future short-term rates movements, as it would be the case if the central tendency was constant. However, since longer-maturity bond prices incorporate information about the central tendency, longer-maturity bond yields can be used to predict future short-term rate movements. We develop a two-factor model of the term-structure which implies that a linear combination of any two rates can be used as a proxy for the central tendency. Based on this central-tendency proxy, we estimate a model of the one-month rate which performs better than models which assume the central tendency to be constant.

Keywords: term structure. **JEL#** G12.

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We thank for helpful comments Ned Elton, David Hait, Peter Jorgensen, Anthony Lynch, Eric Reiner, Frank Skinner, Bruce Tuckman, an anonymous referee, participants at the 1995 Western Finance Association Meetings, the 1995 European Finance Association Meetings, and especially the Editor, John Campbell.

1 Introduction

This paper develops a two-factor model of the term structure of interest rates. We follow the standard tradition in the finance term-structure literature (see, for example, Brennan and Schwartz (1979), Cox, Ingersoll, and Ross (1981, 1985), Longstaff and Schwartz (1992), and Vasicek (1977)), and identify the first factor with the *level* of the short-term rate. The novelty of our analysis is that we identify the second factor with the *central tendency* of the short-term rate. Our model captures the notion that short-term rates display short-lived fluctuations around a time-varying rest level, the central tendency, which, in turn, changes stochastically over time. As a result, current short-term rates are not sufficient to predict future short-term rates movements, as would be the case if the central tendency was constant over time. However, since longer-maturity bond prices incorporate information about the central tendency, longer-maturity bond yields can be used to predict future short-term rates movements.

Our study integrates two strands of literature. The first strand of literature is the traditional regression-based work on the term structure of interest rates, which uses information in longer-term rates to forecast future short-term rates. Examples of this approach are Campbell and Shiller (1992), Fama (1984), Fama and Bliss (1987), Mankiw (1986), Shiller (1979), and Shiller, Campbell, and Schoenholtz (1983). The second strand of literature uses no-arbitrage arguments to derive implications for longer-term yields based on the dynamics of the short-term rate. Classical examples of this second approach include Cox, Ingersoll, and Ross (1985) and Vasicek (1977). We take a middle course between the two approaches in the sense that we show how an equilibrium term-structure model can be used to learn about the dynamics of the short-term rate.

Our analysis gathers some preliminary evidence on the explanatory power of longer-term yields for the short-term rate. We regress the next-period one-month rate on the current one-month rate and a longer-maturity bond yield; we find that the bond yield always has explanatory power, especially for the shorter maturities. This evidence motivates a two-factor term-structure model, which is based on the assumption that the instantaneous riskless rate reverts towards a central tendency, which, in turn, changes stochastically over time. We impose minimal restrictions on the dynamics of the central tendency, other than that its conditional mean should not be affected by the instantaneous riskless rate. The theoretical analysis suggests a proxy for the central-tendency factor which depends on a linear combination of two yields. The central-tendency proxy is then used to estimate the stochastic process for the instantaneous riskless rate, proxied by the one-month Treasury bill rate.

In the empirical analysis we show that models of the one-month rate which allow for a time-varying central tendency perform better than the standard Vasicek (1977) and Cox, Ingersoll, and Ross (1985) models, in which the central tendency is assumed constant. We replicate the analysis for different subperiods. The time variation in the central tendency is especially important for the 1952–1971 and 1982–1993 periods, when the Federal Reserve's operating policy took into account both the volatility of interest rates and the behavior of monetary aggregates.

The approach advocated here complements other two-factor models of the term structure of interest rates such as that of Longstaff and Schwartz (1992), who propose a model in which the second factor is identified with the conditional volatility of the short-term rate. Also, in a spirit similar to ours, Babbs and Webber (1995), Naik and Lee (1994), and Pennacchi and Jagadeesh (1995) postulate a process for the short-term rate where its central tendency changes over time.

The paper is organized as follows. Section 2 presents the data and puts forward the motivating evidence, while Section 3 develops the theoretical model. In Section 4 we estimate various models of the one-month rate and we examine the properties of the central-tendency proxy. Section 5 performs diagnostics of the interest-rate model. Section 6 concludes.

2 Preliminary evidence

This section illustrates the properties of the data, and puts forward some preliminary evidence that longer-maturity yields contain information for future short-term interest rates.

Table 1 presents summary statistics for the interest-rate series used in the empirical analysis. We use monthly U.S. interest rates from June 1952 through December 1993. One-month Treasury-bill rates are from the Center for Research in Security Prices (CRSP) RISK-FREE file, while the discount-bond prices used to compute continuously compounded yields are from the FAMABLIS file also on the CRSP tape. All interest rates are continuously compounded. We consider four separate subperiods, June 1952 – August 1971, September 1971 – September 1979, October 1979 – September 1982, and October 1982 – December 1993.

The subperiods were selected to control for different operating procedures followed by the Federal Reserve: During the 1950s and 1960s the Federal Reserve took the view that monetary policy should be based on the intuitive judgment of money-market conditions. This period also roughly coincides with the time during which the Bretton Woods exchange-rate agreement was in place, which ended in August, 1971. During the 1970s the Federal Reserve made a formal commitment to targeting the growth rate of key monetary aggregates, such as M1 and M2. In practice, this translated into a tight targeting of the federal funds rate at a level perceived to be consistent with the stated quantity targets. The October 1979–October 1982 period witnessed the Federal Reserve's "experiment." The operating target of monetary policy became the amount of non-borrowed reserves with the banking system, and both the level and the volatility of interest rates reached levels never experienced before. The post-October 1982 period is one of de-emphasis of monetary aggregates and of renewed concern for reducing the volatility of interest rates.

While it is the changing institutional setting of post-war monetary policy that suggests the years 1971, 1979, and 1982 as possible break points, the evidence in Table 1 confirms that interest rates behaved quite differently across subperiods. Mean interest rates are highest in the third subperiod, and lowest in the first one; the volatility of the changes of all rates is also highest in the third subperiod, and lowest in the first one. The skewness of the levels of interest rates is highest (and positive) in the first subperiod, while it is lowest (and negative) in the third subperiod. This behavior is consistent with the notion that interest rates cannot turn negative, and that when interest rates are especially high large sudden drops are quite likely. Changes in interest rates also display marked excess kurtosis over the entire period, especially for the shorter-maturity rates, while this excess kurtosis disappears during the individual subperiods. This is consistent with the view that controlling for variations in means and variances across subperiods, as in Naik and Lee (1994) and in this paper, may improve the performance of models with Gaussian innovations. Table 1 also shows that all rates exhibit high first-order autocorrelations for the first subperiod; and the higher-order autocorrelations (not shown in the Table) decay very slowly. Fama (1984, page 515) writes that this "pattern [is] suggestive of mean non-stationarity," as it would be the case with breaks in regimes and changes in the central tendency. In fact, the first-order autocorrelations for the individual subperiods are lower than for the entire period. Given the persistence of the levels of all rates, when appropriate we conduct our analysis both in levels and in first differences. This approach is consistent, for example, with the findings of Campbell and Shiller (1987), who provide strong evidence of unit roots in the monthly series of U.S. interest rates, for the 1959-1978 period.

