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Abstract

We value American options on bonds using the Geske–Johnson (1984) technique as modified by Bunch and Johnson (1992). The method requires the valuation of European options, and options with two and three possible exercise dates. It is shown that a risk-neutral valuation relationship along the lines of Black–Scholes (1973) model holds for options exercisable on multiple dates, even under stochastic interest rates, when the price of the underlying asset is lognormally distributed. The proposed computational procedure uses the maximized value of these options, where the maximization is over all possible exercise dates. The value of American option is then computed by Richardson extrapolation. The volatility of the underlying default-free bond is modelled using a two-factor model, with a short-term and a long-term interest rate factor, where the short-term interest rate is mean-reverting. Simulations show that penny accuracy is achieved with this computationally efficient method.

1 Introduction

The valuation of American-style bond options involves two important aspects that need to be modelled carefully. First, stochastic interest rates influence the volatility of the price of the bond, the underlying asset, in a complex fashion as the bond approaches maturity. The behavior of the volatility over time influences the value of the option if held to maturity, as well as the incremental value of the early exercise (American) feature. Second, the early exercise decision for such options is affected by the term structure of interest rates on future dates, since the live value of the claim on each future date depends on the discount rates on that date.

In this paper, we model the volatility of the default-free bond price using a two-factor model. Hence, the bond's volatility is determined by the volatilities of the two interest rate factors as well as the sensitivity of the bond price to changes in the two factor rates. Such a model allows us to capture the effect of non-parallel shifts in the term structure of interest rates, that may have a significant effect on the volatility of the bond price over time, and hence, on the value of the contingent claims. In analyzing the early exercise decision, it is useful to derive a quasi-analytical model for the value of an option that can be exercised on one of many dates, for speedy computation of option values and hedge parameters. This is possible when (continuously-compounded) interest rates are normally distributed, i.e., when the prices of zero-coupon bonds are lognormally distributed, since, in this case, it can be shown that a risk-neutral valuation relationship (RNVR) along the lines of the Black–Scholes model exists for the valuation of European options, even under stochastic interest rates[1].

Since an American option can be thought of as an option with many exercise dates, where the number of dates becomes very large, it is necessary to establish a similar RNVR for options exercisable on one of many dates. Once such an RNVR is established, American-style options can be valued using an extension of the Geske and Johnson (1984) (GJ) approach, i.e., by extrapolation using a series of options that can be exercised on one of a number of discrete dates. The series consists of a European option, an option exercisable on one of two dates, and so on.

[1] This has been established in the case of a single-factor interest rate model by Jamshidian (1989) and for the general case by Satchell, Stapleton and Subrahmanyam (1995).

GJ apply their methodology to the case of American put options. However, the principle behind the GJ approach can be applied to any American-style option whose value depends upon the underlying asset price as a state variable.[2] In particular, it applies to American options for which early exercise can be generated, for example, by the changing volatility of the underlying asset, or by the nature of the exercise schedule. It can also be applied to the valuation of American options on assets, including bonds, when interest rates are stochastic.[3]

Since the quasi-analytic formulae for American options involve multivariate cumulative-normal density functions, their implementation can be simplified by approximating the normal distribution by n-stage binomial distributions along the lines of Cox, Ross and Rubinstein (1979). However, the method needs to be generalized to handle the changing volatilities of the asset (both conditional and unconditional) over time. Also, the method has to take into account the possibility that the term structure of interest rates and, in turn, bond rices are driven by a multifactor model. The method we use to capture changing volatilities is similar in spirit to that suggested in Nelson and Ramaswamy (1990), generalized by Ho, Stapleton and Subrahmanyam (1995) (HSS).

In the single-variable approach with a constant volatility of the price of the underlying asset, the Cox, Ross and Rubinstein (1979) method involves building a binomial tree centered around the forward price of the asset, rather than around its expected spot price. For a European option, the option payoff is computed at each node of the tree on the expiration date and the expected value of this payoff is discounted to determine the option value. For American options with multiple exercise dates, the procedure is somewhat more complex. First, the method used here entails building a binomial tree whose *conditional* expectation is the forward price for delivery on each of the possible future exercise dates of the option. Next, the contingent exercise decisions on each future state and date are determined and the forward price of the option on each future date is determined. Finally, these forward prices are discounted using the appropriate zero-coupon bond price to determine

[2]Huang, Subrahmanyam and Yu (1996) use an alternative method where the early exercise boundary is first estimated and then the value of American options is determined by extrapolation.

[3]See Ho, Stapleton and Subrahmanyam (1997) for an application of the GJ approach to the general problem of valuation of options on assets when interest rates are stochastic.

the current value of the option.

In the case of American-style options on finite-life, coupon bonds the GJ method has to be adapted somewhat. Since the volatility of a finite-life bond tends to decline over time, with the approach of the bond maturity, an American-style option on the bond is very much a *wasting asset*. In the case of European options, a long-maturity option may have less value than a shorter-maturity option. For this reason we use a GJ type approximation where the European option and the option with two possible exercise dates are chosen so as to maximize the value of the options. Thus our benchmark, or minimum possible, value for the American-style option is the value of the European option with the maximum value; where the maximum is taken with respect to the feasible lives of the option. In the case of a typical ten-year coupon bond, this maximized European option value may be that of a two or three year maturity option. Similarly, in the case of the option with two possible exercise dates, we take the maximum option value taken over all possible pairs of exercise dates.[4] GJ type extrapolation is then performed, using an exponential rather than a linear approximation to generate estimates of the American-option price.[5] We also demonstrate that only a relatively small increase in accuracy is obtained when options exercisable on one of three dates are added to the extrapolation. This small increase can be obtained only with a relatively large amount of computational effort. Simulations show that it is far more important to obtain accurate estimates of the volatility and the forward price inputs, than to consider options exercisable on more than two dates. Thus, our solution provides a rather simple prescription for the answer to a problem of considerable complexity. The method presented in this paper may be applied to the valuation of any American option under stochastic interest rates, given that the distributional assump-

[4]This method was proposed and tested in a somewhat different form by Bunch and Johnson (1992). Since Bunch and Johnson do not value options on finite-lived assets, they can take the first term in the extrapolation as the value of the European option whose maturity equals that of the American option. Along the same lines, they then find the maximum option value with two exercise dates, given that the second exercise date is the final maturity date of the American option. In the case of finite-lived assets such as bonds, the Bunch and Johnson approach has to modified along the lines proposed here since the volatility of the bond declines as it approaches maturity.

[5]Ho, Stapleton and Subrahmanyam (1994) modify the linear Richardson approximation technique used by both GJ and Bunch and Johnson, adapting it for long-maturity option using exponential approximation.

tions are satisfied. It is fully consistent with approaches using a multifactor model of the term structure of interest rates, but is simpler and more efficient than other approaches, because it involves the evaluation of options with only a small number of exercise dates. It is also more general than alternative approaches using a particular factor model for the evolution of the term structure of interest rates, although it uses a two-factor model for generation of the volatility inputs. However, the important restriction, as in the case of the Black and Scholes (1973) model, is that asset prices must follow a multivariate lognormal distribution.

In section 2, we discuss the modification of the GJ approach to the case of American options in the context of other approaches in the literature. In section 3, we present a valuation model for American-style options on bonds and establish the requisite RNVR's. In section 4, we proceed to illustrate the method by applying it to American options on a variety of bonds. We show, using simulations, that for reasonable exercise schedules, the GJ method can be applied in modified form using options exercisable on one date (European options) and on one of two possible dates only.

2 Bond Options and the Use of the Geske–Johnson Methodology

Much of the work in recent years on the valuation of contingent claims on bonds and interest rates uses a factor model to characterize the evolution of the term structure of interest rates. For example, Ho and Lee (1986), Black, Derman and Toy (1990), and Jamshidian (1989) all build a process for the evolution of the term structure based on a single-factor model. Although Heath, Jarrow and Morton (1990a, 1990b, 1992) provide a framework for the pricing of claims using a general multi-factor approach to characterize the term structure, the implementation of this methodology using a binomial lattice becomes difficult when the number of factors increases beyond two, due to the computational problems associated with building a multi-dimensional lattice of bond prices or interest rates.[6] In addition to the cumbersome procedure for building a multidimensional lattice, the problem of valuation of American-style options requires an examination of the optimality of early

[6]See Hull and White (1994) for details of implementation of a two-factor model.

exercise at each node of the lattice, which is even more complex. The computational limits of the multi-factor lattice approaches are illustrated by Amin and Bodurtha (1995) who find even a ten-stage lattice very costly to implement when two or more factors are involved. In contrast, the GJ methodology can be implemented without any restrictive assumptions involving the factor model underlying term structure movements.

