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Ho, T.S., Richard C. Stapleton and Marti G. Subrahmanyam

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The Valuation of American Options with Stochastic Interest Rates: A Generalization of the Geske-Johnson Technique[1]

T.S. Ho, Richard C. Stapleton and Marti G. Subrahmanyam*

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Abstract

The Geske-Johnson approach provides an efficient and intuitively appealing technique for the valuation and hedging of American-style contingent claims. Here, we generalize their approach to a stochastic-interest-rate economy. The method is implemented using options exercisable on one of a finite number of dates. We illustrate how the value of an American-style option increases with interest-rate volatility. The magnitude of this effect depends on the extent to which the option is in the money, the volatilities of the underlying asset and the interest rates, as well as the correlation between them.

Introduction

Stochastic interest rates add a potentially important dimension to the valuation of American-style contingent claims. To value such claims, it is necessary to compare the exercised value of the claim with the "live" value (the unexercised value) on each date. Since the term structure of interest rates affects the live value of the claim on each possible exercise date before expiration, the probability of early exercise and hence, the early-exercise premium will, in general, be affected by the volatility of interest rates. In addition, the correlation between the price of the underlying asset and interest rates is relevant. Essentially, the holder of a contingent claim such as an American call or put option has an additional option when interest rates are stochastic: an option on the interest rate. For instance, if interest rates were to rise, the live value of the American option would fall and, other things being equal, this could trigger early exercise of the option. In order to value American options, it is necessary, therefore, to model the joint evolution of the underlying asset price and interest rates.

Several approaches to the valuation and hedging of American-style options have been suggested in the literature. These can be classified into three main types of approach: the finite-difference method, the binomial-lattice method, and various analytical methods.[1] There are significant difficulties however, in extending these methods to the case of stochastic interest rates because the state-space becomes multidimensional. In the case of the finite-difference method or the binomial method the lattice has to be built with at least two state variables: representing the underlying asset and the interest rate(s). Similarly, in the context of analytical approaches using the optimal exercise boundary, the computation becomes complicated by the fact that the boundary itself is multidimensional.

The search for rapid computational procedures and an analytical solution to the American-style option valuation problem motivated Geske and Johnson (1984)(GJ) to propose an approach based on a series of options exercisable on one of a finite number of dates.[2] The GJ method uses Richardson extrapolation to estimate the price of the American-style claim using, at most, an option with three possible exercise points. This method is attractive from a computational viewpoint and has the potential to be extended to

the context of stochastic interest rates, since the number of stochastic variables can be limited without making restrictive assumptions regarding the processes generating the variables.[3]

In this paper we derive a valuation model that is in the spirit of Merton's (1973) stochastic-interest-rate option-pricing model for options with multiple exercise dates. Merton (1973) shows that European-style options can be priced using a forward-adjusted martingale measure. Following Jamshidian (1991), we derive a risk-neutral valuation relationship in which the option with several possible exercise dates can be valued using conditional forward measures. We then adapt the GJ approach to American-option valuation in a stochastic-interest-rate environment.[4] The model is implemented using a multivariate-binomial approximation.

Section I presents a general valuation framework for the valuation of contingent claims in an economy with stochastic interest rates. We establish a risk-neutral valuation relationship for options exercisable on any one of n dates.[5] In Section II, we discuss the implementation of these valuation relationships using a multivariate-binomial lattice. In Section III, we report results of computations using the modified GJ prediction, and show the sensitivity of option prices to changes in the volatility of interest rates and to the correlation between interest rates and the asset price. Section IV concludes.

V The Valuation Model

We consider an American-style contingent claim, on a non-dividend paying asset, whose price at time t is S_t.[6] The expiration date of the claim is time T and its payoff function, if exercised at time t, is $g(S_t) \geq 0$, $t \in [0, T]$. The "live" value of the claim, i.e., its market value if not exercised at or before time t, is C_t and its value, just prior to the exercise decision at t is

$$\max[g(S_t), C_t], \qquad t \in [0, T]. \tag{18}$$

Following Geske-Johnson, we divide the interval $[0, T]$ into n sub-intervals of size h. We assume that the claim is exercisable at any one of the n dates in the set $(h, 2h, ..., T)$. The value of this claim at time t is denoted $C_{n,t}$. It