With a time-varying central tendency, current short-term rates do not adequately summarize the information in the bond market about future short-term rates. Hence, in addition to the current short-term rate, we should use longer-maturity yields. To explore this idea we regress the future short-term rate on the current short-term rate and current longer-term yields:

$$r_{t+1} = \beta_0 + \beta_1 r_t + \beta_2 \, y_t(\tau) + \text{error}_{t+1}, \tag{1}$$

where $y(\tau)$ is the yield on a discount bond of maturity $\tau = 1, 2, 3, 4,$ and 5 years, and r is the one-month interest rate. This regression is performed for both the levels and the first differences of the interest rates. If interest rates were driven by a single factor following an autoregressive process as in Cox, Ingersoll, and Ross (1985) or Vasicek (1977), current short-term rates would completely summarize the information about future short-term rates, and we should find that $\beta_2 = 0$.

Table 2 reports the results of estimating the statistical model (1), both in levels and in first differences, for the entire period and for the different subperiods. The main finding is that for the entire period the estimates of all coefficients are significant. In particular β_2, the coefficient of the long-term yield, is significant both for the regression in levels and in first differences. This finding confirms that there is information in the longer-maturity yields that can be used to explain future short-term rates variations. Longer-term rates are more significant in the first and fourth subperiods. Thus, we expect a model with a time-varying central tendency to perform better in these subperiods than in the second and third one. Also, the coefficient of the long-term yield, β_2, tends to decrease with the maturity of the yield used in the regression, especially for the regression in levels. This finding is consistent with the notion that the shorter the maturity of the bond yield, the higher the information on proximate short-term rate changes. When we consider the regression in the levels, we also find that the coefficient of the short-term rate, β_1, tends to increase with the maturity. In fact, in all regressions the sum of β_1 and β_2 is approximately equal to one. This is consistent with the evidence that interest rates of all maturities are very persistent, as it is the case with integrated series, and with the notion that interest rates may be cointegrated. In fact, Campbell and Shiller (1987) provide evidence of a cointegrating relationship between short-term and long-term rates, as implied by the expectations hypothesis.

3 Yields with a time-varying central tendency

The evidence presented in the previous section suggests that longer-term bond yields help forecast future short-term rates, even when we control for information contained in the current level of the short-term rate. Hence, in this section we develop an equilibrium model of the term structure of interest rates, in which longer-term yields depend on the time-varying central tendency of the short-term rate. The resulting theory suggests that an appropriate combination of bond yields proxies for the central tendency of the postulated short-term rate process.

The behavior of the instantaneous riskless rate r is described by the stochastic differential equation[1]

$$dr = k(\theta - r)dt + \sqrt{\sigma_0^2 + \sigma_1^2 r}\, dZ, \tag{2}$$

where k, σ_0, and σ_1 are constants, and Z is a standard Brownian motion; θ is the central tendency towards which the instantaneous rate reverts. As in Pearson and Sun (1994), the diffusion term of (2) generalizes the square-root process of Cox, Ingersoll, and Ross (1985) by allowing the lower bound for the instantaneous interest rate to be different from zero.

In turn, θ evolves over time according to the stochastic differential equation

$$d\theta = m(\theta)dt + s(\theta)dW, \tag{3}$$

where, like Z, W is a standard Brownian motion. We assume that the conditional drift $m(\theta)$ and the conditional volatility $s(\theta)$ are "smooth" (continuous with continuous derivatives)

[1] We use the same notation to indicate the instantaneous riskless rate and the one-month rate, and in the empirical applications of Section 4 we use the one-month rate to proxy for the instantaneous riskless rate.

functions of θ alone. Specifically, θ may evolve according to

$$d\theta = (m_0 + m_1\theta)dt + \sqrt{s_0^2 + s_1^2\theta}\, dW. \tag{3'}$$

The covariance between the two factors, $drd\theta/dt = \sigma_{r\theta}$, is assumed constant.

We assume the premium required to compensate investors for the risk of fluctuations of r to be linear in r, $\lambda_0 + \lambda_1 r$, with λ_0, λ_1 constant, while the premium required to compensate investors for the risk of fluctuations of θ, $l(\theta)$, is a smooth function of θ alone. Under the assumptions above, the price of a risk-free discount bond of maturity τ, $P = P(r, \theta; \tau)$ satisfies the partial differential equation (see Cox, Ingersoll, and Ross (1985))

$$E(\mathcal{D}P) - rP - P_r(\lambda_0 + \lambda_1 r) - P_\theta l(\theta) = 0, \tag{4}$$

where $\mathcal{D}$ denotes the Dynkin differential operator.[2]

We assume that there is a solution for the fundamental valuation equation (4); the solution is of the form

$$P(r, \theta; \tau) = e^{-A(\theta;\tau) - B(\tau)r}, \tag{5}$$

where $B(\tau)$ is a constant that depends only on the maturity τ, while $A(\theta; \tau)$ is a smooth function of θ, for any given maturity τ. In fact one can write equation (4) as

$$B(\tau)[k\theta - \lambda_0 - (k + \lambda_1)r] + B^2(\tau)[\sigma_0^2 + \sigma_1^2 r + \sigma_2^2\theta]/2 + A_\theta(\theta; \tau)[m(\theta) - l(\theta)]$$
$$+[A_{\theta\theta}(\theta; \tau) + A_\theta^2(\theta; \tau)]s(\theta)^2/2 + B(\tau)A_\theta(\theta; \tau)\sigma_{r\theta} - B_\tau(\tau)r - A_\tau(\theta; \tau) - r = 0,$$

and verify that for the previous equation to be uniformly satisfied over the domain of r we need

$$-B(\tau)(k + \lambda_1) + B^2(\tau)\sigma_1^2/2 - B_\tau(\tau) - 1 = 0.$$

The solution for $B(\tau)$, subject to the initial condition that $B(0) = 0$, is well-known (see Cox, Ingersoll, and Ross (1985)):

$$B(\tau) = \frac{2(e^{\delta\tau} - 1)}{(\lambda_1 + \delta + k)(e^{\delta\tau} - 1) + 2\delta}, \quad \delta = \sqrt{(\lambda_1 + k)^2 + 2\sigma_1^2}, \tag{6}$$

which reduces to $B(\tau) = (1 - e^{-k\tau})/k$ when $\lambda_1 = \sigma_1^2 = 0$ (see Vasicek (1977)). If $m(\theta)$, $s(\theta)^2$ and $l(\theta)$ were linear functions of θ as postulated in equation (3′), yields would be linear in r and θ, and bond prices would have the familiar form

$$P(r, \theta; \tau) = e^{-C(\tau) - B(\tau)r - D(\tau)\theta}, \tag{7}$$

[2]Equation (4) can be written in the more intutive excess-returns form

$$\frac{E(\mathcal{D}P)}{P} - r = \frac{P_r}{P}(\lambda_0 + \lambda_1 r) + \frac{P_\theta}{P}l(\theta),$$

where $\frac{P_r}{P}$ can be interpreted as the short-term rate "beta," and $\frac{P_\theta}{P}$ as the central-tendency beta. By analogy with standard linear pricing models, $(\lambda_0 + \lambda_1 r)$ and $l(\theta)$ are the factor risk premia associated with r and θ, respectively.

where $A(\theta;\tau) = C(\tau) + D(\tau)\theta$. Conditions for yields to be linear in the underlying state variables are given in Cox, Ingersoll, and Ross (1981), and Duffie and Kan (1993).