In view of the limitations of the lattice-based approaches, it is worthwhile to explore the possibility of deriving a quasi-analytical model and using the GJ methodology to value American options. GJ originally suggested the use of the Richardson approximation to extrapolate the value of an American option from the values of a series of options: a European option, an option with two possible exercise dates, an option with three possible exercise dates, and so on. A number of subsequent papers have extended and modified the basic GJ approach. For example, Omberg (1987) and Breen (1991) approximate the distribution of the price of the underlying asset with a binomial process. However, Omberg (1987) shows that there could be problems of non-uniform convergence in some cases. Essentially, in these cases, the computed value of the American option is not monotonic in the number of number of options considered for the Richardson extrapolation. Hence, one has to be careful in choosing the number of options for extrapolation. Bunch and Johnson (1992) modify the GJ method by showing that it may be more efficient to compute the prices of all options with two exercise dates and select the one with the maximum value. In this manner, one can obtain the best approximation with the extrapolation. HSS (1994) point out that the accuracy of the GJ technique can be improved, particularly in the case of long-term options, such as warrants and bond options, by using an exponential rather than a linear approximation in the extrapolation. In addition, HSS (1997) show that the GJ technique can be extended successfully to the multi-dimensional case where interest rates as well as the price of the underlying asset are stochastic.

In the present paper, we use all these extensions and modifications of the GJ technique, and apply them to the problem of valuation of bond options. First, we use the binomial methodology of Omberg (1987) and Breen (1991), but avoid the non-convergence problem by using a two-point extrapolation on the lines of Bunch and Johnson (1992). We also use the exponential approximation proposed by HSS (1994) to improve the results for long-term options. Also, since we necessarily have to address the issue of stochastic rates when valuing bond options, we use the results in HSS (1997) where it is shown that a risk neutral valuation relationship exists for the pricing of claims even in this case.

3 The Valuation Model

We are interested in valuing American-style options on bonds, given the exercise schedule, i.e., the relationship between the exercise price of the option and the exercise date.[7] The options could, in principle, be standard call or put options or more complex exotic options whose characteristics are defined by the respective payoff functions. The exercise schedule is defined as

$$K_{t_i} = K(t_i), \; i = 1, 2, \ldots, J, \tag{1}$$

where t_i are the exercise dates, t_1 is the earliest date on which the option can be exercised, $t_J = T$ is the maturity date of the option, and J is the number of dates between the current date, 0, and the maturity date t_J on which the option can be exercised.

The value of the underlying bond at time t_i is defined as S_{t_i}. Thus, the "live" value of the option, i.e., its market value if it is not exercised at or before time t_J, is C_{t_i}, and its value, just prior to the exercise decision at time t_i is

$$\max[g(S_{t_i}), C_{t_i}], \; i = 1, 2, \ldots, J, \tag{2}$$

where $g(S_{t_i})$ is the payoff function of the option. Since we are concerned here with the possible early exercise of such options, the price of the option on intermediate dates between 0 and t_J is relevant. We denote the price of the option at time t_i, with J possible exercise dates over its life, as

$$C_{t_i}(t_1, t_2, \ldots, t_J, K(t_i)), \; i = 1, 2, \ldots, J. \tag{3}$$

Similarly, the value of the option at time 0 is defined as $C_0(t_1, t_2, \ldots, t_J, K(t_i))$, i = 1, 2, ..., J.

In general, the GJ approach to the valuation of American options estimates the American options by Richardson extrapolation from the values of a series of options, with $1, 2, \ldots, J$ exercise dates. We denote the estimated

[7] The exercise schedule, which represents the changing exercise price of the option over its time to maturity, is specified as part of the bond option contract. It is a feature of many bond option contracts, particularly those that are embedded as part of the bond.

American option prices as $\hat{C}(J, K(t_i))$, where J refers to the maximum number of exercise dates of the option in the series used. For instance, in this notation $\hat{C}(2, K(t_i))$ is the estimated price of the American option using the values of two options: the corresponding European option and an option with at most two exercise dates.

We first establish conditions under which options can be valued using formulae analogous to those of Black and Scholes (1973). The central idea here is the concept of a risk neutral valuation relationship, which can be defined as follows for European options:

Definition 1: *A Risk-Neutral Valuation Relationship (RNVR) exists for a European option if it can be valued by taking the expected value of its payoff using a distribution for the asset price which is identical to the true distribution but with the mean shifted to equal the forward price of the asset.*

The Black and Scholes (1973) model can be thought of as a RNVR, under the assumption of continuous trading (or a lognormal pricing kernel) and a lognormal distribution for the asset price on the expiration date of the option. As shown by Merton (1973) and extended by several others including Satchell, Stapleton and Subrahmanyam (1995), Turnbull and Milne (1991) and Heath, Jarrow and Morton (1990a), this result can be extended to the case of stochastic interest rates. In the case of American-style options under stochastic interest rates, the definition of a RNVR has to be broadened somewhat along the following lines:

Definition 2: *A Risk-Neutral Valuation Relationship (RNVR) exists for the valuation of an option that has multiple exercise dates, if the option can be valued by taking the expected values of its payoff using distributions of the asset price at the various exercise dates, and discounting them using the relevant zero-coupon bond prices. The distributions are identical to those of the true distributions except for a mean shift which makes the conditional expected value of each of the prices equal to their respective forward prices.*

The concept of a RNVR for European options can thus be generalized to American options. The key aspect of the RNVR for American options is that it yields a valuation model based only on the forward price of the asset for delivery at various future dates before the expiration date of the option and the corresponding volatilities. We now define the implications of a RNVR more precisely and then establish conditions under which the price of an option with two possible exercise dates, $C_0(t_1, t_2, K(t_1), K(t_2))$, can be found, if we know: (a) the forward price at time 0 of the asset for delivery at t_1, (b) the conditional forward price of the asset at time t_1 for delivery at t_2, (c) the forward price at time 0 of the zero-coupon bond for delivery at t_1 which pays one unit of currency at t_2, plus all the relevant volatilities. More formally,

Proposition 1 *If a RNVR relationship exists for the valuation of an option with two possible exercise dates, then*

$$C_0(t_1, t_2, K_{t_1}), K_{t_2}) = B_{0,t_1} E[Y_{t_1}], \tag{4}$$

where

$$Y_{t_1} = \max[S_{t_1} - K_{t_1}, C_{t_1}(t_2, K_{t_2})],$$

(5)

and where

$$C_{t_1}(t_2, K_{t_2}) = B_{t_1,t_2} E_{t_1}[Y_{t_2}], \tag{6}$$

$$Y_{t_2} = \max[S_{t_2} - K_{t_2}, 0], \tag{7}$$

and all the relevant conditional distributions of the three random variables, S_{t_1}, S_{t_2}, and B_{t_1,t_2}, have means equal to their respective forward prices.

Proof: Y_{t_1} and Y_{t_2} are the option values (or cash flows accruing to the holders of the option) at times t_1 and t_2. A positive cash payoff occurs at t_1 if the value of the option at t_1 if not exercised, $C_{t_1}(t_2, K_{t_2})$, is less than the payoff from early exercise. The positive payoff Y_{t_2} occurs only if the early exercise condition at t_1 is not fulfilled and the option ends up in-the-money at t_2.

From the definition of a RNVR, we know that the option value is the value discounted at B_{0,t_1} and B_{t_1,t_2} of the expected payoffs on the option. Hence (4) is correct if expectations are taken with respect to the shifted distributions of S_{t_1}, S_{t_2} and B_{t_1,t_2}. Also, if the RNVR holds, the exercise decision at t_1 can be taken by valuing the option at t_1, using (6) and (7). Note that there are two random variables at t_1 that affect this decision, the price of the underlying bond, S_{t_1}, and the zero-coupon bond price, B_{t_1,t_2}.