follows that

$$\lim_{n\to\infty} C_{n,t} = C_t. \tag{19}$$

We first derive a general valuation relationship for American options that includes the effect of stochastic interest rates. We do so without making assumptions about the stochastic processes generating asset and bond prices. The current value of the n exercise-date claim, $C_{n,0}$ depends upon a set of pricing kernels $(\phi_{0,h}, \phi_{h,2h}, ..., \phi_{T-h,T})$ and a set of zero-coupon bond prices $(B_{0,h}, B_{h,2h}, ..., B_{T-h,T})$ that can be used to price any security with multiperiod payoffs in a no-arbitrage economy. Here, $\phi_{t,\tau}$ is the pricing kernel relevant for valuation at t of cash flows that arise at time $\tau > t$, and $B_{t,\tau}$ is the zero-coupon bond price at t for a bond paying one dollar at time τ. E_t denotes the expectations operator, conditional on the information set at time t. In the case of our American-style contingent claim, it follows from successive substitution and the no-arbitrage principle that[7]

$$C_{n,0} = E_0[\max\{g(S_h), E_h[\max\{g(S_{2h}), E_{2h}[...]\tilde{B}_{2h,3h}\}\phi_{h,2h}]\tilde{B}_{h,2h}\}\phi_{0,h}]B_{0,h}. \tag{20}$$

In this formulation, the tilde on the bond price is added to emphasize the fact that the future zero-coupon bond prices are stochastic. In (20), the stochastic bond prices and the correlation of these prices with the asset prices affect the value $C_{n,0}$ in a complex manner. Even if bond prices are non-stochastic, as in GJ, the influence of the term structure is not straightforward. This can be seen by taking the special cases of (20) where $n = 1, 2$. Here, we have the two option prices

$$C_{1,0} = E_0[g(S_T)\phi_{0,T}]B_{0,T}, \tag{21}$$

$$C_{2,0} = E_0\left[\max\left\{g(S_{\frac{T}{2}}), E_{\frac{T}{2}}\left[g(S_T)\phi_{\frac{T}{2},T}\right]\tilde{B}_{\frac{T}{2},T}\right\}\phi_{0,\frac{T}{2}}\right]B_{0,\frac{T}{2}}. \tag{22}$$

It is easy to see that in the case of n=3 or larger, the whole term structure of interest rates on future dates would affect the current value of the option. For an option that is exercisable on one of two dates, the interest rate at the first date is in general relevant to the options' valuation, since it determines the time value of money on the exercise price. However, if the

option is so much in the money that it is highly likely to be exercised early, then, for this particular option, the stochastic interest rate at the first date has only a small effect.

A. Valuation of the Options Assuming Lognormal Bond Prices and Pricing Kernels

So far, we have used general no-arbitrage-based arguments to highlight the possible effects of stochastic interest rates on the American value of a contingent claim. However, implementation of this approach requires the estimation or elimination of the preference-related pricing kernels. Fortunately, as in the GJ case, the $\phi_{t,t+h}$ terms drop out if we assume that the S_{t+h} and $\phi_{t,t+h}$ are joint lognormally distributed.[8] We now assume that both the $\phi_{t,t+h}$ and $B_{t+h,t+2h}$ are joint lognormally distributed with S_{t+h} for $t = 0, h, ..., T-h$. In this case, equations (21) and (22) can be written in terms of the risk-neutral distributions of S_t and $B_{t,t+h}$. We have, in place of equation (3),

$$C_{n,0} = \hat{E}_0[\max\{g(S_h), \hat{E}_h[\max\{g(S_{2h}), \hat{E}_{2h}[...]\tilde{B}_{2h,3h}\}]\tilde{B}_{h,2h}\}]B_{0,h}, \qquad (23)$$

where $\hat{E}$ is the expectation under the risk-neutral distribution and where the variables S_t and $B_{t,t+h}$ are lognormally distributed under the risk-neutral distribution, with conditional means equal to the respective conditional forward prices, and volatilities equal to the exogenously given volatilities.

The proof of the risk-neutral relationship (23) is given in the Appendix for the case where $n = 2$. The proof in the general case of n possible exercise points follows a similar argument. For the European option, with $n = 1$, (23) is just the Black-Scholes equation:

$$C_{1,0}(S_0, B_{0,T}) = \hat{E}_0[g(S_T)]B_{0,T}, \qquad (24)$$

since the expectation is under the lognormal distribution with the property

$$\hat{E}_0(S_T) = \frac{S_0}{B_{0,T}} = F_{0,T}, \qquad (25)$$

where $F_{t,\tau}$ is the forward price at t for delivery at τ of the underlying asset. In other words, the expectation of S_T under the risk-neutral distribution is the asset's forward price at time 0, for delivery at time T, given that the asset pays no dividends. Equation (23) can be appreciated by considering the special case of $C_{2,0}$[9], with two equally spaced exercise points, for which we have