In the following we show how, by choosing appropriate weights, one can build a linear combination of two bond yields which is independent of r and thus mimics the variation of the second factor only. This implication that yields of different maturities can be used as instruments for unobservable factors is well known (Cox, Ingersoll, and Ross (1985), pp.398-401), and is the basis of a number of empirical studies, *e.g.*, Stambaugh (1988) and Sun (1992), before ours. Formally, consider two bonds of maturity τ_1 and τ_2, respectively. From equation (5), the corresponding yields are

$$y(r,\theta;\tau_i) = -\ln P(r,\theta;\tau_i)/\tau_i = [A(\theta;\tau_i) + B(\tau_i)r]/\tau_i, \qquad \text{for } i = 1, 2. \tag{8}$$

Solving for r from the first yield, substituting into the second, and rearranging we obtain:

$$\tau_1 B(\tau_2)y(r,\theta;\tau_1) - \tau_2 B(\tau_1)y(r,\theta;\tau_2) = B(\tau_2)A(\theta;\tau_1) - B(\tau_1)A(\theta;\tau_2). \tag{9}$$

Note that this quantity does not depend on r. Thus, we can use an affine function of the variable on the left-hand side of equation (9) as a proxy for θ. In fact, if the drift and diffusion of the process for θ were also linear in θ, then prices would be of the form in equation (7), and the second factor θ could be written as

$$\theta = \frac{B(\tau_2)[\tau_1 y(r,\theta;\tau_1) - C(\tau_1)] - B(\tau_1)[\tau_2 y(r,\theta;\tau_2) - C(\tau_2)]}{B(\tau_2)D(\tau_1) - B(\tau_1)D(\tau_2)}. \tag{10}$$

While this is the case only when θ is an exact affine function of the variable on the left-hand side of equation (9), equation (10) justifies a *proxy* for θ which is used in the empirical analysis of the next section. We denote this proxy with $\hat{\theta}$:

$$\hat{\theta} = a_0 + a_1[B(\tau_2)\tau_1 y(r,\theta;\tau_1) - B(\tau_1)\tau_2 y(r,\theta;\tau_2)]. \tag{11}$$

Using the proxy (11) rather than the variable (10) is convenient in that even if m and s are not linear in θ, still we would expect (11) to be a reasonable approximation of the true, unknown, functional form relating θ to any two bond yields. At this point, we may also note that if the conditional mean and variance of the process for θ were in fact linear in θ, all proxies estimated from any bond-maturity pair (τ_1, τ_2) would be the *same*.

4 Empirical analysis

This section uses the previous theoretical findings to show how an equilibrium term-structure model can be used to learn about the dynamics of the short-term rate.

An important issue in estimation is that of identifying empirical counterparts to the factors r and θ, which are in principle unobservable. We follow a standard practice (see, for example, Chan, Karolyi, Longstaff, and Sanders (1992), and Longstaff and Schwartz (1992))) and treat the one-month Treasury bill rate as a proxy for the instantaneous rate r. While this practice is convenient, the one-month rate is only an approximation of the instantaneous rate of interest, because yields on any finite-maturity bond depend on *both* factors, r and θ, as well as on their risk premia, $\lambda_0 + \lambda_1 r$ and $l(\theta)$ (see, *e.g.*, Pearson and Sun (1994)). Second, we proxy θ with the quantity proposed in equation (11) of the previous section. The approximation depends on the fact that *only* when m and s are linear, is θ a linear function of the yields in (11).

A second issue is that we need to discretize the stochastic differential equation describing dr to implement it on discretely sampled data. When there is only one factor as, for example, when θ is constant, we could estimate an *exact* stochastic difference equation for the instantaneous rate implied by the differential equation for dr. However, since we allow θ to vary, we shall use the Euler discretization often used in the literature (see, for example, Chan, Karolyi, Longstaff, and Sanders (1992)),

$$r_{t+h} - r_t \approx kh(\theta_t - r_t) + \sqrt{\sigma_0^2 + \sigma_1^2 r}\, \epsilon_{t+h}. \tag{12}$$

Since we use monthly observations, we set $h = 1/12$ and estimate our model by maximizing the log-likelihood function

$$-.5 \sum \left[\ln \sigma_{t+1}^2 + (r_{t+1} - \bar{r}_{t+1})^2 / \sigma_{t+1}^2\right], \tag{13}$$

where

$$\bar{r}_{t+1} = (1 - (k/12))r_t + (k/12)\hat{\theta}_t \tag{14}$$
$$\sigma_{t+1}^2 = (\sigma_0^2/12) + (\sigma_1^2/12)r_t, \tag{15}$$

and $\hat{\theta}_t$ is given by

$$\hat{\theta}_t = a_0 + a_1[B(\tau_2)\tau_1 y(r_t, \theta_t; \tau_1) - B(\tau_1)\tau_2 y(r_t, \theta_t; \tau_2)], \tag{11}$$

where

$$B(\tau) = \frac{2(e^{\delta\tau} - 1)}{(\lambda_1 + \delta + k)(e^{\delta\tau} - 1) + 2\delta}, \quad \delta = \sqrt{(\lambda_1 + k)^2 + 2\sigma_1^2}. \tag{6}$$

We estimate the model (6), (11), (13), (14), and (15) using monthly observations. The estimates presented here are obtained using one- and two-year bond yields to construct the proxy in equation (11): $\tau_1 = 1$ and $\tau_2 = 2$.[3] The estimation period goes from July 1952 to December 1993. Results are presented in Table 3.

The Table shows parameter estimates for four different models:

[3]We experimented with different maturity pairs as well, with very similar, if slightly worse, results. The robustness of our results to the choice of maturity pairs is further discussed in Section 5.

1. Vasicek: $a_1 = \sigma_1 = 0$;
2. CIR: $a_1 = \sigma_0 = 0$;
3. Vasicek*: $\sigma_1 = 0$;
4. CIR*: $\sigma_0 = 0$;

In "Vasicek" and "CIR" we restrict $\hat{\theta}$ to be a constant. The first model corresponds to the Vasicek (1977) Ornstein-Uhlenbeck process, while the second model corresponds to the Cox, Ingersoll, and Ross (1985) square-root process. In "Vasicek*" and "CIR*" the mean of the square-root process is also affected by $\hat{\theta}$.

By equations (11) and (6) the coefficients relating the central-tendency proxy to bond yields depend on the parameter λ_1 which, in turn, describes the sensitivity of the market price of r-risk to r. In the empirical implementation, though, we found the log-likelihood to be largely insensitive to different values of λ_1: the Vasicek*, and CIR* models were estimated imposing $\lambda_1 = 0.0, -.5, -1.0, -2.0$, with slight decreases in likelihood value, the larger λ_1 in absolute value. Parameter estimates and the extracted $\hat{\theta}$ series were also essentially the same. Table 3 reports the estimation results for the different models assuming $\lambda_1 = 0$.

The full-period estimation results indicate that in the Vasicek* and CIR* models the estimate of a_1 is negative and significant: the central-tendency proxy is indeed time-varying. It is also apparent that the level of the one-month rate affects its conditional volatility: in the CIR* model the estimate of σ_1 is positive and significant. Also, it is worth noting that the assumption of a constant central tendency (Vasicek and CIR models) leads to lower estimates of the mean-reversion parameter k. This is consistent with the intuition that interest rates should converge more quickly towards a central tendency which is time-varying than towards a constant one.