The latter affects the spot price of the option (if unexercised). Equation (6) values the option at t_1 with a RNVR. The expectation of S_{t_2}, as of time t_1, is the forward price of S_{t_2} at time t_1. □

Corollary *If a RNVR exists for the valuation of an option with exercise dates, t_1 and t_2, a RNVR exists for the valuation of European options with exercise dates t_1 and t_2, respectively.*

Proof: If we make $K_{t_2} = \infty$, (4) becomes

$$C_0(t_1, t_2, K_{t_1}, K_{t_2}) = B_{0,t_1} E[Y_{t_1}],$$

that is

$$C_0(t_1, t_2, K_{t_1}, K_{t_2}) = B_{0,t_1} E[\max(S_{t_1} - K_{t_1}, 0)], \tag{8}$$

since

$$C_{t_1}(t_2, K_{t_2}) = 0.$$

This confirms the RNVR, for a European option of maturity t_1.
Also, if we make $K_{t_1} = \infty$, (4) becomes

$$C_0(t_1, t_2, K_{t_1}, K_{t_2}) = B_{0,t_1} E[B_{t_1,t_2} \max(S_{t_2} - K_{t_2}, 0)],$$

that is

$$C_0(t_1, t_2, K_{t_1}, K_{t_2}) = B_{0,t_2} E[\max(S_{t_2} - K_{t_2}, 0)]. \tag{9}$$

This confirms the RNVR, for a European option of maturity t_2. □

The implications of *Proposition 1* for the computation of the C_2 price are illustrated in Figure 1. There are n_1 states at time t_1 where a state is defined as a pair of values of the asset price (S_{t_1}) and the zero coupon bond price (B_{t_1,t_2}). The expected value of each variable is its respective forward price. In each state a call price is computed using (6). This is compared with the early exercise payoff, $S_{t_1} - K_{t_1}$. In Figure 1, states 0 to h_1 indicate states in which early exercise occurs. In all other states the option is not exercised at t_1. In the states where exercise occurs, Y_{t_1} is equal to $S_{t_1} - K_{t_1}$. In all other states $Y_{t_1} = C_{t_1}$. If the option is not exercised at t_1 it may pay off at t_2. This occurs in states h_2 to n_2 at t_2. Note that the probability of the Y_{t_1} values occurring are joint probabilities over the pair of variables (S_{t_1}, B_{t_1,t_2}). The probability of the payoff $Y_{t_2} = \max[S_{t_2} - K_{t_2}, 0]$ values occurring are joint probabilities over the triplet of variables $(S_{t_1}, B_{t_1,t_2}, S_{t_2})$. *Proposition 1* implies that the expected values of Y_{t_1} and Y_{t_2} can be computed using distributions of the three random variables each with a conditional mean equal to its forward price. The call price can then be computed by discounting the time t_1 payoff or option value at the zero-coupon bond prices B_{0,t_1}.

HSS (1997) establish sufficient conditions for the existence of a RNVR relationship to exist for the valuation of an option exercisable on one of many dates when the asset prices on a future date are joint-lognormally distributed. Specifically, this involves the derivation of conditions that are strong enough to guarantee that the risk-neutral distributions of the underlying asset price, $\{S_{t_i}, i = 1, 2, \ldots, J\}$ are joint lognormal with their conditional means being equal to the respective forward prices. These conditions are that the price process for the underlying asset and the (conditional) pricing kernels, at time 0 for cash flows at time t_1, and at time t_1 for cash flows at time t_2, ψ_{t_1} and ψ_{t_1,t_2}, respectively, are joint-lognormally distributed.[8] The result holds for the general case with J exercise dates. However, to avoid cumbersome notation, we state the following proposition for the case of options that are exercisable on one of two dates:

Proposition 2 *Suppose that the prices of an asset at t_1 and t_2, S_{t_1} and S_{t_2}, and the price at t_1 of the zero-coupon bond which matures at t_2, B_{t_1,t_2} are joint lognormally distributed. Then, if there exist joint lognormally distributed pricing variables ψ_{t_1}, ψ_{t_1,t_2}, which satisfy*

$$F_{0,t_1} = E(S_{t_1}\psi_{t_1}),\ E(\psi_{t_1}) = 1, \tag{10}$$

$$F_{t_1,t_2} = E(S_{t_2}\psi_{t_1,t_2}),\ E_{t_1}(\psi_{t_1,t_2}) = 1, \tag{11}$$

and if a RNVR holds for all European call options, then a RNVR exists for the valuation of an option with two possible exercise points.

Proof: See HSS(1997). □

Since, by *Proposition 2*, a RNVR exists for the option with two exercise dates, it follows from *Proposition 1* that the option can be valued given appropriate forward price and volatility inputs. The same argument applies to the case of an option exercisable on one of J dates.

[8]The pricing kernel can be thought of as a (state-dependent) random variable that adjusts for the risk aversion in the economy.

4 The Application of the Geske–Johnson Technique to Bond Options

In HSS (1996), we extend the GJ methodology to the case of American options with stochastic interest rates. In its simplest form, the GJ technique estimates the value of an American option by Richardson extrapolation as

$$\hat{C}(2, K(t_i)) = C_0(t_1, t_2, K(t_i)) + [C_0(t_1, t_2, K(t_i)) - C_0(t, K(t_i))], \quad (12)$$

using options with just one and two exercise dates, and as

$$\begin{aligned} \hat{C}(3, K(t_i)) &= C_0(t_1, t_2, t_3, K(t_i)) + \frac{7}{2}[C_0(t_1, t_2, t_3, K(t_i)) - C_0(t_1, t_2, K(t_i))] \\ &\quad -\frac{1}{2}[C_0(t_1, t_2, K(t_i)) - C_0(t, K(t_i))], \end{aligned} \quad (13)$$

using options with one, two, and three exercise dates. For simplicity of notation, we write C_J for the time 0 value of an option with J exercise dates and $\hat{C}_J$ as the estimated price of the American option using J options[9].

In applying this technique to the case of options on bonds, this procedure needs to be modified because of the changing volatility of the underlying asset. To see this, consider the case of stock options to which the GJ technique was first applied. The reason why the simple GJ technique works quite well for stock options is that the (non-annualized) volatility of the underlying asset increases with time in this case. Hence, in this case, the European option C_1 with an expiration date T has the highest value of any of the European options with maturities in the range $[0, T]$. Similarly, the C_2 option with the highest value is, at least approximately, the one with exercise dates at $T/2$ and T, and the C_3 option with the highest value is close to the one with exercise dates $T/3, 2T/3$ and T.

The pattern of volatility of a bond price over the life of the bond is quite different from that for stock prices because of the finite life of the bond. A

[9] In simplified notation, equations (12) and (13) are as follows:

$$\begin{aligned} \hat{C}_2 &= C_2 + [C_2 - C_1], \\ \hat{C}_3 &= C_3 + \frac{7}{2}[C_3 - C_2] - \frac{1}{2}[C_2 - C_1]. \end{aligned}$$

default-free bond with a finite maturity of N years tends to have declining (annualized) volatility over its life, with the volatility declining to zero at maturity. This means that the (non-annualized) variance of the bond price, as a function of time, rises and then eventually falls to zero, at maturity. The changing volatility of the bond price creates a problem in applying the GJ technique, since it is no longer clear which values of options exercisable on a finite number of dates (C_1, C_2, C_3) should be used in the extrapolation in equations (12) and (13) above. For example, suppose the American option that we wish to value has an expiration date of $T \leq N$, where N is the maturity date of the underlying bond. Now, consider a European option on the bond with the same expiration date, T. The value of the European option C_1 depends on the expiration date, since the volatility of the underlying bond price changes over time, depending on the value of T. In the extreme case, where $T = N$, the volatility is zero, and the option C_1, therefore, has zero value. Similarly, when T is very small in relation to N, the time to expiration of the option is too low for the option to have much value. However, if T is somewhere in between, say at $N/2$, the value is likely to be much higher[10].

A practical solution to this problem is to use the "maximizing" modification of Bunch and Johnson (1992) to the basic GJ technique. Under this modification, the C_1, C_2, C_3 values that are used are the *maxima* over all possible exercise dates. Thus, C_1^* is the value of the European option with the highest value, where C_1 is maximized over all possible exercise dates in the range $[0, T]$[11]. Similarly, C_2^* is maximized over all possible pairs of exercise dates, and C_3^* over all possible triplets of exercise dates[12]. The Bunch and

[10]This highlights an important difference between the option on a finite life bond and an option on an infinite life asset, such as a stock. In the case of stock options, the longest life call option is the one with the highest value.

[11]The European option with the highest time 0 value is

$$C_1^* = C_1(t^*) = \max_t [C_1(t)], t \in (0, T],$$

where T is the final maturity date of the American option.