$$C_{2,0} = \hat{E}_0 \left[\max \left\{ g(S_{\frac{T}{2}}), \hat{E}_{\frac{T}{2}} \left[g(S_T) \right] \tilde{B}_{\frac{T}{2},T} \right\} \right] B_{0,\frac{T}{2}}, \tag{26}$$

where the $\hat{}$ distributions are lognormal with

$$\hat{E}_0(S_{\frac{T}{2}}) = \frac{S_0}{B_{0,\frac{T}{2}}} = F_{0,\frac{T}{2}}, \tag{27}$$

$$\hat{E}_0(B_{\frac{T}{2},T}) = \frac{B_{0,T}}{B_{0,\frac{T}{2}}}, \tag{28}$$

$$\hat{E}_{\frac{T}{2}}(S_T) = \frac{S_{\frac{T}{2}}}{B_{\frac{T}{2},T}} = F_{\frac{T}{2},T}, \tag{29}$$

and variances equal to the actual variances. A number of points can be noted from equations (24) and (26). First, even if bond prices are non-stochastic, $C_{1,0}$ and $C_{2,0}$ depend upon the term structure of zero bond prices at time 0. Second, if future zero bond prices are non-stochastic, the values of the claims, in the special case of put options, are the same as those of GJ. Third, equation (24) for European-style contingent claims is consistent with the formula devised by Merton (1973) using similar assumptions regarding the distribution of bond prices. Finally, note that two different risk-neutral distributions are required for the valuation. In the case of the European option, $C_{1,0}$, the mean of S_T in equation (25) is the forward price, as of $t = 0$, for delivery of the stock at T. However, in the case of $C_{2,0}$, the mean of $S_{\frac{T}{2}}$ in equation (27) is its forward price and the conditional mean in equation (29) is the conditional forward price at $T/2$ for delivery of the asset at T. If $B_{\frac{T}{2},T}$ is stochastic, the unconditional mean of S_T under the risk-neutral distribution is not, in general, equal to its forward price.[10]

In order to obtain the correct conditional mean at $T/2$ we need to model S_T with an unconditional mean

$$\hat{E}_0(S_T) = \hat{E}_0[\hat{E}_{\frac{T}{2}}(S_T)] = \hat{E}_0[F_{\frac{T}{2},T}]. \tag{30}$$

In the Appendix, we show the relationship between this expected spot price, under the risk-neutral distribution, and the asset forward price for delivery at T. The adjustment depends on the covariance of the asset price and the zero bond price at $T/2$. We have, given spot-forward parity, joint lognormality of the asset price and the zero-coupon bond price, and the no-arbitrage condition, the following relationship[11]

$$\hat{\mathrm{E}}_0[S_T] = F_{0,T} \exp\{-[\sigma_{S_{\frac{T}{2}},B_{\frac{T}{2},T}} - \sigma^2_{B_{\frac{T}{2},T}}]\}, \tag{31}$$

where σ_X^2 and σ_{XY} are respectively the variance of $\ln X$ and the covariance of $\ln X$ and $\ln Y$.[12] This adjustment takes the observable asset forward price and converts it into an expectation that is akin to the futures price of the asset. The adjustment depends on the covariance of the asset price and the zero-coupon bond price. However, note that the resulting price is the futures price which the asset would have if the futures contract were marked to market at intervals of $T/2$, rather than daily as is the case for the usual traded futures contract.

B. Application of the Geske-Johnson Method

The purpose of computing $C_{n,0}$, $n = 1, 2, ...$ is to obtain a good approximation for the continuous-exercise value, $C_{\infty,0}$. As in GJ, $C_{1,0}$, $C_{2,0}$, $C_{3,0}$, ... define a sequence, whose limit is the American value. The first few values in the sequence can be used, via Richardson extrapolation, to predict the American option value. For example, using just $C_{1,0}$ and $C_{2,0}$

$$\hat{C}_{\infty,0} = C_{2,0} + (C_{2,0} - C_{1,0}). \tag{32}$$

Using the first three options values, $C_{1,0}$, $C_{2,0}$ and $C_{3,0}$, the GJ approximation is

$$\hat{C}_{\infty,0} = C_{3,0} + \frac{7}{2}(C_{3,0} - C_{2,0}) - \frac{1}{2}(C_{2,0} - C_{1,0}), \tag{33}$$

where $C_{1,0}$ and $C_{2,0}$ are given by equations (7) and (9), and $C_{3,0}$ is given by solving (6) for $n = 3$.