Turning now to the subperiod evidence, we find that during the two periods September 1971 – September 1979 and October 1979 – September 1982, the measure of the central tendency captures little if any time-variation in the interest rate. This evidence is in agreement with the findings presented in Table 2, which shows that for the second and third subperiod the level and the changes of longer-maturity yields have little explanatory power for the future level and changes of the one-month rate, respectively. A possible interpretation for these findings is that during the September 1971 – September 1979 period the one-month rate itself was changing very smoothly, making the very concept of a central tendency of little use, at least at the monthly frequency. Also, the Federal Reserve's aggressive management of non-borrowed reserves during the October 1979 – September 1982 period resulted in such a high volatility of interest rates that the notion of a smoothly-changing time-varying central tendency is again useless. For this period, it is only the conditional volatility parameters to be estimated with reasonable precision, and the data show a slight preference for a model where volatility is tied to the level of the interest rate (CIR and CIR*) relative to a model with constant volatility (Vasicek and Vasicek*). On the other hand, for the first and the last subperiod the central-tendency proxy plays an important role in explaining the conditional mean of the one-month rate. These two periods have in common a somewhat "eclectic" approach of the Federal Reserve to monetary policy, where both interest-rate volatility and the growth of monetary aggregates were taken into account when formulating operating targets.

Using the parameter estimates for the CIR* model of Table 2, for the entire period, we obtain a time series of $\hat{\theta}$. Figure 1 plots $\hat{\theta}$ and the one-month rate. The visual evidence of Figure 1 confirms that $\hat{\theta}$ indeed behaves as a central tendency, and it is less volatile than the one-month rate of interest.

Table 4, Panel A, presents some summary statistics for the series $\hat{\theta}$ estimated for the entire period and for the four subperiods, both in levels and first differences. The comparison between Table 4 and Table 1 shows that $\hat{\theta}$ has essentially the same mean as the one-month rate, while its variability is always lower (both for the levels and for the first differences). This is not surprising since the central tendency should fluctuate around the same long-run mean as the one-month rate, and should exhibit a smoother behavior than the rate that reverts towards it. Other indications that $\hat{\theta}$ displays a smoother behavior than the one-month rate are as follows. The autocorrelation coefficient of $\hat{\theta}$ is always higher than that of the one-month rate. Moreover, when we consider the entire period, the kurtosis of the first differences of $\hat{\theta}$ is much lower than that of the first differences of the one-month rate: we may conclude that jumps in the central tendency are unlikely and our estimation of $\hat{\theta}$ acts as a "filter" of the one-month rate, eliminating some of the outliers in that series.

Table 4, Panel B, calculates the correlations between $\hat{\theta}$ series extracted using different maturity pairs using the CIR* model. We may recall that if the process for θ were in fact linear (that is, with $m(\theta)$ and $s(\theta)$ linear in equation (3)), all proxies estimated from any pair of bond yields would be the same. The correlations for the levels are all very close to one, showing that different maturity pairs lead to very similar $\hat{\theta}$ series. Still, the correlations for the first differences of the $\hat{\theta}$ series are lower, which is consistent, for example, with departures from linearity in the process for θ.

5 Diagnostics of the model

In this section we perform some tests of the appropriate specification of our model. First, we test whether the spread $\hat{\theta} - r$ indeed helps predict changes in r, as implied by our model. Second, we test whether the one-month rate helps predict future changes in the central-tendency proxy, which should not be the case according to the assumed recursive structure of the model. Third, we compare the forecast of changes in the one-month rate based on our model to an unrestricted forecast.

In Table 5, Panel A, we estimate the model

$$r_{t+1} - r_t = \beta_0 + \beta_1 \hat{\theta}_t + \beta_2 r_t + \text{error}_{t+1}, \tag{16}$$

where $\hat{\theta}$ is obtained from the estimation of the CIR* model (see Table 3). This model differs from the CIR* model in that we do not impose the restriction that $\beta_1 = -\beta_2 = k/12$. Moreover, we are not imposing a specific structure for the conditional volatility of the innovation. The standard errors reported in the Table are adjusted for heteroskedasticity of unknown form of the residuals. The evidence of Panel A is supportive of our analysis. The parameters are precisely estimated for the entire sample and in the last subperiod. Also, with the exclusion of the first subperiod, we cannot reject the restriction that $\beta_1 = -\beta_2$. This is consistent with the prediction of our model that the spread between the estimated central-tendency proxy and the one-month rate, $\hat{\theta} - r$, has explanatory power for future changes in the one-month rate.

The second test is reported in Table 5, Panel B, where we estimate the model

$$\hat{\theta}_{t+1} - \hat{\theta}_t = \beta_0 + \beta_1 \hat{\theta}_t + \beta_2 r_t + \text{error}_{t+1}. \tag{17}$$

According to the assumed recursive structure of our model (see equation (2) and (3)), r should not help predict future changes in $\hat{\theta}$. Panel B provides evidence that this assumption is not always consistent with the data. While in the last two subperiods β_2 is close to zero and imprecisely esimated, which is consistent with our model, the estimate of β_2 is positive and precise for the entire period and for the first subperiod. The fit of the model (17), as measured by the adjusted R^2, is best for the third period, when, as shown in Table 4, θ displays the lowest serial correlation.

Table 6 presents one last diagnostic of our model: a comparison between the optimal forecast of $r_{t+1} - r_t$ implied by the theoretically-motivated CIR* model, and an unrestricted forecast. Based on our model, we have

$$\begin{aligned} E_t r_{t+1} - r_t &= -(k/12) r_t + (k/12)\hat{\theta}_t \\ &= a_0(k/12) + (-k/12) r_t + a_1 (k/12) B(2) y_t(\tau_1) + (-a_1)(k/12) B(1) 2 y_t(2) \\ &= w_0 + w_r r_t + w_{y(1)} y_t(1) + w_{y(2)} y_t(2). \end{aligned}$$

These weights are compared to the estimated coefficients of the model

$$r_{t+1} - r_t = \beta_0 + \beta_1 r_t + \beta_2\ y_t(1) + \beta_3\ y_t(2) + \text{error}_{t+1}. \tag{18}$$

Panel A of Table 6 presents the estimates of w_r, $w_{y(1)}$, and $w_{y(2)}$. For the periods in which the time-varying central tendency matters most, namely the entire period and the first and last subperiod, the estimate of $w_{y(1)}$ is negative, while the estimate of $w_{y(2)}$ is positive, and larger in absolute size. This is consistent with the notion that a measure of the slope of the yield curve, $y(2) - y(1)$, should anticipate changes in the one-month rate in the same direction. Also, we calculate the sum of the three weights, and we find that it is quite close to zero for the entire period, and for the second and third subperiod. This is not surprising, since we are using a linear combination of three series, r, $y(1)$, and $y(2)$, which are highly persistent (see Table 1), to explain the changes in the one-month rate, which instead exhibit almost no serial correlation (see Table 1). Hence, the coefficients of the linear combination need to sum up to zero. Also, if we believe that the three interest-rate series contain a unit root, this would be evidence that they are *cointegrated*, where the weights w represent the coefficients of the cointegrating vector. Panel B of Table 6 presents the estimates of the parameters of the unrestricted model (18). While the estimates of the coefficients β_1, β_2, and β_3 are quite different from those of the weights w, the general message from the unrestricted model is similar to that of the restricted one. First, the estimates of β_3 are not significant, which implies that the one-month rate and one longer-maturity yield are sufficient to model the conditional expectation of $r_{t+1} - r_t$. Hence, this confirms that two factors (interest rates) are sufficient to capture the dynamics of the short-term rate. Note, though, that the estimates of β_1 are larger (in absolute value) than the estimates of the weight w_r. This is further evidence that the short-term rate contributes to explain the time-varying central tendency θ, contrary to the assumed recursive structure of our model. Second, the estimates of β_1 are negative, while the estimates of β_2 are positive. This is again consistent with the notion that a measure of the slope of the yield curve, this time $y(1) - r_t$, should anticipate in the same direction changes in the one-month rate. Third, the sum of the three β coefficients is always close to zero, which is the same result that we obtained for the restricted model.