[12]The "mid-Atlantic" option with two exercise dates which has the highest value at time 0 is worth

$$C_2^* = C_2(t_1^*, t_2^*) = \max_{t_1, t_2} [C_2(t_1, t_2)], t_1 \leq t_2, t_1, t_2 \in (0, T],$$

where T is the final maturity date of the American-style option. C_3^* is defined analogously

Johnson (1992) technique, which provides only a marginal improvement in accuracy in the case of stock options is, therefore, essential in applying the GJ methodology to bond options.[13]

In HSS (1994), a further modification of the GJ methodology is suggested. It is shown that, for long-dated options, the accuracy of the GJ approach can be improved by assuming an exponential relationship between the prices of options with different numbers of exercise dates. Combining the ideas of the HSS (1994) "exponential" technique and the Bunch and Johnson (1992) "maximization" technique, we use the following predictor of the value of an American option. Using just C_1^* and C_2^* values, the approximation for the value of the American option is given by

$$\hat{C}_2 = [C_2^*/C_1^*]C_2^*. \tag{14}$$

The value of the American option is the asymptotic value of the series of maximized option values. The methodology is illustrated in Figure 2, which shows a plot of option values as a function of the number of exercise points. When there is only one exercise point, the option values lie in the range A to A', the highest value being at A. Similarly, for two and three exercise points, the maximum values are at D and E, respectively. Using the values at A, D and E, the asymptotic value at B is obtained by extrapolation.

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by

$$C_3^* = C_3(t_1^*, t_2^*, t_3^*) = \max_{t_1,t_2,t_3}[C_3(t_1, t_2, t_3)], t_1 \le t_2 \le t_3, t_1, t_2, t_3 \in (0, T].$$

[13]Bunch and Johnson (1992) found that the increased accuracy produced by their maximization technique meant that inclusion of options with more than two exercise dates was unnecessary (except for deep-in-the-money options). We conducted a similar test using options with three exercise points and using the prediction formula (33) with $J = 3$. The maximization procedure is more complex with three exercise points, so the time taken to compute the prices is considerably increased. We found that the prices were very similar from the two models, showing that penny accuracy (i.e., to within 1%) was produced by the model with just two possible exercise dates. Details of the simulations are given in Ho, Stapleton and Subrahmanyam (1991).

Inputs required for the calculation of $\hat{C}_1$, $\hat{C}_2$ and $\hat{C}_3$

In HSS (1995), we describe a method which can be used to construct a multivariate-binomial approximation to a joint-lognormal distribution. This approximation can be used to value an option with two possible exercise dates. The key step in this methodology is the construction of a binomial tree with the required mean, variance and covariance characteristics. In this section we describe the required inputs for the model.

The important inputs required for the calculation of option prices are the forward prices of the asset for each exercise date, and volatility of the asset price over the relevant time periods. For example, since we need the maximum European option price, we need the forward price and volatilities for all possible future exercise dates. In the examples that follow we maximize the option prices by calculating the prices of options with maturities that increase by six-monthly intervals. Similarly, when calculating C_2 and C_3 values, we consider a set of possible exercise dates on a grid of six-monthly spaced points. We also consider bonds with semi-annual interest payments. Therefore, in the examples, we simply take the forward price of the bond, F_{0,t_i}, to be a constant. In general, however, the forward prices need to be computed in the usual way by compounding the spot prices of the bond up to the exercise date and adjusting for the value of any intermediate coupon interest payments.[14]

The model first requires volatility inputs for computing the European

[14]In the case of a bond, the forward price of the underlying asset for delivery at time t_i, F_{0,t_i}, depends upon the coupon-interest payments on the bond. If the bond pays no interest then by spot-forward parity the forward price would be

$$F_{0,t_i} = \frac{S_0}{B_{0,t_i}}, \quad i = 1, 2, \ldots, J.$$

However, given semi-annual coupon payments of $\frac{c}{2}$ paid at $\tau = \frac{1}{2}, 1, 1\frac{1}{2}, \ldots, N$, this simple relationship has to be modified as follows using spot-forward parity:

$$F_{0,t_i} = \left[S_0 - \sum_{\tau=\frac{1}{2}}^{N} \frac{\frac{c}{2} B_{0,\tau}}{2} \right] / B_{0,t_i},$$

where $\frac{c}{2}$ is the semi-annual coupon and N is the maturity date of the bond. Note that coupons paid after time t_i are deducted from the bond price.

option prices, for all maturities $t_i \in (0, T]$, where T is the final maturity date of the American option. As discussed earlier, the price of the underlying bond has a time-dependent volatility due to its fixed final maturity date. For the valuation of the options with two and three possible exercise dates, we require both unconditional and conditional volatilities on the relevant dates. For example, if we wish to value an option with two exercise dates, t_1 and t_2, we need the unconditional volatilities σ_{0,t_1} and σ_{0,t_2} and the conditional volatility σ_{t_1,t_2}.[15]

A number of approaches to estimating these volatilities are possible. First, the volatilities could simply be assumed to be given exogenously. Second, we could generate the volatilities using a factor model. Thirdly, we could build a model of the evolution of the term structure of interest rates, value bonds given these interest rates, and then price the options using these prices.

The first approach has been used in many practical applications of the Black and Scholes (1973) model to the pricing of European options on bonds. The second approach was employed by Brennan and Schwartz (1979) and Schaefer and Schwartz (1987), for pricing bond options. The former paper uses a two-factor model, with the long rate and the spread between the short and long rate as factors. The latter paper uses a one-factor duration model to generate bond volatilities. The third approach builds a no-arbitrage term structure and was first used by Ho and Lee (1986) and then by Heath, Jarrow and Morton (1990a, 1990b, 1992). In this paper, we use the second of the approaches outlined above, for the following reasons.

First, we need so many volatility inputs that the first approach is somewhat impractical when a large number of simulations are to be performed. The third approach on the other hand, which was used by Jamshidian (1989) to value bonds options, is extremely complicated to apply, except in the case of one-factor models. Thus, there is a tradeoff between the number of factors used to describe the movements in the term structure and the level of detail in defining the evolution over time. We, therefore, use the second approach and assume that an exogenously given two-factor model of interest rates generates the yields on bonds. In such a model, we run the risk of

[15]Since conditional and unconditional volatilities are required for any combination of exercise dates, we need to ensure consistency between the volatility estimates. The bond volatility, for example, should be a declining function of time, as the maturity of the bond approaches. This is roughly analogous to ensuring consistency between spot and forward interest rates.

not satisfying the basic no-arbitrage conditions of a complete term structure model. However, at a practical level, this risk is perhaps worth taking, given the computational effort that would be required to build a full, arbitrage-free two-factor model of the term structure. The volatility of a bond over a specified period depends on the volatility of the term structure of interest rates. Here, we assume that the term structure is generated by two factors, a short-term rate factor x_t and an orthogonal second factor y_t. The second factor can be thought of as a spread between the short-term interest rate and the long-term interest rate. The i–th interest rate at time t is given by the linear relationship

$$r_{t_i} = a_i x_t + b_i y_t,\ \ i = 1, 2, \ldots, J, \tag{15}$$

where $a_1 = 1$, $b_1 = 0$, and hence, $r_{t_1} = x_t$. We further assume that the short-term interest rate factor follows a mean-reverting process of the form

$$x_t = \mu + (x_{t-1} - \mu)(1 - \alpha_x) + \epsilon_t, \tag{16}$$

where μ is the long-run mean of the process, α_x is the periodic mean reversion and ϵ_t is a white noise error term. In this discrete version of the Vasicek-type model, the (non-annualized) variance of x_t over any period $(0, t)$ is

$$\mathrm{var}_0(x_t) = \mathrm{var}_{t-1}(x_t)[1 - (1 - \alpha_x)^{2t}]/[1 - (1 - \alpha_x)^2]. \tag{17}$$

Equation (17) shows the relationship between the degree of mean reversion of the short-term interest rate factor and its volatility over a finite time-period. If the short rate mean-reverts strongly, the volatility will be a steeply-declining function of time. Thus, on an annualized basis, the volatility of the short-term interest rate over a long period will be significantly less than its volatility looked at over a short period. On the other hand, we assume here that the long-rate spread factor, y_t, follows a random walk. This implies that the long-rate factor has a constant volatility, looked at over different time intervals, $(0, t)$.