Equation (33) is the GJ approximation formula given estimates of the value of $C_{1,0}$ (the European option with maturity T), the value of $C_{2,0}$ (the

option exercisable either at $T/2$ or at T), and the value of $C_{3,0}$ (the option exercisable at any one of the three dates, $T/3$, $2T/3$ and T). GJ found the approximation (33) to be an accurate predictor of the American price in the case of non-stochastic interest rates.

VI Implementation of the Model Using a Multivariate Binomial Approximation

In order to obtain numerical values of the option prices $C_{n,0}, (n = 1, 2, ..)$ and an estimate of the American option value, we construct a multivariate binomial approximation of the underlying asset and the zero-coupon bond prices. Since the binomial distributions must have the characteristic that the conditional expected values of the prices equal the forward prices at every point in time and at every node, it is numerically efficient to construct a tree of the underlying asset and zero-coupon bond forward prices rather than of spot prices.[13] Given the asset forward prices, for delivery at the final maturity date T, together with the zero-coupon bond prices, the spot prices relevant for making the optimal exercise decision can be calculated using the spot-forward parity relationship. In the case of $C_{2,0}$ we require a binomial distribution of $S_T, S_{\frac{T}{2}}$, and of the zero-coupon bond price $B_{\frac{T}{2},T}$. In the case of $C_{3,0}$ we need the joint distribution of the six variables, $S_{\frac{T}{3}}$, $S_{\frac{2T}{3}}$, S_T, $B_{\frac{T}{3},\frac{2T}{3}}$, $B_{\frac{T}{3},T}$ and $B_{\frac{2T}{3},T}$.

In the following computations we restrict the estimates to the two-point GJ predictor for the following three reasons. First, since there are three relevant stochastic variables in the two-point estimate case, and six variables in the three-point estimate case, we need to use binomial approximation techniques that are a generalization of Breen (1991). The calculations of the option values $C_{0,1}$, $C_{0,2}$ and $C_{0,3}$ are therefore made with errors.[14] However, the GJ estimation has the effect of magnifying these errors. It turns out that the two-point estimates are in this case more accurate than the three-point estimate.[15] Secondly, in the original GJ computations, the two-point estimates are, in fact, remarkably accurate, and we have no reason to believe that this would change with the addition of stochastic interest rates.[16] Finally,

the optimal number of options to be included in a GJ estimate clearly is a balance between computational efficiency and the accuracy of the estimate. Adding a second determining variable, in this case stochastic interest rates, increases the computational cost significantly. It is likely, therefore, that the balance will shift to the inclusion of fewer options in the series. For all these reasons, the simulations below use the two-point GJ method.

Therefore, having limited the number of relevant variables to three, i.e., $S_{\frac{T}{2}}$, S_T and $B_{\frac{T}{2},T}$, we approximate their joint distribution using a joint binomial distribution.[17] We choose the method developed by Ho, Stapleton and Subrahmanyam (1995). The required inputs are the forward prices from (27), the expected forward price from (13) and (14) and the volatilities. In order to construct the distribution with the correct volatilities we compute the variance of the logarithm of the forward price, given the spot-rate volatilities. This follows from spot-forward parity as follows:

$$\sigma^2_{F_{\frac{T}{2},T}} = \sigma^2_{S_{\frac{T}{2}}} + \sigma^2_{B_{\frac{T}{2},T}} - 2\sigma_{S_{\frac{T}{2}},B_{\frac{T}{2},T}}. \tag{34}$$

The volatility inputs for (17) are the exogenously given spot volatilities for the asset and the zero-coupon bond.

VII Simulations of the Generalized Geske and Johnson Valuation Model

We now illustrate the use of the extended Geske-Johnson technique and test the effect of stochastic interest rates on a range of American put prices reported previously by GJ for the case of non-stochastic interest rates.[18] We then introduce some examples of longer-maturity put options where the effect of stochastic interest rates is more important. In order to be able to compare directly with the results of GJ, we assume that the asset price S_t follows a geometric random walk with a constant volatility σ. Also, the asset pays no dividends and hence has a forward price for delivery at time t of $S_0/B_{0,t}$.

[Table I here]

In Table I,we show the effect of stochastic interest rates in the case of twelve put options valued by GJ and previously by Parkinson (1977). The options are all at-the-money American puts on a non-dividend-paying stock with a price $S_0 = 1$. Columns (a) to (g) are from GJ Table I (1984, p. 1519). Column (h) shows our binomial approximation, using a European option and an option with two exercise points. A comparison of the estimates in columns (f), (g) and (h) shows that these estimates are as close to the numerical method computation of Parkinson (1977) as the GJ estimates. The estimates of the stochastic-interest-rate American model are shown in column (i). These are estimated using the same method as for column (h) and are hence directly comparable. The effect of stochastic interest rates on the option values is generally small. However, it is significantly higher, in absolute as well as relative terms, in the cases where the volatility of the underlying asset is low.