6 Conclusions

In one-factor models, such as Cox, Ingersoll, and Ross (1985) or Vasicek (1977), the conditional mean of the instantaneous rate changes with its current level. This papers gathers evidence that the conditional mean of the one-month rate is explained by bond yields of different maturities, even after controlling for the effect of the current level of the one-month rate. This suggests the presence of a second factor driving the conditional mean of the one-month rate, other than the level of the one-month rate itself: we refer to this second factor as the *central tendency*. The above idea is captured in a two-factor model of the term structure, where the instantaneous rate fluctuates around a stochastic central tendency. We then build a proxy for the central-tendency factor based on the information contained in the term structure of interest rates. Based on this central-tendency proxy, we estimate a model of the one-month rate which performs better than models which assume the central tendency to be constant.

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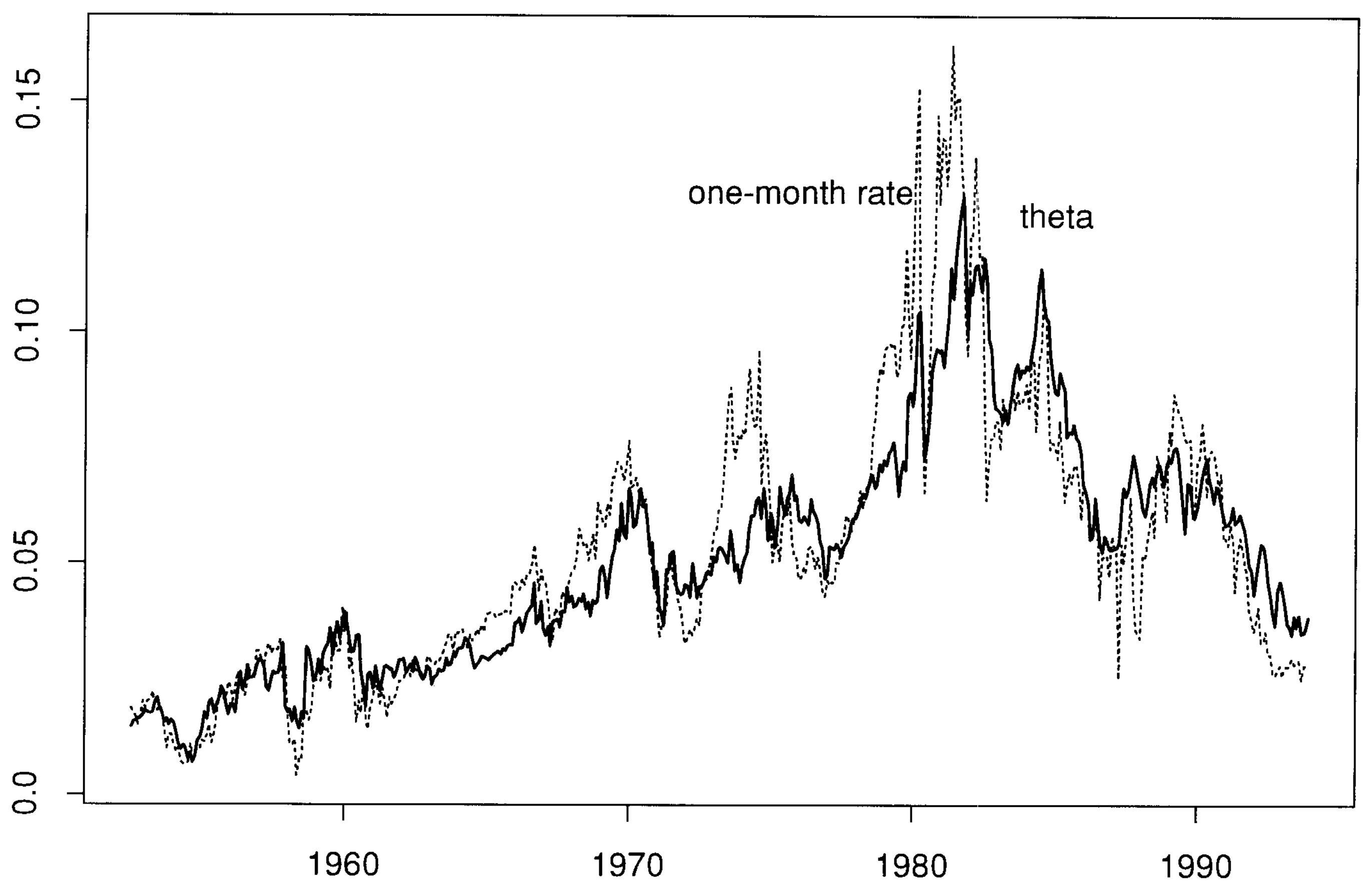
Figure 1.



Plot of $\hat{\theta}$ (solid line) and the one-month rate (dashed line). The proxy for the central tendency $\hat{\theta}$ is calculated as in equation (11) using the one- and two-year bond yields and the estimates of the model parameters reported in Table 3.

Table 1. Summary statistics of selected U.S. interest rates: Levels

The data are monthly, annualized continuously-compounded yields on zero-coupon U.S. government fixed-income instruments. Observations are monthly, June 1952 though December 1993; data on the one-month Treasury-bill rate are from the Center for Research in Security Prices (CRSP) RISKFREE file, while the discount-bond prices used to calculate continuously-compounded yields on longer-maturity instruments are from the FAMABLIS file, also on the CRSP tape. MEAN is the sample mean, ST DEV the standard deviation, and AUTO the first-order autocorrelation. SKEWNESS and KURTOSIS denote the standardized measures of third and fourth moments; both are zero for the normal distribution.

Jun 1952 – Dec 1993

Maturity	MEAN	ST DEV	SKEWNESS	KURTOSIS	AUTO
1 month	5.113	2.936	0.983	1.059	0.966
1 year	5.954	3.057	0.745	0.351	0.982
2 years	6.138	3.022	0.685	0.135	0.986
3 years	6.293	2.968	0.645	0.040	0.987
4 years	6.408	2.951	0.606	−0.051	0.988
5 years	6.479	2.923	0.584	−0.160	0.989

Jun 1952 – Aug 1971

Maturity	MEAN	ST DEV	SKEWNESS	KURTOSIS	AUTO
1 month	3.153	1.566	0.551	−0.298	0.967
1 year	3.788	1.675	0.446	−0.267	0.977
2 years	3.864	1.576	0.534	−0.056	0.976
3 years	3.997	1.531	0.566	−0.067	0.977
4 years	4.069	1.497	0.591	−0.006	0.976
5 years	4.121	1.446	0.650	0.036	0.976

Sep 1971 – Sep 1979

Maturity	MEAN	ST DEV	SKEWNESS	KURTOSIS	AUTO
1 month	6.110	1.836	0.489	−0.721	0.917
1 year	6.909	1.588	0.462	−0.551	0.912
2 years	7.043	1.281	0.367	−0.462	0.917
3 years	7.118	1.084	0.268	−0.560	0.909
4 years	7.199	0.997	0.152	−0.632	0.906
5 years	7.257	0.928	0.087	−0.720	0.917