The price of a default free bond, with principal amount of \$1, coupon rate c, and final maturity date N, at time t is modelled as the linear sum of the discounted cash flows. We denote the discount factor for the bond cash flows that occur at time $t + t_i, t_i = (\frac{1}{2}, 1, 1\frac{1}{2}, ..., N - t)$ as $B_{t,t+t_i}$. Time is counted in half-years since we model the price of a bond paying semi-annual coupons. Assuming that time t is a coupon-payment date, the ex-coupon price of the coupon bond at time t, denoted by $B^c_{t,N}$ is

$$B^c_{t,N} = \sum_{t_i=\frac{1}{2}}^{N-t} \frac{c}{2} B_{t,t+t_i} + B_{t,N}, \tag{18}$$

where

$$B_{t,t+t_i} = e^{-r_{t_i} t_i}, \tag{19}$$

and where r_{t_i} is given by the two-factor model in equation (15). We can now model the volatility of the coupon bond price as a function of the volatilities of the two interest rate factors x_t and y_t. First, we invoke the following approximation[16]

$$\text{var}[f(x_t, y_t)] \approx \left(E\left[\frac{\partial f(x_t, y_t)}{\partial x_t}\right]\right)^2 \text{var}(x_t) + \left(E\left[\frac{\partial f(x_t, y_t)}{\partial y_t}\right]\right)^2 \text{var}(y_t), \tag{20}$$

given that x_t and y_t are independent. To apply this relationship in the case of our two-factor model, we first define

$$f(x_t, y_t) = \ln B^c_{t,N}, \tag{21}$$

and then derive

$$\frac{\partial f(x_t, y_t)}{\partial x_t} = \frac{\partial \ln B^c_{t,N}}{\partial x_t} = -\frac{\sum_{t_i=\frac{1}{2}}^{N-t} t_i \frac{c}{2} a_i B_{t,t+t_i} + (N-t) a_N B_{t,N}}{B^c_{t,N}}, \tag{22}$$

and

$$\frac{\partial f(x_t, y_t)}{\partial y_t} = \frac{\partial \ln B^c_{t,N}}{\partial y_t} = -\frac{\sum_{t_i=\frac{1}{2}}^{N-t} t_i \frac{c}{2} b_i B_{t,t+t_i} + (N-t) b_N B_{t,N}}{B^c_{t,N}}. \tag{23}$$

Note that the expectation in (20) in our case is the expectation under the risk-neutral measure where the mean is the forward price of the asset. It

[16]See Stuart and Ord (1987), p. 324.

follows, therefore, that we can use the following approximation for the mean of the partial derivatives:

$$E\left[\frac{\partial \ln B^c_{t,N}}{\partial x_t}\right] \simeq -\frac{\sum_{t_i=\frac{1}{2}}^{N-t} t_i \frac{c}{2} a_i F_{0,t,t+t_i} + (N-t) a_N F_{0,t,N}}{F_{0,t}}, \tag{24}$$

$$E\left[\frac{\partial \ln B^c_{t,N}}{\partial y_t}\right] \simeq -\frac{\sum_{t_i=\frac{1}{2}}^{N-t} t_i \frac{c}{2} b_i F_{0,t,t+t_i} + (N-t) b_N F_{0,t,N}}{F_{0,t}}, \tag{25}$$

where $F_{0,t}$ is the forward price of the coupon bond and $F_{0,t,t+t_i}$ is the forward price for delivery at t of a zero-coupon bond with final maturity $t+t_i$.[17] We now, for convenience, define the "duration"-type terms as follows:

$$D_x = \sum_{t_i=\frac{1}{2}}^{N-t} t_i \frac{c}{2} a_i F_{0,t,t+t_i} + (N-t) a_N F_{0,t,N}, \tag{26}$$

$$D_y = \sum_{t_i=\frac{1}{2}}^{N-t} t_i \frac{c}{2} b_i F_{0,t,t+t_i} + (N-t) b_N F_{0,t,N}. \tag{27}$$

It follows, after substituting in equation(20), that the variance of the logarithm of the coupon-bond price is:

$$\text{var}_{0,t}[\ln B^c_{t,N}] = D_x^2 \text{var}_{0,t}(x) + D_y^2 \text{var}_{0,t}(y). \tag{28}$$

Finally, we have the expression for the coupon-bond volatility in terms of the annualized volatilities of x_t and y_t:

$$\sigma_{0,t} = \sqrt{D_x^2 \sigma^2_{0,t,x} + D_y^2 \sigma^2_{0,t,y}}. \tag{29}$$

In order to price options with two possible exercise dates, t_1 and t_2, we require unconditional volatilities from (29) and also the conditional volatilities. The

[17]The approximation in (24) and (25) ignores the effect of non-linearity due to Jensen's inequality. In particular, the effect of the covariances of $F_{0,t,t+t_i}$ and $F_{0,t}$ are ignored. This would have the effect of understating the volatilities to some extent by ignoring the effects of second-order (convexity) and higher-order effects.

conditional volatilities are computed from the same model, simply recognizing the maturity of the underlying bond at time t_1. Hence, the "duration" terms become

$$D_x' = \frac{\sum_{t_i=\frac{1}{2}}^{N-t_2} t_i \frac{c}{2} a_i F_{t_1,t_2,t_2+t_i} + (N-t_2) a_N F_{t_1,t_2,N}}{F_{t_1,t_2,N}}, \tag{30}$$

$$D_y' = \frac{\sum_{t_i=\frac{1}{2}}^{N-t_2} t_i \frac{c}{2} b_i F_{t_1,t_2,t_2+t_i} + (N-t_2) b_N F_{t_1,t_2,N}}{F_{t_1,t_2,N}}, \tag{31}$$

and the conditional volatility is

$$\sigma_{t_1,t_2} = \sqrt{(D_x')^2 \sigma_{t_1,t_2,x}^2 + (D_y')^2 \sigma_{t_1,t_2,y}^2}. \tag{32}$$

Estimation of American Option Values

The computational efficiency of the method is achieved by predicting the value of an American option using a European option and an option with two possible exercise date[18]. However, as illustrated in Figure 2, it is only the maximized option prices denoted

$$\begin{aligned} C_1^* &\equiv C_1(t^*, K) = \max_t [C_1(t)], t \in (0, T], \\ C_2^* &\equiv C_2(t_1^*, t_2^*, K) = \max_{t_1,t_2} [C_2(t_1, t_2)],\ t_1 \le t_2, t_1, t_2 \in (0, T], \end{aligned}$$

for simplicity, that are relevant. In Figure 2, the options with one exercise point are the European options. Point A denotes the option with price C_1^*.

In HSS (1994) we argued that an exponential relationship could be assumed to exist between the American option value and the number of possible exercise points. Applying the exponential relationship in this case we have

$$\hat{C}(\infty) = C^*(J) \exp\left(\frac{k}{J}\right), \tag{33}$$

where $\hat{C}(\infty)$ is the American-option price, $C^*(J)$ is the maximum value of the function $C_0(t_1, t_2, \ldots, t_J, K)$, and k is a constant. We define $\hat{C}_2$ as the predicted value of the American-style option, using just C_1^* and C_2^* as shown in equation (14). In the following section, we examine the comparative statics of the predicted value of the American option.

[18] Breen (1991) shows the efficiency of the GJ approximation in the binomial case.

5 Comparative Statics of the Model

In this section, we examine the characteristics of the American bond option prices generated by our model in some detail. We demonstrate that the model values American bond options to "penny accuracy" using only the prices of European options and options with two exercise dates. We consider two types of simulations of our model:

5.1 A. Sensitivity analysis of the computational method

Here, we examine the effect of two key inputs to the algorithm. One is the size of the binomial lattice (i.e., the number of binomial stages, n) and the other is effect of using three rather than two option prices in the extrapolation.

5.2 B. Comparative statics and analysis of key input parameters

The parameters we consider are the exercise price, volatility, and time to expiration. In the simulations reported below, the parameters used in the base case are:

Maximum size of binomial lattice, $n = 60$.
Maturity of the underlying bond, $N = 10$ years.
Annual coupon rate of bond, $c = 10.8\%$.
Time-grid size for the underling bond = 0.5 years.
Short term interest rate volatility, $\sigma_{0,t,x} = 0.0055$.[19]
Long term interest rate volatility, $\sigma_{0,t,y} = 0.0040$.
Mean reversion coefficient, $\alpha_x = 0.05$.
Exercise price, $K = 100$.[20]

[19]The interest rate volatility numbers, $\sigma_{0,t,x}$ and $\sigma_{0,t,y}$ are chosen so that they provide reasonable estimates for bond price volatility when multiplied by the "duration"-type terms in equation (29).

[20]Although it is possible to make the strike price a function of t we simply choose

$$K(t_i) = K,$$

a constant, in the following simulations.