The comparisons above with the GJ simulations give the impression that the effect of stochastic interest rate is of minor importance. However, this is partly because the options considered by GJ are all of short maturity, and are options on assets with relatively high volatility. In Table II we show the result of calculating the value of options that have two possible exercise dates: $T/2$ and T, for the long maturity ($T = 5$ years) options with varying volatility and depth-in-the-money. The results show that the absolute and the percentage effects of stochastic interest rates are significantly higher for options on low-volatility assets. In the case of $\sigma = 0.20$ the effect is swamped by the volatility of the underlying asset, as it is in many of the examples in Table I. When the asset volatility is low, on the other hand, the effect of stochastic interest rates is quite large. The effect also generally increases as the put option goes out of the money. For the low-volatility options ($\sigma = 0.05$) the effect is clearly higher for the out-of-the-money options. However, for the high-volatility options ($\sigma = 0.20$), the effect is highest for the at-the-money options.

[Table II here]

In all the calculations reported in Tables I and II, we assume that the correlation between the asset price and the zero-coupon bond price at time $T/2$ is $\rho = 0.3$. However, in the case of underlying assets that are sensitive to interest rates, the correlation may well be higher. For example,

in the case of bond options, we might expect this to be the case. In Table III we show the results of a simulation where the correlation ranges from -0.3 to $+0.6$. The effect of higher correlation on the value of the put option is generally positive, except in the case of low bond volatility where there is little discernable effect.

[Table III here]

In summary, the effect of stochastic interest rates on the values of American options is particularly noticeable for long-term options on assets with relatively low volatility and relatively high correlation with bond prices, particularly with high interest rates. The reason for this can be explained in intuitive terms by relating it to the cause of rational premature exercise. Early exercise of American put options is likely to occur when the time value of money on the strike price exceeds the insurance value of the option. This, in turn, happens when interest rates are high, when the volatility of the underlying asset is low, and when the asset and bond prices are both low.

VIII Concluding Comments

In this paper, we have established a valuation model for options exercisable on one of several exercise dates, under conditions of stochastic interest rates. The method used is essentially a generalization of Merton's (1973) model for European-style options. We have then applied the pricing model to estimate the price of the American-style contingent claim using the Geske and Johnson (1984) methodology. With European options and options exercisable on any one of two (and possibly three) dates, we can use Richardson extrapolation to estimate the American-claim price. Hence, our results lead to an extension of the computationally efficient GJ methodology to a stochastic-interest-rate environment.

The extension of the GJ methodology to the case of stochastic interest rates is potentially useful for solving a number of problems in option valuation. First, it could be used to value long-maturity options such as equity warrants where the stochastic nature of interest rates could be an important influence on the valuation even if the correlation between the interest

rate and the asset price is low. Second, the approach could improve the computational efficiency, both speed and accuracy, of methods for valuing American-style foreign-exchange options such as those suggested by Amin and Bodurtha (1995). Third, the approach could be used in the special case of bond options and swap options to provide more rapid calculations of option values and hedge ratios. Finally, although it may be possible to calculate option hedge ratios and other risk-management parameters using numerical methods, the GJ approach allows the analytic computation of these values. Our extension to the case of stochastic interest rates may allow more accurate hedge strategies to be evaluated.

In our simulations we have restricted consideration to American-style put options. The same method could be used to value American call options on dividend-paying stocks or other more complex options. Results reported here for American puts show significant effects of stochastic interest rates, which are particularly important when the underlying asset has low volatility, and when the options are out of the money.

Appendix

A. Proof of the Valuation Relationship for the Option with Two Possible Exercise Dates.

The exercise dates for this option are $T/2$ and T. Since S_T and $\phi_{\frac{T}{2},T}$ are joint lognormal, the live option value at $T/2$ is given by the Black-Scholes relationship

$$C_{2,\frac{T}{2}}(S_{\frac{T}{2}}, B_{\frac{T}{2},T}) = \hat{E}_{\frac{T}{2}}[g(S_T)]B_{\frac{T}{2},T}, \qquad (A1)$$

where the risk-neutral distribution is lognormal with a mean equal to the forward price, at time $T/2$, and variance equal to $\sigma^2_{S_T}$.

