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A Two-Factor No-Arbitrage Model of the Term Structure of Interest Rates [1]

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Abstract

We derive a no-arbitrage model of the term structure in which any two futures rates act as factors. The term structure shifts and tilts as the factor rates vary. The cross-sectional properties of the model derive from the solution of a two-dimensional ARMA process for the short rate which exhibits mean reversion and a lagged memory parameter. We show that the correlation of the factor rates is restricted by the no-arbitrage conditions of the model. Hence in a multiple-factor model it is not valid to independently choose both the mean reversion, volatility and correlation parameters. The term-structure model, derived here, can be used to value options on bonds and swaps or to generate term structure scenarios for the risk management of portfolios of interest-rate derivatives.

1 Introduction

The term structure of nominal interest rates exhibits several patterns of changes over time. In some periods it shifts up or down, perhaps in response to higher expectations of future inflation. In other periods, it tilts, with short rates rising and long rates falling, perhaps in response to a tightening of monetary policy. Sometimes its shape changes to an appreciable extent. Models of the term structure are of interest to practitioners and financial academics alike, both to price interest-rate sensitive derivative contracts, and to measure and manage the interest-rate risk arising from portfolios of these contracts. A desirable feature of these models is that they should capture at least the shifts and tilts of the term structure.

One early, intuitively appealing two-factor model which captured the above features of the empirical term structure was the long rate-spread model of Brennan and Schwartz (1979). Although this model has the attractive feature of modelling term structure movements in terms of two key rates, it is not presented in the "no-arbitrage" setting first proposed by Ho and Lee (1986). Today, it is recognised that a highly desirable, if not a necessary condition for a model to satisfy is the no-arbitrage condition. In this paper, we develop a model that is consistent with the principle of no arbitrage and which yields a two-factor model similar to that of Brennan and Schwartz.

Fundamentally, the no-arbitrage condition, when applied to the term structure requires the price of a long bond to be related to the expected value (under the 'risk-neutral' probability density) of the future relevant short bond prices. This requirement links the cross-sectional properties of the term structure at each point in time to the time series properties of bond prices and interest rates. This point is discussed extensively in a one-factor Gaussian interest model context by Backus and Zin (1993). In the context of our two-factor model we are able to show that if the short rate follows a mean-reverting two-dimensional process (a process generated by two state variables) then the no-arbitrage condition implies a short rate-long rate model of the term structure not dissimilar to that of Brennan and Schwartz. Also, in this model the correlation between the long and short rates is restricted by the degree of mean reversion of the short rate and the relative volatilities of the long and short rates.

We suggest a time series model in which the conditional mean of the short rate follows an ARMA (autoregressive, moving average) process. The short rate itself is ARMA plus an independent white noise term. This assumption allows us to nest the popular AR(1) single-factor model as a special case. It is also general enough to produce stochastic no-arbitrage term structures with shapes that capture most of those observed. A similar model in which the conditional mean of the short rate is stochastic has been suggested by Balduzzi, Das and Foresi (1995).

Recent literature, mainly inspired by the practical need to price various interest rate derivative contracts, has produced a bewildering variety of term structure models. In section 2 of this paper we discuss this literature, relate our model to previously proposed models and attempt to assess the incremental contribution of our work. One of the most difficult aspects of term structure modelling is notation and definition of the relevant variables and parameters. For this reason we devote section 3 to a description of the set up of the problem, the variables and our notation. We also, in this section, specify the ARMA process which the conditional mean of the short is assumed to follow.