Oct 1979 – Sep 1982

Maturity	MEAN	ST DEV	SKEWNESS	KURTOSIS	AUTO
1 month	11.545	2.647	−0.419	−0.640	0.708
1 year	12.671	1.909	−0.651	−0.045	0.686
2 years	12.593	1.716	−0.487	−0.038	0.751
3 years	12.493	1.657	−0.419	−0.349	0.795
4 years	12.401	1.650	−0.301	−0.524	0.817
5 years	12.341	1.514	−0.391	−0.741	0.826

Oct 1982 – Dec 1993

Maturity	MEAN	ST DEV	SKEWNESS	KURTOSIS	AUTO
1 month	6.043	2.058	−0.158	−0.886	0.919
1 year	7.190	2.168	−0.193	−0.549	0.969
2 years	7.664	2.153	0.015	−0.454	0.970
3 years	7.980	2.081	0.123	−0.427	0.966
4 years	8.248	2.025	0.237	−0.413	0.967
5 years	8.397	1.986	0.344	−0.360	0.967

Table 1. Summary statistics of selected U.S. interest rates: First differences

Jul 1952 – Dec 1993

Maturity	Mean	St Dev	Skewness	Kurtosis	Auto
1 month	0.002	0.735	-0.863	10.072	-0.089
1 year	0.003	0.538	-1.083	13.411	0.116
2 years	0.004	0.462	-0.724	10.161	0.154
3 years	0.005	0.421	-0.177	7.562	0.109
4 years	0.006	0.405	-0.210	5.323	0.060
5 years	0.006	0.375	-0.330	5.031	0.071

Jul 1952 – Aug 1971

Maturity	Mean	St Dev	Skewness	Kurtosis	Auto
1 month	0.011	0.378	-0.009	1.925	-0.200
1 year	0.014	0.319	-0.042	3.153	0.163
2 years	0.015	0.299	0.022	4.526	0.090
3 years	0.015	0.279	-0.304	4.620	0.033
4 years	0.016	0.268	-0.197	4.017	-0.003
5 years	0.017	0.253	-0.165	4.026	0.014

Sep 1971 – Sep 1979

Maturity	Mean	St Dev	Skewness	Kurtosis	Auto
1 month	0.059	0.583	-1.614	8.372	-0.137
1 year	0.056	0.486	-0.572	1.670	-0.007
2 years	0.046	0.379	-0.068	0.288	0.016
3 years	0.040	0.338	0.087	0.594	-0.026
4 years	0.036	0.332	0.244	1.355	-0.075
5 years	0.032	0.282	0.023	0.692	-0.068

Oct 1979 – Sep 1982

Maturity	Mean	St Dev	Skewness	Kurtosis	Auto
1 month	-0.085	1.883	-0.535	0.592	0.036
1 year	-0.012	1.482	-0.701	1.126	0.101
2 years	0.031	1.207	-0.676	1.034	0.163
3 years	0.045	1.050	-0.283	0.653	0.127
4 years	0.053	0.972	-0.404	0.122	0.054
5 years	0.064	0.866	-0.522	0.441	0.072

Nov 1982 – Dec 1993

Maturity	Mean	St Dev	Skewness	Kurtosis	Auto
1 month	-0.030	0.764	0.213	2.857	-0.245
1 year	-0.048	0.393	0.103	0.418	0.134
2 years	-0.050	0.388	0.130	0.019	0.209
3 years	-0.049	0.386	0.306	-0.012	0.138
4 years	-0.045	0.396	0.309	0.145	0.099
5 years	-0.045	0.394	0.100	0.136	0.071

Table 2. Long yields' information on the future short-term rate

We estimate the following model

$$r_{t+1} = \beta_0 + \beta_1 r_t + \beta_2 y_t(\tau) + \text{error}_{t+1}, \qquad (1)$$

where $y(\tau)$ is the yield on a discount bond of maturity $\tau = 1, 2, 3, 4$, and 5 years, and r is the one-month interest rate. The model is estimated both in levels and in first differences

$$\Delta r_{t+1} = \beta_0 + \beta_1 \Delta r_t + \beta_2 \Delta y_t(\tau) + \text{error}_{t+1}. \qquad (1')$$

ADJ. R–SQ is the adjusted R^2 and DW is the Durbin-Watson statistic. Standard errors, adjusted for heteroskedasticity (White (1980)), are in parenthesis.

Levels

Jul 1952 – Dec 1993

Maturity	β_1	β_2	ADJ. R–SQ	DW
1 year	0.624	0.338	0.944	1.849
	(0.080)	(0.075)		
2 years	0.763	0.207	0.941	1.925
	(0.065)	(0.059)		
3 years	0.823	0.152	0.940	1.980
	(0.059)	(0.052)		
4 years	0.847	0.129	0.940	1.996
	(0.054)	(0.047)		
5 years	0.865	0.111	0.939	2.019
	(0.051)	(0.044)		

Jul 1952 – Aug 1971

Maturity	β_1	β_2	ADJ. R–SQ	DW
1 year	0.662	0.297	0.948	2.088
	(0.073)	(0.068)		
2 years	0.733	0.245	0.946	2.156
	(0.066)	(0.066)		
3 years	0.788	0.196	0.945	2.198
	(0.054)	(0.056)		
4 years	0.830	0.156	0.944	2.225
	(0.050)	(0.054)		
5 years	0.827	0.165	0.944	2.206
	(0.051)	(0.057)		

Sep 1971 – Sep 1979

Maturity	β_1	β_2	ADJ. R–SQ	DW
1 year	0.721	0.317	0.904	2.080
	(0.123)	(0.143)		
2 years	0.873	0.179	0.902	2.084
	(0.094)	(0.113)		
3 years	0.904	0.161	0.902	2.101
	(0.077)	(0.109)		
4 years	0.931	0.123	0.901	2.116
	(0.072)	(0.108)		
5 years	0.949	0.093	0.900	2.128
	(0.062)	(0.098)		

Oct 1979 – Sep 1982

Maturity	β_1	β_2	ADJ. R–SQ	DW
1 year	0.726	0.083	0.549	1.623
	(0.226)	(0.313)		
2 years	0.771	0.015	0.548	1.657
	(0.188)	(0.280)		
3 years	0.792	−0.028	0.548	1.681
	(0.171)	(0.255)		
4 years	0.756	0.055	0.549	1.634
	(0.167)	(0.250)		
5 years	0.785	−0.017	0.548	1.674
	(0.162)	(0.273)		

Oct 1982 – Dec 1993

Maturity	β_1	β_2	ADJ. R–SQ	DW
1 year	0.447	0.499	0.900	2.023
	(0.110)	(0.099)		
2 years	0.554	0.400	0.893	2.067
	(0.099)	(0.090)		
3 years	0.626	0.342	0.887	2.123
	(0.089)	(0.084)		
4 years	0.684	0.291	0.882	2.154
	(0.079)	(0.077)		
5 years	0.711	0.268	0.880	2.188
	(0.075)	(0.074)		

Table 2. Long yields' information on the future short-term rate (ctd.)