Moving back to time 0, the option has a value

$$C_{2,0} = E_0\left[\max[g(S_{\frac{T}{2}}), C_{2,\frac{T}{2}}(S_{\frac{T}{2}}, B_{\frac{T}{2},T})]\phi_{0,\frac{T}{2}}\right] B_{0,\frac{T}{2}}. \qquad (A2)$$

Since the option payoff is a deterministic function of the two state variables $S_{\frac{T}{2}}$ and $B_{\frac{T}{2},T}$, and since the triplet of variables $(S_{\frac{T}{2}}, B_{\frac{T}{2},T}, \phi_{0,\frac{T}{2}})$ are joint lognormally distributed, it follows directly from Stapleton and Subrahmanyam (1984) that

$$C_{2,0}(S_0, B_{0,\frac{T}{2}}, B_{0,T}) = \hat{E}_0\left[\max[g(S_{\frac{T}{2}}), C_{2,\frac{T}{2}}(S_{\frac{T}{2}}, B_{\frac{T}{2},T})]\right] B_{0,\frac{T}{2}}, \qquad (A3)$$

where the distribution $\hat{}$ is joint lognormal with the means of the variables given by their forward prices and log variances equal to the actual log variances. Hence, it follows that we can write $C_{2,0}$ as a function of the three time-0 variables $S_0, B_{0,\frac{T}{2}}$, and $B_{0,T}$.

□

B. Derivation of the Unconditional Expectation of the Asset Prices.

For the case where $n = 2$, we can write, using spot-forward parity, the no-arbitrage condition and the definition of covariance,

$$\hat{\mathrm{E}}_0[F_{\frac{T}{2},T}] = \frac{B_{0,\frac{T}{2}}}{B_{0,T}}\left[F_{0,\frac{T}{2}} - \hat{\mathrm{C}}\mathrm{ov}_0[F_{\frac{T}{2},T}, B_{\frac{T}{2},T}]\right], \qquad (A4)$$

where $\hat{\mathrm{C}}\mathrm{ov}[.]$ is the covariance under the risk-neutral distribution.

From the assumption of joint lognormality of $S_{\frac{T}{2}}$ and $B_{\frac{T}{2},T}$ and hence of $F_{\frac{T}{2},T}$ and $B_{\frac{T}{2},T}$, we can write

$$\hat{\mathrm{C}}\mathrm{ov}_0[F_{\frac{T}{2},T}, B_{\frac{T}{2},T}] = \hat{\mathrm{E}}_0[F_{\frac{T}{2},T}]\hat{\mathrm{E}}_0[B_{\frac{T}{2},T}][\exp\{\rho\sigma_{F_{\frac{T}{2},T}}\sigma_{B_{\frac{T}{2},T}}\} - 1], \qquad (A5)$$

where ρ is the coefficient of correlation between $S_{\frac{T}{2}}$ and $B_{\frac{T}{2},T}$, since the covariances of the variables are the same under the true and the risk neutral distribution. Substitution of $(A5)$ in $(A4)$ and spot-forward parity yields

$$\hat{\mathrm{E}}_0[F_{\frac{T}{2},T}] = F_{0,T}\exp\left\{-\left(\sigma_{S_{\frac{T}{2}},B_{\frac{T}{2},T}} - \sigma^2_{B_{\frac{T}{2},T}}\right)\right\}, \qquad (A6)$$

where σ^2_X and σ_{XY} are respectively the variance of $\ln X$ and the covariance of $\ln X$ and $\ln Y$.

□

Table I
American Put Option Prices: Stochastic and Non-Stochastic Interest Rates

[Notes to Table I here]

(a) r	(b) K	(c) σ	(d) T	(e) P_E	(f) P_{GJ}	(g) P_{PK}	(h) P_{NSR}	(i) P_{SR}	(j) %
0.125	1.0	0.5	1	0.1327	0.1476	0.148	0.1479	0.1485	0.4
0.080	1.0	0.4	1	0.1170	0.1258	0.126	0.1250	0.1258	0.6
0.045	1.0	0.3	1	0.0959	0.1005	0.101	0.0990	0.1006	1.6
0.020	1.0	0.2	1	0.0694	0.0712	0.071	0.0705	0.0716	1.6
0.005	1.0	0.1	1	0.0373	0.0377	0.038	0.0373	0.0395	5.6
0.090	1.0	0.3	1	0.0761	0.0859	0.086	0.0883	0.0889	0.7
0.040	1.0	0.2	1	0.0600	0.0640	0.064	0.0644	0.0660	2.5
0.010	1.0	0.1	1	0.0349	0.0357	0.036	0.0357	0.0379	6.2
0.080	1.0	0.2	1	0.0442	0.0525	0.053	0.0532	0.0538	1.1
0.020	1.0	0.1	1	0.0304	0.0322	0.033	0.0326	0.0346	6.1
0.120	1.0	0.2	1	0.0317	0.0439	0.044	0.0455	0.0457	0.4
0.030	1.0	0.1	1	0.0263	0.0292	0.030	0.0295	0.0311	5.4