A. Sensitivity analysis of the computational method

The effect of changing the size of the binomial lattice(n)

Table 1 shows the estimated values of the option, $\hat{C}_2$, based on two option prices in extrapolation as a function of the number of binomial stages, n. For example, for $n = 60$, the maximum European option price is estimated with $t^* = 2.5$ years, resulting in a value of $C_1^* = 0.7948$. The combination of (t_1^*, t_2^*) which gives the maximum value of $C_2^* = 0.9580$, is $t_1^* = 1.5$ years and $t_2^* = 5.0$ years. The estimated $\hat{C}_2$ in this case is 1.1548. The model values exhibit the normal fluctuations associated with the binomial lattice method as a function of n, which get dampened as n gets larger. These values and other simulations not shown here with different exercise prices show that the values in the range of $n = 11$ to $n = 15$ provide a reasonable approximation to the asymptotic $\hat{C}_2$ value. The advantage of using a relative small n is the obvious computational efficiency in relation to competing methods that use numerical (polynomial approximations) for bivariate and trivariate normal distributions.

The effect of using three option prices in the extrapolation ($\hat{C}_3$)

In Table 2, we investigate the effect of adding options exercisable at one of three dates in the extrapolation to value the American call options, i.e., using equation (13) instead of (12). Again, the option price used is the maximum of the values across exercise dates, where the three exercise dates are chosen with $t_1 \leq t_2 \leq t_3$. The principal finding is that only a marginal increase in accuracy is obtainable by considering options exercisable on three dates.

The simulation results show that the American call option price using options with one and two exercise dates, $\hat{C}_2$, is 1.4758 for $n = 12$, whereas the corresponding price using options exercisable on one, two and three dates, $\hat{C}_3$, is 1.5005. Using $n = 30$ and $n = 60$, the American call price for $\hat{C}_2$ are 1.4826 and 1.4796, respectively. The corresponding $\hat{C}_3$ prices are 1.5070 and 1.5106, respectively. The differences between $\hat{C}_3$ and $\hat{C}_2$ for $n = 12, 30, 60$ are small, about 2 or 3 pennies (about 1–2 %). The pricing differences appear to increase with the size of the binomial lattice, n.

The $\hat{C}_3$ model requires a far more complex calculation and optimization procedure than the $\hat{C}_2$ model, since the value of the option must be maximized over combinations of three different exercise dates. We feel that the marginal increase in accuracy obtained may not be justified by the increase in computational time.

B. Sensitivity analysis of key input parameters

We now consider the effect of changing three key input parameters, the exercise price, the maturity of the underlying bond and the volatility inputs.

Sensitivity of option prices to changes in exercise price (K)

We next investigate the impact of the change in the exercise price on value of the American-style option, $\hat{C}_2$. This has the effect of investigating the valuation characteristic of the model for options which are deep-in-the-money to options which are deep-out-of-the-money. Because of the convergence of the option prices when the option is very deep-in-the-money and deep-out-of-the-money, the results reported for Simulation 2 are tabulated in Table 3 for exercise prices of $K = 95$ to $K = 110$ only.

The simulations show that as the call option is further out-of-the-money, the value of $\hat{C}_2$ approaches zero. Using the case where $\sigma_{0,t,x} = 0.0055$ and $\sigma_{0,t,y} = 0.0040$ as the call option gets deep-in-the-money the value of $\hat{C}_2$ increases from an at-the-money ($K = 100$) price of 1.1585 to a price of 4.9457 for $K = 95$. The well-behaved characteristics of the option prices, which are quite similar to those found in the Black-Scholes model, are clearly depicted in Figure 3.

In addition, Figure 3 shows that as $\sigma_{0,t,x}$ and $\sigma_{0,t,y}$ increases the value of the call option also increases. The call values are therefore shown to be sensitive to the forward prices (as represented by changing the exercise price, K) and the estimates of $\sigma_{0,t,x}$ and $\sigma_{0,t,y}$. The sensitivity, however, is more pronounced for at-the-money options.

Sensitivity of option prices to the maturity of the underlying bond (N)

The next comparative statics exercise investigates the pricing characteristics of the $\hat{C}_2$ estimate for the valuation of options on 10.8% coupon bonds with maturities of 5, 10, 15 and 20 years. The other parameters used in the model are listed in Table 4. It can readily be seen from the table that the price of $\hat{C}_2$ increases with bond maturities for a given estimate of the volatility of the short-term ($\sigma_{0,t,x}$) and long-term ($\sigma_{0,t,y}$) interest rate factors.

Sensitivity of $\hat{C}_2$ to volatility inputs ($\sigma_{0,t,x}$), ($\sigma_{0,t,y}$)

Lastly, we investigate whether the results above, on the accuracy of the $\hat{C}_2$ estimation, is sensitive to the volatility used. The results tabulated in Table 5 show as expected that $\hat{C}_2$ increases with increase in the volatilities of the short and long rates, i.e., $\sigma_{0,t,x}$ and $\sigma_{0,t,y}$.

6 Conclusions

An American option can be thought of as the limit of a series of options exercisable on one of many exercise dates. However, in the case of an option with a general exercise schedule, on an asset with an arbitrary volatility structure, the limit is one of a series of maximized option prices. We propose a model which uses just a European and an option exercisable on one of two dates. We show in the simulations of the model, that a binomial version of the model, with just 12 stages in the binomial process is sufficient for penny accuracy. Also we show, using simulations of bond option prices, that the model has characteristics which are similar to those of the Black and Scholes (1973) model with respect to changes in strike prices and volatility.

Predictions using just the European and an option exercisable on one of two dates are tested by comparing them with a prediction using a European and an option exercisable on one of three dates. Again we show that using the European and an option exercisable on one of two dates, leads to penny accuracy.

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Figure 1
Computation of $C_0(t_1, t_2, K(t_i))$, the early exercise decision and payoffs from exercise.

The figure illustrates the computation of the value of the option exercisable at time t_1 and t_2, using equations (3) to (6). E and NE indicate, respectively, the states in which early exercise does or not occur at time t_1. States 0 to h_1 indicate states at time t_2, which follow from states at time t_1, when early exercise occurs. In states 0 to h_2, it is worthwhile to exercise at time t_2. The option is valued by discounting the payoffs period-by-period, taking the optimal exercise decision into account, and using the discount factors in each state. The value on each date is the expectation of the discounted payoffs under the risk-neutral distribution.

Figure 2
Approximating American call option values using "maximized" values of European options and options exercisable on one, two and three dates.

The range $A - A'$ shows the European option values for different feasible maturities. C_1^* is the maximum European option value. The range $D - D'$ shows the option values for options with two possible exercise dates. C_2^* is the maximum of these option values. Similarly, C_3^* is the maximum value of options exercisable on three possible dates. The asymptotic point B is derived by using the following equation

$$\hat{C}_3(\infty) = C^*(J)\exp(k/J),$$

after solving for the constant k using C_1^* and C_2^*. $\hat{C}_3(\infty)$ is the predicted value of the American option.

Figure 3
Sensitivity of American call option values to changes in the exercise price for different volatilities of the short-term and long-term interest rates factors.

The graph plots American call option values against exercise prices for fixed volatilities of the short- and long-term interest rate factors, $\sigma_{0,t,x} = \sigma_{0,t,y} = 0.002, 0.0055, 0.008, 0.01$. The other parameters used in the calculations of option values are as follows: The size of the binomial lattice, n, is 12, the grid size is 0.5 years, the mean reversion coefficient, α_x, is 0.05, the bond maturity, N, is 10 years with an annual coupon, c, of 10.8%. In the graph, $\hat{C}_2(\infty)$ is the exponential estimate of the American call option value.

Table 1
American call option values as function of the size of the binomial lattice.

The table shows the estimated American call option value for different sizes of the binomial lattice, n. The grid size used in the maximization process is 0.5 years, the mean-reversion coefficient, α_x, is 0.05, the volatilities of the short- and long-term interest rate factors are, respectively, $\sigma_{0,t,x} = 0.0055$ and $\sigma_{0,t,y} = 0.0040$, the bond maturity, N, is 10 years with an annual coupon, c, of 10.8%, the exercise price of the option, K, is 100. In the table, t^* is the maturity at which the maximum is obtained for C_1^* the maximum valued European option value, where the maximum is taken over all possible option maturities. C_2^* is the maximum value of all options with two possible exercise dates where the maximum is taken over all possible pairs of exercise dates, t_1 and t_2. The pair of dates for the maximum is (t_1^*, t_2^*). $\hat{C}_2(\infty)$ is the exponential estimate of the American call option value.