2 Term Structure Models : The Literature

One fundamental decision that has to be made in term structure modelling is the choice of the assumption about the distributional properties of interest rates (and hence bond prices). One classification of the literature is according to whether interest rates are normally distributed or lognormally distributed and whether they evolve in discrete time or continuously. Gaussian interest rate models of the type first derived by Vasicek (1977) have been developed extensively by Jamshidian (1989), Hull and White (1993), Turnbull and Milne (1990) and applied to the valuation of a variety of interest rate and bond options. Also, the no-arbitrage models of Ho and Lee (1986) and Heath,Jarrow and Morton (1990a,1990b) (HJM) are discrete time additive binomial models whose interest rates limit to normally distributed variables. An objection that has often been raised against this whole class of models is that they allow nominal rates to be negative, with positive probability. However, perhaps from a practical point of view, a more important drawback is that interest rates have higher variance when they are high than when they are low. Empirical evidence provided by Chan et al (1994) rejects the assumptions of this class of models in favour of the alternative assumption that variance is level dependent.

In this paper we propose a model in which the rate of interest is lognormally distributed. This assumption has the advantage that the variance is dependent on the level of the rate. Thus rates are skewed to the right in our model. In practice many traders use the Black (1976) model to price interest rate caps, a model that also assumes lognormal interest rates. Also as discrete approximations, the Black, Derman and Toy (1990)(BDT) and Black and Kirinski (1990) models have similar assumptions. Our incremental contribution to this literature is that we provide a particularly simple two factor extension of the BDT model. We also provide a set of sufficient conditions for the cross-sectional two-factor model to hold in a no-arbitrage setting.

Another categorization of models in the literature is that between equilibrium models and no-arbitrage models of the term structure. The former include Cox, Ingersoll and Ross (1985) and the extension to a two-factor model with stochastic volatility by Longstaff and Schwartz (1992). In contrast there are the no-arbitrage models of Ho and Lee (1986), BDT, HJM and many others. Our model is in the no-arbitrage model category. The addition to the literature in this case is that we show that the no-arbitrage condition restricts the correlation of the factor interest rates in a multi-factor model. Closely connected to the no-arbitrage models, in fact a sub-category, are recent theories based on the pricing kernel. Constantinides (1992) assumes a process for the kernel and derives a single-factor model of the term structure. Backus and Zin (1993) develop a model in a Gaussian one-factor framework and show the relationship between the time series process of the pricing kernel, the process for the short interest rate and the term structure. Backus and Zin use a discrete time ARMA model of the pricing kernel. In this paper we directly model the 'risk-neutral' density of the short rate, rather than the pricing kernel. Hence our approach is somewhat different from theirs. However, in one aspect we extend their approach by using a vector ARMA process which leads, given no-arbitrage, to a two-factor characterization of the term structure.

A multi-factor model for the term structure has been proposed recently by Duffie and Kan (1994). Duffie and Kan analyse a class of 'affine' or linear models, assuming a vector process for the yields on zero-coupon bonds. Their non-stochastic volatility example reduces to a multi-variable Gaussian model in which any two rates can be interpreted as factors. In our model we derive a somewhat similar result. In our case any two forward rates can be employed as factors. However, our no-arbitrage model restricts the correlation of these chosen factor rates. Lastly, in Longstaff and Schwartz (1992) a two-factor model is derived in which the volatility of interest rates is the second factor. This model is capable of explaining the term risk premium. In this sense it is similar in spirit to the model proposed here. However, in our model two factors explain the term structure even when either the local expectations hypothesis holds or when volatility is non-stochastic. If stochastic volatility is an important explanatory variable, it may act in addition to our two factors. It could therefore be added as a third factor in a possible extension. In our model the term structure shifts and tilts perhaps in response to expectations of future real interest rates and inflation rates. It does so even in a risk neutral world. Hence our incremental contribution is to derive a different set of conditions for a two-factor model to those of Longstaff and Schwartz.

3 Assumptions, Notation, and the Spot Rate Process

As in HJM we denote $P(t,T)$ as the time t price of a zero-coupon bond paying \$1 with certainty at time T. Today is time 0 and t and T are measured in years. At time t the price of an m year zero coupon bond is revealed. This price is $P(t,t+m)$. We define the spot interest rate for m year money at time t by the linear relationship

$$i_t = [1 - P(t,t+m)]/m \tag{1}$$

Note that the interest rate is defined on a bankers' discount (or T Bill basis). This has considerable analytical advantages over the conventional definition where the rate is defined on a continuously compounded basis. The other difference between the spot rate defined by (1) and the HJM interest rate is that m is not necessarily a very short (instantaneous) period. However, as in HJM, m does not vary.