Table II
The Effect of Stochastic Interest Rates on the Prices of Long-Maturity Put Options with Two Possible Exercise Dates

[Notes to Table II here]

(a) Asset Price	(b) σ	(c) r	(d) T	(e) K	(f) P_{NSR}	(g) P_{SR}	(h) % change
1.00	0.05	0.03	5	1	0.0081	0.0088	8.6
1.00	0.05	0.06	5	1	0.0007	0.0008	14.3
1.00	0.10	0.03	5	1	0.0392	0.0410	4.6
1.00	0.10	0.06	5	1	0.0155	0.0156	0.6
1.00	0.20	0.03	5	1	0.1141	0.1169	2.5
1.00	0.20	0.06	5	1	0.0723	0.0744	2.9
1.05	0.05	0.03	5	1	0.0025	0.0029	13.0
1.05	0.05	0.06	5	1	0.0001	0.0001	18.2
1.05	0.10	0.03	5	1	0.0264	0.0275	4.2
1.05	0.10	0.06	5	1	0.0091	0.0095	4.4
1.05	0.20	0.03	5	1	0.0999	0.1017	1.8
1.05	0.20	0.06	5	1	0.0626	0.0631	0.8
0.95	0.05	0.03	5	1	0.0213	0.0224	5.2
0.95	0.05	0.06	5	1	0.0035	0.0037	7.2
0.95	0.10	0.03	5	1	0.0567	0.0586	3.4
0.95	0.10	0.06	5	1	0.0253	0.0262	3.6
0.95	0.20	0.03	5	1	0.1309	0.1335	2.0
0.95	0.20	0.06	5	1	0.0887	0.0897	1.1

Table III
The Effect of the Correlation between the Asset Price and the Zero-Coupon Bond Price on the Value of an Option

[Notes to Table III here]

Volatility of zero-coupon bond	Coefficient of correlation			
	−0.3	0	0.3	0.6
0.05	0.0299	0.0307	0.0313	0.0314
0.03	0.0290	0.0293	0.0295	0.0294
0.01	0.0281	0.0281	0.0281	0.0281

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Footnotes

* The first two authors are from The Management School, Lancaster University. The third author is from Leonard N. Stern School of Business, New York University. Earlier versions of this paper have been presented at the European Institute for Advanced Studies in Management, and at the European Finance Association. The authors thank Jing-zhi Huang and John Chang for able research assistance.

1. Various analytical methods have been suggested by Geske and Johnson (1984), Barone-Adesi and Whaley (1987), and others.

2. This technique, developed under non-stochastic interest rates, has since been refined by Omberg (1987) and Bunch and Johnson (1992) and used in a binomial context by Breen (1991).

3. Some recent work on foreign-exchange options under stochastic interest rates is reported by Amin and Bodurtha (1995).

4. Although their model does not deal with interest-rate uncertainty, GJ note the potential importance of the term structure of interest rates in the case of American options. They point out "if one were to introduce uncertainty about future interest rates, then term structure effects could be important. ... the duplicating portfolio for out-of-the-money puts is skewed toward longer maturity bonds, while for in-the-money puts it is skewed toward shorter maturities".

5. We assume that asset prices and zero-coupon bond prices are joint normally distributed. Our assumptions are similar to those used by Jamshidian (1991) in the context of bond options, except that we are able to generalize the covariance structure. A well known drawback of these assumptions is that interest rates are Gaussian, and hence, can become negative. The approach could, however, be adapted to the case of lognormally distributed interest rates to avoid this problem.

6. If the underlying asset pays a non-stochastic dividend, it would be simple, in principle, to modify the analysis that follows by changing the mean of the distribution of the underlying asset price appropriately, i.e., by using spot-forward parity for dividend-paying assets.

7. See for example Cox and Ross (1976) and Harrison and Kreps (1979).

8. Sufficient conditions for the pricing kernels to be lognormally distributed are either that the asset price follows a continuous diffusion process with stationary parameters, or that there is a representative-investor economy in which the investor has constant-proportional-risk-aversion preferences. See, for example, Bick (1987).

9. The equations determining $C_{3,0}$, $C_{4,0}$, ... can be written down in a similar manner. The only difference is that we need to compute the option values and bond prices at the intermediate dates.

10. In fact, in the limit as $n \to \infty$ the unconditional mean is the futures price. This equals the forward price if asset prices and the zero bond prices are uncorrelated, or if interest rates are non-stochastic.