Size of Binomial Lattice, n	Maturity, t^*	Maximum European Option Value, C_1^*	Maturity, t_1^*	t_2^*	Maximum Twice–Exercisable Option, C_2^*	Exponential American Option Value, $\hat{C}_2(\infty)$
5	2.5	0.7996	1.5	4.0	0.9216	1.0622
6	2.5	0.7988	2.0	5.0	0.9665	1.1693
7	2.5	0.7984	1.5	4.0	0.9420	1.1114
8	2.5	0.7980	2.0	5.5	0.9651	1.1672
9	2.5	0.7977	1.5	4.0	0.9507	1.1331
10	2.5	0.7975	2.0	5.5	0.9630	1.1630
12	2.5	0.7971	2.0	5.5	0.9609	1.1585
14	2.5	0.7968	2.0	6.0	0.9590	1.1542
16	2.5	0.7965	2.0	6.0	0.9575	1.1510
18	2.5	0.7963	1.5	4.0	0.9470	1.1263
20	2.5	0.7962	1.5	4.0	0.9498	1.1331
25	2.5	0.7958	1.5	5.0	0.9597	1.1572
30	2.5	0.7956	1.5	4.5	0.9576	1.1527
35	2.5	0.7954	1.5	5.0	0.9571	1.1517
40	2.5	0.7952	1.5	4.5	0.9590	1.1565
45	2.5	0.7951	1.5	4.5	0.9546	1.1461
50	2.5	0.7950	1.5	5.0	0.9587	1.1561
55	2.5	0.7949	1.5	4.5	0.9572	1.1526
60	2.5	0.7948	1.5	5.0	0.9580	1.1548

Table 2
American call option values estimated with values of the European option and options with two and three possible exercise dates.

The table shows the estimated American call option values for varying size of the binomial lattice, n. The volatilities of the short- and long-term interest rate factors are: $\sigma_{0,t,x} = \sigma_{0,t,y} = 0.0055$ The grid size used in the maximization process is 0.5 years, the mean-reversion coefficient, α_x, is 0.05, the bond maturity, N, is 10 years with an annual coupon, c, of 10.8%, the exercise price of the option, K, is 100. In the table, t^* is the maturity at which the maximum is obtained for the European option, C_1^* is the maximum European option value, where the maximum is taken over all possible option maturities. C_2^* is the maximum value of all options with two possible exercise dates, where the maximum is taken over all possible pairs of exercise dates, t_1 and t_2. The pair of dates for the maximum is (t_1^*, t_2^*). $\hat{C}_2(\infty)$ is the exponential estimate of the American call option value. C_3^* is the maximum value of all options with three possible exercise dates where the maximum is taken over all possible combinations of exercise dates, t_1, t_2 and t_3. The three dates for the maximum is (t_1^*, t_2^*, t_3^*). $\hat{C}_3(\infty)$ is the corresponding exponential estimate of the American call option value with three possible exercise dates.

Size of Binomial Lattice, n	Maturity, t^*	Maximum European Option Value, C_1^*	Maturity, t_1^*	t_2^*	Maximum Twice-Exercisable Option, C_2^*	Exponential American Option Value, $\hat{C}_2(\infty)$
12	2.5	1.0432	1.5	4.5	1.2408	1.4758
30	2.5	1.0416	1.5	4.0	1.2427	1.4826
60	2.5	1.0407	1.5	4.5	1.2409	1.4796

Size of Binomial Lattice, n	Maturity, t_1^*	t_2^*	t_3^*	Maximum Thrice-Exercisable Option, C_3^*	Exponential American Option Value, $\hat{C}_3(\infty)$	$\hat{C}_3(\infty) - \hat{C}_2(\infty)$
12	1.5	3.0	6.0	1.3292	1.5005	0.0247
30	1.5	3.0	5.5	1.3324	1.5070	0.0244
60	1.5	3.0	6.0	1.3343	1.5108	0.0312

Table 3
American call option values for different values of the exercise price.

The table shows the estimated American call option value for different values of the exercise price, K. The size of the binomial lattice, n, is 12, the grid size is 0.5 years, the mean-reversion coefficient, α_x, is 0.05, the volatility of the short-term interest rate factor, $\sigma_{0,t,x}$, is 0.0055, volatility of the long-term interest rate factor, $\sigma_{0,t,y}$, is 0.0040, the bond maturity, N, is 10 years with an annual coupon, c, of 10.8%. In the table, t^* is the maturity at which the maximum is obtained for the European option value, C_1^* is the maximum-valued European option value where the maximum is taken over all possible option maturities, C_2^* is the maximum value of all options with two possible exercise dates where the maximum is taken over all possible pairs of exercise dates, t_1 and t_2. The combinations of dates for the maximum are (t_1^*, t_2^*). $\hat{C}_2(\infty)$ is the exponential estimates of the American call option values.

Exercise Price, K	Maturity, t^*	Maximum European Option Value, C_1^*	Maturity, t_1^*	t_2^*	Maximum Twice–Exercisable Option, C_2^*	Exponential American Option Value, $\hat{C}_2(\infty)$
95	1.0	4.4090	0.5	4.5	4.6696	4.9457
96	1.0	3.5189	0.5	4.5	3.7549	4.0068
97	1.0	2.6584	0.5	4.5	2.8878	3.1371
98	1.0	1.8611	1.0	6.0	2.1413	2.4637
99	2.0	1.2373	1.5	5.0	1.4533	1.7071
100	2.5	0.7971	2.0	5.5	0.9609	1.1585
101	3.0	0.4863	2.0	4.5	0.5977	0.7345
102	3.0	0.2765	2.5	5.0	0.3514	0.4466
103	3.5	0.1495	2.0	4.0	0.1892	0.2395
104	3.0	0.0722	2.5	4.5	0.0925	0.1185
105	3.5	0.0336	2.0	3.5	0.0391	0.0453
106	3.0	0.0129	2.0	3.5	0.0150	0.0174
107	3.5	0.0052	1.5	3.5	0.0053	0.0054
108	3.5	0.0018	1.0	3.5	0.0018	0.0018
109	3.0	0.0005	1.0	3.0	0.0005	0.0005
110	3.0	0.0001	0.5	3.0	0.0001	0.0001

Table 4
American call option values for different bond maturities.

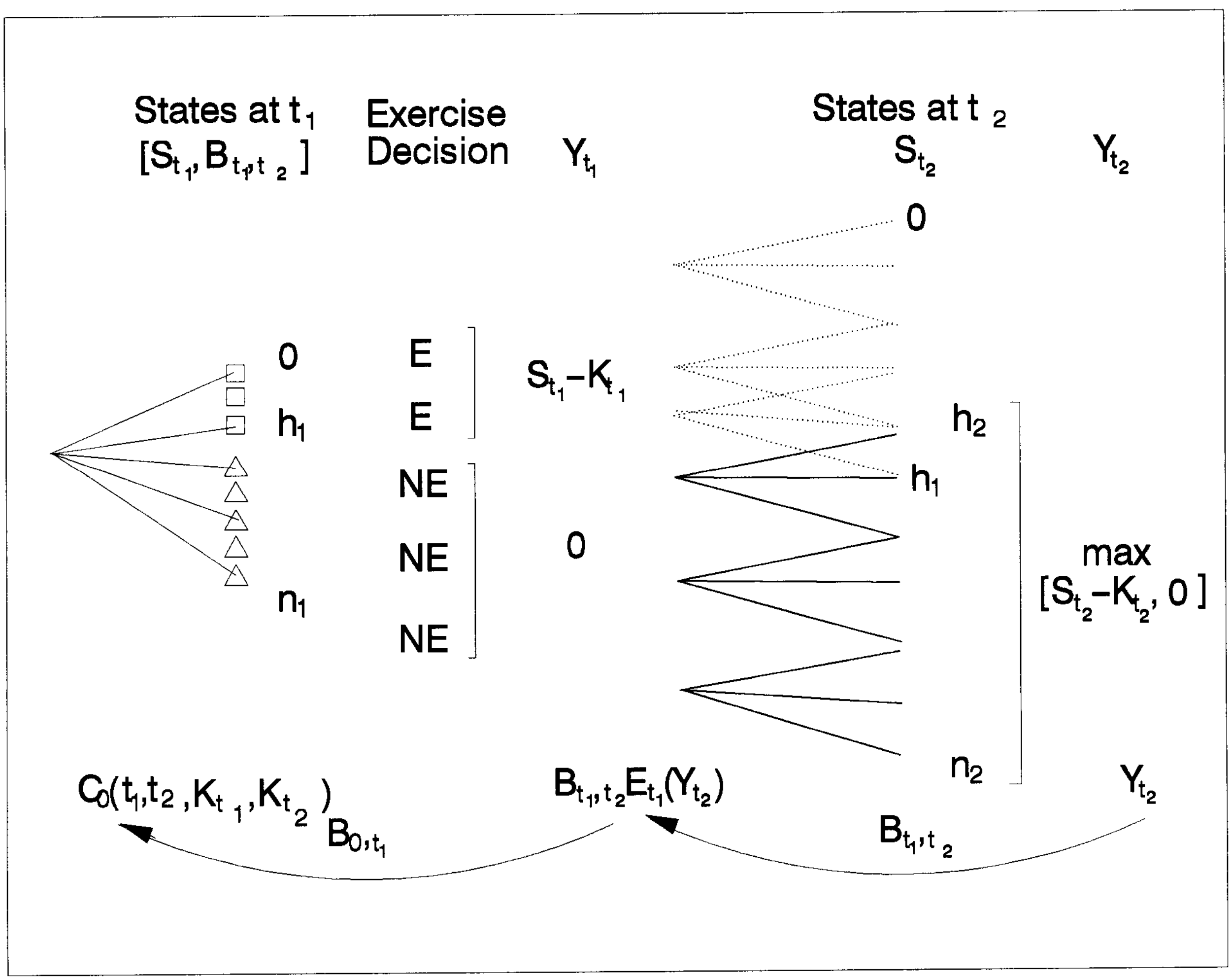
The table shows the estimated American call option value for different maturities of the underlying bond. The size of the binomial lattice, n, is 12, the grid size is 0.5 years, the mean-reversion coefficient, α_x, is 0.05, the volatility of the short-term interest rate factor, $\sigma_{0,t,x}$, is 0.0055, volatility of the long-term interest rate factor, $\sigma_{0,t,y}$, is 0.0040, the bond maturity, N, varies from 5 to 20 years, with an annual coupon, c, of 10.8%, the exercise price of the option, K, is 100. In the table, t^* is the maturity at which the maximum is obtained for the European option, C_1^* is the maximum European option value, where the maximum is taken over all possible option maturities, $C^*(2)$ is the maximum value of all options with two possible exercise dates where the maximum is taken over all possible pairs of exercise dates, t_1 and t_2. The pair of dates for the maximum is (t_1^*, t_2^*). $\hat{C}_2(\infty)$ is the exponential estimate of the American call option value.