First differences

Aug 1952 – Dec 1993				
Maturity	β_1	β_2	ADJ. R–SQ	DW
1 year	−0.356	0.607	0.131	1.951
	(0.081)	(0.104)		
2 years	−0.301	0.645	0.124	1.952
	(0.077)	(0.113)		
3 years	−0.272	0.645	0.107	1.949
	(0.083)	(0.142)		
4 years	−0.237	0.641	0.107	1.962
	(0.081)	(0.139)		
5 years	−0.223	0.616	0.085	1.945
	(0.080)	(0.142)		

Aug 1952 – Aug 1971				
Maturity	β_1	β_2	ADJ. R–SQ	DW
1 year	−0.401	0.479	0.156	2.153
	(0.083)	(0.096)		
2 years	−0.355	0.406	0.111	2.127
	(0.087)	(0.102)		
3 years	−0.345	0.417	0.107	2.135
	(0.082)	(0.098)		
4 years	−0.328	0.398	0.095	2.117
	(0.084)	(0.104)		
5 years	−0.317	0.438	0.105	2.112
	(0.084)	(0.115)		

Sep 1971 – Sep 1979				
Maturity	β_1	β_2	ADJ. R–SQ	DW
1 year	−0.348	0.359	0.036	2.006
	(0.188)	(0.155)		
2 years	−0.263	0.376	0.036	1.983
	(0.192)	(0.175)		
3 years	−0.270	0.497	0.056	1.980
	(0.191)	(0.213)		
4 years	−0.233	0.424	0.041	1.968
	(0.189)	(0.220)		
5 years	−0.249	0.553	0.050	1.975
	(0.182)	(0.232)		

Table 2. Long yields' information on the future short-term rate (ctd.)

Oct 1979 – Sep 1982

Maturity	β_1	β_2	ADJ. R–SQ	DW
1 year	−0.441	0.809	0.127	1.603
	(0.212)	(0.242)		
2 years	−0.373	0.930	0.137	1.567
	(0.180)	(0.250)		
3 years	−0.327	0.974	0.112	1.547
	(0.212)	(0.386)		
4 years	−0.258	1.006	0.134	1.567
	(0.182)	(0.362)		
5 years	−0.243	0.990	0.076	1.564
	(0.179)	(0.401)		

Oct 1982 – Dec 1993

Maturity	β_1	β_2	ADJ. R–SQ	DW
1 year	−0.315	0.557	0.125	2.098
	(0.097)	(0.197)		
2 years	−0.305	0.554	0.123	2.086
	(0.095)	(0.199)		
3 years	−0.295	0.472	0.102	2.082
	(0.098)	(0.211)		
4 years	−0.283	0.458	0.103	2.082
	(0.096)	(0.201)		
5 years	−0.279	0.398	0.088	2.075
	(0.100)	(0.205)		

Table 3. Maximum likelihood estimation

We maximize the log-likelihood function

$$-.5\sum\left[\ln\sigma_{t+1}^2+(r_{t+1}-\bar{r}_{t+1})^2/\sigma_{t+1}^2\right], \tag{13}$$

where

$$\bar{r}_{t+1}=(1-(k/12))r_t+(k/12)\hat{\theta}_t \tag{14}$$

$$\sigma_{t+1}^2=(\sigma_0^2/12)+(\sigma_1^2/12)r_t, \tag{15}$$

and $\hat{\theta}_t$ is given by

$$\hat{\theta}_t=a_0+a_1[B(2)y_t(1)-B(1)2y_t(2)] \tag{11}$$

where

$$B(\tau)=\frac{2(e^{\delta\tau}-1)}{(\lambda_1+\delta+k)(e^{\delta\tau}-1)+2\delta},\quad \delta=\sqrt{(\lambda_1+k)^2+2\sigma_1^2}. \tag{6}$$

We use one- and two-year bond yields to construct the proxy in equation (11). We set the market price of interest rate risk $\lambda_1=0$. We report the value of the parameter estimates, the maximized log likelihood, and standard errors (in parenthesis).

Jul 1952 – Dec 1993

Model	k	a_0	a_1	σ_0	σ_1	log lik.
Vasicek	0.383	0.051		0.025		2202
	(0.107)	(0.013)		(0.000)		
CIR	0.296	0.051			0.102	2291
	(0.096)	(0.011)			(0.001)	
Vasicek*	1.778	0.000	-2.068	0.024		2210
	(0.228)	(0.007)	(0.216)	(0.000)		
CIR*	2.122	−0.001	-2.269		0.099	2305
	(0.246)	(0.002)	(0.159)		(0.002)	

Jul 1952 – Aug 1971

Model	k	a_0	a_1	σ_0	σ_1	log lik.
Vasicek	0.354	0.035		0.013		1169
	(0.180)	(0.008)		(0.000)		
CIR	0.489	0.034			0.082	1158
	(0.211)	(0.008)			(0.002)	
Vasicek*	2.064	-0.004	-2.479	0.012		1173
	(0.521)	(0.004)	(0.359)	(0.000)		
CIR*	2.616	-0.003	-2.769		0.079	1166
	(0.524)	(0.002)	(0.387)		(0.002)	

Table 3. Maximum likelihood estimation (subperiods, contd.)

Sep 1971 – Sep 1979

Model	k	a_0	a_1	σ_0	σ_1	log lik.
Vasicek	0.370	0.079		0.019		452
	(0.497)	(0.024)		(0.000)		
CIR	0.238	0.089			0.075	461
	(0.412)	(0.057)			(0.003)	
Vasicek*	0.389	0.044	-1.642	0.019		452
	(0.930)	(0.084)	(7.100)	(0.000)		
CIR*	0.243	0.095	0.369		0.075	461
	(0.510)	(0.072)	(5.234)		(0.004)	

Oct 1979 – Sep 1982

Model	k	a_0	a_1	σ_0	σ_1	log lik.
Vasicek	2.790	0.112		0.061		127
	(1.535)	(0.013)		(0.007)		
CIR	2.463	0.112			0.180	128
	(1.475)	(0.014)			(0.024)	
Vasicek*	2.381	0.160	1.130	0.060		127
	(1.669)	(0.159)	(3.497)	(0.007)		
CIR*	1.924	0.189	1.670		0.180	128
	(1.496)	(0.181)	(3.575)		(0.024)	

Oct 1982 – Dec 1993

Model	k	a_0	a_1	σ_0	σ_1	log lik.
Vasicek	0.737	0.055		0.026		593
	(0.410)	(0.010)		(0.001)		
CIR	0.772	0.055			0.113	587
	(0.458)	(0.011)			(0.004)	
Vasicek*	3.986	-0.011	-3.580	0.023		604
	(0.620)	(0.008)	(0.597)	(0.001)		
CIR*	4.467	-0.013	-4.108		0.102	602
	(0.518)	(0.008)	(0.612)		(0.004)	

Table 4. Summary statistics of the estimated central tendency $\hat{\theta}$

Panel A. In this panel, the proxy for the central tendency $\hat{\theta}$ is calculated as in equation (11) using the one- and two-year bond yields and the estimates of the model parameters reported in Table 3. MEAN is the sample mean, ST DEV the standard deviation, and AUTO the first-order autocorrelation. SKEWNESS and KURTOSIS denote the standardized measures of third and fourth moments; both are zero for the normal distribution.

Levels

Sample Period	MEAN	ST DEV	SKEWNESS	KURTOSIS	AUTO
Jun 1952 – Nov 1993	5.092	2.522	0.634	−0.076	0.985
Jun 1952 – Jul 1971	3.178	1.359	0.700	0.362	0.957
Aug 1971 – Aug 1979	8.931	0.210	0.534	−0.128	0.832
Sep 1979 – Aug 1982	11.114	1.113	0.211	−0.522	0.780
Sep 1982 – Nov 1993	5.988	1.954	0.231	−0.394	0.961

First differences

Sample Period	MEAN	ST DEV	SKEWNESS	KURTOSIS	AUTO
Jul 1952 – Nov 1993	0.000	0.003	−0.150	2.963	0.013
Jul 1952 – Jul 1971	0.000	0.003	0.180	4.029	−0.178
Sep 1971 – Aug 1979	0.000	0.001	−1.119	7.045	−0.298
Oct 1979 – Aug 1982	-0.000	0.006	0.543	0.049	0.121
Oct 1982 – Nov 1993	-0.000	0.003	0.135	0.255	0.215

Panel B. Correlation of $\hat{\theta}$ extracted using the maturity pairs 1–2 years, 1–3 years, 1–4 years, and 1–5 years.