11. Note that the variances under the risk-neutral and the true distribution are the same, given lognormality. See, for example, Brennan (1979).

12. Note that σ_X^2 and σ_{XY} are not annualized and hence already include the time to maturity.

13. Our procedure is similar to the technique used by Heath, Jarrow and Morton (1992) in the case of bond and interest-rate options.

14. The errors reduce as the grid size in the binomial approximation increases. However, given feasible node numbers, significant errors remain.

15. The three-point GJ estimate is

$$\hat{C}_3 = C_{3,0} + \frac{7}{2}(C_{3,0} - C_{2,0}) - \frac{1}{2}(C_{2,0} - C_{1,0}).$$

Suppose that $C_{2,0}$ is estimated with error ϵ_2 and $C_{3,0}$ with error ϵ_3. Then the error in $\hat{C}_3$ is

$$\epsilon_3 + \frac{7}{2}\epsilon_3 - \frac{7}{2}\epsilon_2 - \frac{1}{2}\epsilon_2 = \frac{9}{2}\epsilon_3 - 4\epsilon_2.$$

In the two-point GJ estimate

$$\hat{C}_2 = C_{2,0} + C_{2,0} - C_{1,0},$$

and the error in $\hat{C}_2$ is

$$2(\epsilon_2).$$

Unless the errors ϵ_3 and ϵ_2 are correlated, then the error in $\hat{C}_3$ is likely to exceed the error in $\hat{C}_2$. In simulations carried out by the authors using $C_{3,0}$, errors (compared to Cox-Ross-Rubinstein (1979) option values) are significantly larger for $\hat{C}_3$ than for $\hat{C}_2$.

16. See Ho, Stapleton and Subrahmanyam (1994) for a demonstration of the accuracy of the two-point GJ estimator.

17. As mentioned above, this method is extendable to the estimate of $C_{3,0}$.

18. Since this example has been studied by other researchers, we can relate directly to previous results in the literature reported by Parkinson (1977) and Geske and Johnson (1984).

Notes to Table I

Table I shows the difference between American put option prices with and without stochastic interest rates. The first seven columns are from Geske and Johnson's Table I (1984, p. 1519). Columns (a) to (d) represent the parameter input for r, the continuously compounded risk-free rate, K, the option strike price, σ, the volatility of the underlying asset, and T, the time to expiration. The stock price in all cases is \$1. The remaining columns refer to put option prices in dollars. Column (e) shows the European put option values, P_E. Column (f) shows the GJ American put option values, P_{GJ}. Column (g) indicates the American put option values computed by the Parkinson numerical method, P_{PK}. Column (h) reports the results of our modified GJ approximation using the multivariate binomial distribution approach of Ho, Stapleton and Subrahmanyam (1995), assuming interest rates are non-stochastic, P_{NSR}. Column (i) shows the results of our American put option prices, P_{SR}, which incorporate stochastic interest rates, where the volatilities of the bonds are 2 percent for bonds with a maturity of $\frac{1}{2}$ year and the coefficient of correlation between the (log) asset price and the (log) zero-coupon bond price is 0.3. All prices in columns (h) and (i) are computed using binomial distributions with twenty stages. Column (j) shows the percentage increase in price due to stochastic interest rates.

Notes to Table II

Table II shows the difference between a put option with two exercise dates $T/2$ and T valued with and without stochastic interest rates. Column (a) shows the asset price. Column (b) is the volatility of the underlying asset price, column (c) is the continuously compounded interest rate, column (d) is the time to maturity of the option, and column (e) is the strike price. The first six options in the table are at-the-money puts. The next six options are out-of-the-money puts where $K < S_0$. The final six options are in-the-money puts where $K > S_0$. Column (f) shows the value of the put options computed using the binomial approximation method of Ho, Stapleton and Subrahmanyam (1995) with the number of binomial stages n_1 and n_2 equal to 12, assuming non-stochastic interest rates. Column (g) shows the value assuming stochastic interest rates with a volatility of the zero-coupon bond of 3% and coefficient of correlation between the (log) asset price and the (log) bond price of 0.3. Column (h) shows the percentage change in

the option price when the effect of stochastic interest rates is included in the calculation. The values in columns (f) and (g) are rounded to four decimal places whereas the percentage change in column (h) is based on the unrounded values.

Notes to Table III

The option price C_2 is computed using a binomial approximation with twenty time steps. The option is a put at a strike price $K = 1$ on an asset whose current price is $S_0 = 1$. Volatility is 10% and the risk-free rate is 3%. The maturity of the option is $T = 1$ year and the option is exerciseable at either time $T/2$ or at time T.