Bond Maturity, N	Maturity, t^*	Maximum European Option Value, C_1^*	Maturity, t_1^*	t_2^*	Maximum Twice–Exercisable Option, C_2^*	Exponential American Option Value, $\hat{C}_2(\infty)$
5	1.5	0.4081	1.0	3.0	0.4909	0.5904
10	2.5	0.7971	2.0	5.5	0.9609	1.1585
15	3.0	1.1507	2.5	6.5	1.3737	1.6399
20	4.0	1.3989	3.0	8.0	1.6678	1.9885

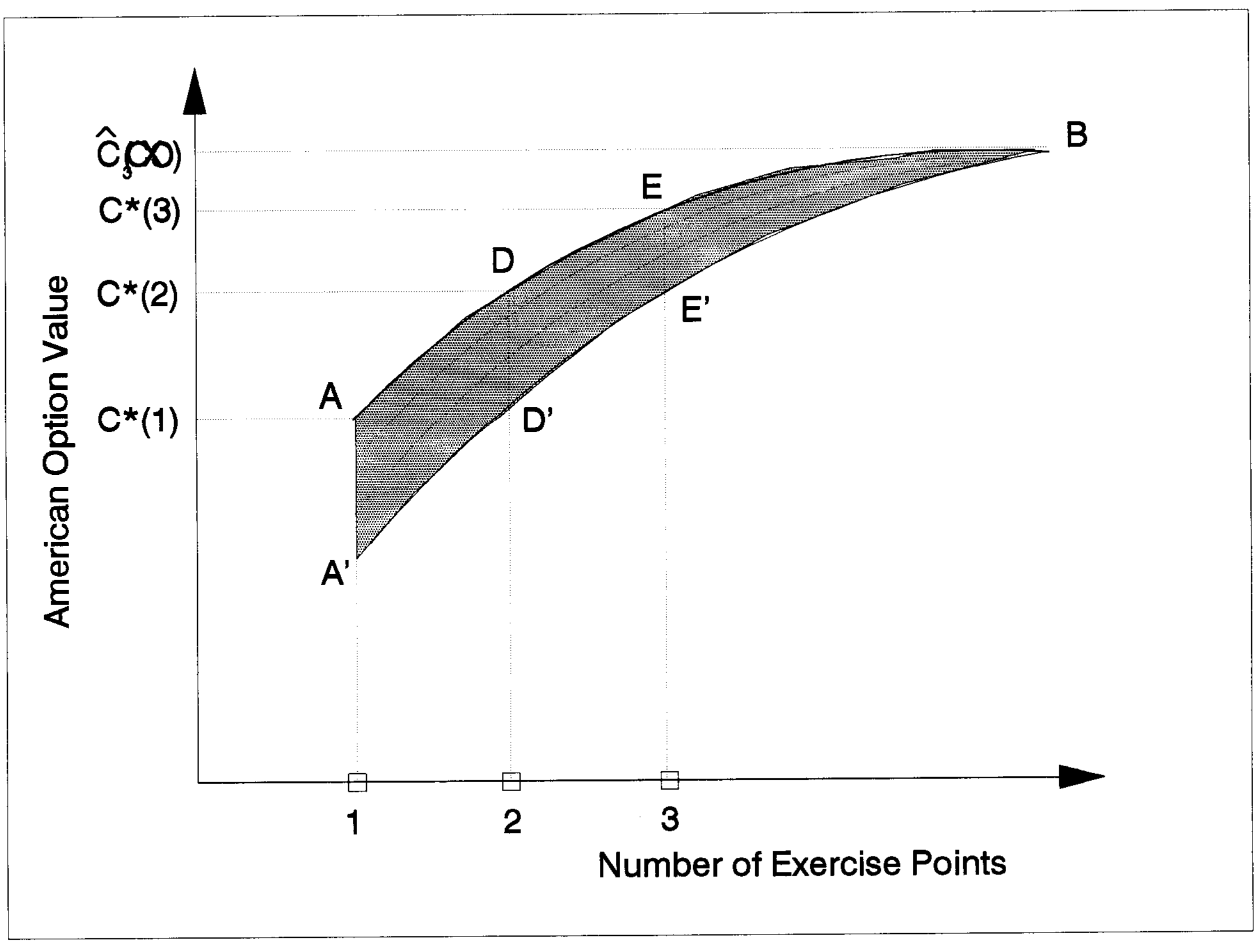
Table 5
American call option values for varying short and long interest rate volatilities.

The table shows the estimated American-style bond option values for varying volatilities of the short- and long-term interest rate factors, $\sigma_{0,t,x}$ and $\sigma_{0,t,y}$, respectively. The size of the binomial lattice, n, is 12, the grid size is 0.5 years, the mean-reversion coefficient, α_x, is 0.05, the bond maturity, N, is 10 years with an annual coupon, c, of 10.8%, the exercise price of the option, K, is 100. In the table, t^* is the maturity at which the maximum is obtained for the European option, C_1^* is the maximum European option value, where the maximum is taken over all possible option maturities, C_2^* is the maximum value of all options with two possible exercise dates where the maximum is taken over all possible pairs of exercise dates, t_1 and t_2. The pair of dates for the maximum is (t_1^*, t_2^*). $\hat{C}_2(\infty)$ is the exponential estimate of the American call option value.

Short and Long Rate Factors Volatility, $\sigma_{0,t,x} = \sigma_{0,t,y}$	Maturity, t^*	Maximum European Option Value, C_1^*	Maturity, t_1^*	t_2^*	Maximum Twice–Exercisable Option, C_2^*	Exponential American Option Value, $\hat{C}_2(\infty)$
0.0020	2.5	0.3582	2.0	5.0	0.4277	0.5107
0.0040	2.5	0.7494	1.5	4.5	0.8917	1.0611
0.0055	2.5	1.0432	1.5	4.5	1.2408	1.4758
0.0080	2.5	1.5336	1.5	4.5	1.8215	2.1634
0.0100	2.5	1.9266	1.5	4.5	2.2850	2.7100



States at t1
[St1, Bt1,t2]
Exercise
Decision
Yt1
States at t 2
St2
Yt2
0
0
E
E
h1
St1-Kt1
NE
NE
NE
0
h2
h1
n1
max
[St2-Kt2, 0]
n2
C0(t1,t2 ,Kt1,Kt2)
B0,t1
Bt1,t2Et1(Yt2)
Bt1,t2
Yt2



American Option Value
Ĉ(∞)
C*(3)
C*(2)
C*(1)
A
A'
D
D'
E
E'
B
1
2
3
Number of Exercise Points