Levels, June 1952 – November 1993

	$\hat{\theta}_{1,2}$	$\hat{\theta}_{1,3}$	$\hat{\theta}_{1,4}$	$\hat{\theta}_{1,5}$
$\hat{\theta}_{1,2}$	1.000			
$\hat{\theta}_{1,3}$	0.995	1.000		
$\hat{\theta}_{1,4}$	0.992	0.997	1.000	
$\hat{\theta}_{1,5}$	0.990	0.996	0.998	1.000

First differences, July 1952 – November 1993

	$\hat{\theta}_{1,2}$	$\hat{\theta}_{1,3}$	$\hat{\theta}_{1,4}$	$\hat{\theta}_{1,5}$
$\hat{\theta}_{1,2}$	1.000			
$\hat{\theta}_{1,3}$	0.834	1.000		
$\hat{\theta}_{1,4}$	0.804	0.888	1.000	
$\hat{\theta}_{1,5}$	0.792	0.872	0.903	1.000

Table 5.

Panel A. Does $(\hat{\theta} - r)$ explain dr?

We estimate the following model

$$r_{t+1} - r_t = \beta_0 + \beta_1 \hat{\theta}_t + \beta_2 r_t + \text{error}_{t+1}; \qquad (16)$$

the proxy for the central tendency $\hat{\theta}$ is calculated as in equation (11) using the one- and two-year bond yields and the estimates of the model parameters reported in Table 3. ADJ. R–SQ is the adjusted R^2 and DW is the Durbin-Watson statistic. Standard errors, adjusted for heteroskedasticity (White (1980)), are in parenthesis. In the last column, $\chi^2(1)$ is the Wald test of the restriction that $\beta_1 = -\beta_2$; the statistic is distributed chi–square with one degree of freedom, with p-value indicated in parenthesis below.

Sample Period	β_1	β_2	ADJ. R–SQ	DW	$\chi^2(1)$
Jul 1952 – Dec 1993	0.128	-0.133	0.044	2.012	0.053
	(0.049)	(0.049)			(0.816)
Jul 1952 – Aug 1971	0.152	0.000	0.047	2.256	4.505
	(0.056)	(0.021)			(0.033)
Sep 1971 – Sep 1979	-0.120	-0.022	-0.010	2.206	0.099
	(0.454)	(0.030)			(0.752)
Oct 1979 – Sep 1982	0.121	-0.201	0.047	1.733	0.032
	(0.319)	(0.155)			(0.856)
Oct 1982 – Dec 1993	0.314	-0.325	0.162	2.151	0.172
	(0.079)	(0.079)			(0.677)

Panel B. Do $\hat{\theta}$ and r explain $d\hat{\theta}$?

We estimate the following model

$$\hat{\theta}_{t+1} - \hat{\theta}_t = \beta_0 + \beta_1 \hat{\theta}_t + \beta_2 r_t + \text{error}_{t+1}; \qquad (16)$$

the proxy for the central tendency $\hat{\theta}$ is calculated as in equation (11) using the one- and two-year bond yields and the estimates of the model parameters reported in Table 3. ADJ. R–SQ is the adjusted R^2 and DW is the Durbin-Watson statistic. Standard errors, adjusted for heteroskedasticity (White (1980)), are in parenthesis.

Sample Period	β_1	β_2	ADJ. R–SQ	DW
Jul 1952 – Nov 1993	−0.072	0.055	0.031	1.942
	(0.021)	(0.020)		
Jul 1952 – Jul 1971	−0.158	0.119	0.044	2.204
	(0.044)	(0.040)		
Sep 1971 – Aug 1979	−0.054	0.003	0.020	2.551
	(0.033)	(0.008)		
Oct 1979 – Aug 1982	−0.259	−0.042	0.126	1.475
	(0.087)	(0.046)		
Oct 1982 – Nov 1993	−0.036	0.017	−0.000	1.550
	(0.037)	(0.035)		

Table 6. Restricted and unrestricted forecasts of short-term rate changes

According to equations (14) and (11), the optimal forecast of short-term rate changes is a linear combination of the short-term rate and two longer-maturity bond yields

$$\begin{aligned} E_t r_{t+1} - r_t &= -(k/12) r_t + (k/12)\theta_t \\ &= \underbrace{a_0(k/12)}_{w_0} + \underbrace{(-k/12)}_{w_r} r_t + \underbrace{a_1(k/12)B(\tau_2)\tau_1}_{w_{y(1)}} y_t(\tau_1) + \underbrace{(-a_1)(k/12)B(\tau_1)2}_{w_{y(2)}} y_t(\tau_2) \\ &= w_0 + w_r r_t + w_{y(1)} y_t(\tau_1) + w_{y(2)} y_t(\tau_1) \end{aligned}$$

where w_x denotes the estimate of the weight (loading) on the explanatory variable x. Each weight is a function of the estimated parameters and thus an estimate itself. Hence, we compute standard errors on w_x using a first-order Taylor series expansion around the ML estimates, and the estimated variance-covariance matrix of the ML estimates of the parameters of the model. We contrast the estimates of the weights w with the estimates of the regression parameters of the unrestricted model

$$r_{t+1} - r_t = \beta_0 + \beta_1 r_t + \beta_2 y_t(\tau_1) + \beta_3 y_t(\tau_1) + \text{error}_{t+1}.$$

Standard errors for this regression, in parenthesis, are adjusted for heteroskedasticity (White (1980)).

Sample Period	w_r	$w_{y(1)}$	$w_{y(2)}$	$\sum w$
Jul 1952 – Dec 1993	−0.176	−0.186	0.332	−0.030
	(0.020)	(0.056)	(0.027)	(0.033)
Jul 1952 – Aug 1971	−0.218	−1.052	1.961	0.691
	(0.046)	(0.146)	(0.078)	(0.077)
Sep 1971 – Sep 1979	−0.020	0.583	−0.655	−0.092
	(0.042)	(0.157)	(0.186)	(0.032)
Oct 1979 – Sep 1982	−0.160	0.844	−1.476	−0.092
	(0.124)	(0.155)	(0.623)	(0.468)
Oct 1982 – Dec 1993	−0.372	−0.919	1.817	0.526
	(0.043)	(0.188)	(0.154)	(0.063)
Sample Period	β_1	β_2	β_3	$\sum \beta_j$
Jul 1952 – Dec 1993	−0.436	0.673	−0.283	−0.046
	(0.079)	(0.160)	(0.125)	(0.018)
Jul 1952 – Aug 1971	−0.337	0.361	−0.068	−0.044
	(0.072)	(0.152)	(0.146)	(0.020)
Sep 1971 – Sep 1979	−0.318	0.469	−0.161	−0.009
	(0.168)	(0.486)	(0.379)	(0.045)
Oct 1979 – Sep 1982	−0.371	0.748	−0.639	−0.261
	(0.241)	(0.897)	(0.771)	(0.144)
Oct 1982 – Dec 1993	−0.568	0.797	−0.288	−0.060
	(0.108)	(0.256)	(0.215)	(0.026)