The futures interest rate at time t for delivery at time T is denoted $F(t,T)$. Again, the rate is defined on a bankers' discount basis. Hence, in relation to the futures price of an m-maturity bond, at t for delivery at T, $P(t,T,T+m)$

$$F(t,T) = [1 - P(t,T,T+m)]/m \tag{2}$$

we now denote the logarithm of the futures rate as

$$f(t,T) = \ln[F(t,T)] \tag{3}$$

Note that under this notation, which is broadly consistent with HJM, $F(t,t) = i_t$ and $f(t,t) = \ln(i_t)$.

In Table 1 we summarize the notation used in the paper. The mean and annualized standard deviation of the (logarithm) of the spot rate are denoted

Table 1
Notation for the Mean and Volatility of Spot and Futures Rates

	(1)		(2)		(3)	
Time Period	0		t		T	
Spot prices and interest rates for m-year money	$\mu(0,t,t)$	Unconditional logarithmic mean of i_t	$P(t,t+m)$	Zero bond price at t for delivery of \$1 at $(t+m)$	$P(T,T+m)$	Zero bond price at time T for delivery of \$1 at time $T+m$
	$\sigma(0,t,t)$	Unconditional (annualised) volatility of i_t	$i_t = F(t,t)$	m–year interest rate at time t	$i_T = F(T,T)$	m–year interest rate at time T
Futures interest rates for bonds maturing at time $T+m$	$\mu(0,t,T)$	Mean of $f(t,T)$				
	$\sigma(0,t,T)$	Unconditional (annualised) volatility of $f(t,T)$	$F(t,T)$	futures interest rate at t for delivery at T (m-year money)		
			$f(t,T)$	Logarithm of $F(t,T)$		
			$\mu(t,T,T)$	Conditional mean of $f(T,T)$		
			$\sigma(t,T,T)$	Conditional (annualised) volatility of $f(T,T)$		

In Table 1, above, m is a constant. In the simulations of the model, $m = 91/365$. Hence, in this case, i_t can be representative of the three–month LIBOR rate. All interest rates used are on an annualised basis. Note: all means and volatilities are under the risk neutral measure .

$$\mu(t, T, T) = \mathrm{E}_t[f(T, T)] \tag{4}$$

$$\sigma(t, T, T) = [\mathrm{var}_t[f(T, T)]/(T - t)]^{\frac{1}{2}} \tag{5}$$

respectively. Also in the case of futures rates, we define

$$\mu(0, t, T) = \mathrm{E}_0[f(t, T)] \tag{6}$$

$$\sigma(0, t, T) = [\mathrm{var}_0[f(t, T)]/t]^{\frac{1}{2}} \tag{7}$$

Note that the mean and variance of the spot rate in (5) are statistics of a time T measurable random variable. In (7) the statistics relate to a time t measurable random variable.

Our main assumptions are as follows:-

1. Prices of zero-coupon bonds are determined in a no-arbitrage economy.
2. The process for the short rate is lognormal under the equivalent martingale measure.
3. The logarithm of the short rate follows a two-dimensional process with the characteristics:-
   (a) the mean of $f(t, t)$ follows an autoregressive moving average (ARMA) process
   (b) innovations in $f(t, t)$ are mean reverting, i.e., they follow an autoregressive (AR) process.

These assumptions imply first that an equivalent martingale measure exists for the pricing of zero-coupon bonds. Second, the process for the logarithm of the short rate under this measure is mean reverting and of the form

$$f(t,t) = \mu(0,t,t) + [f(t-1,t-1) - \mu(0,t-1,t-1)](1-c) + \sum_{\tau=1}^{t-1} \nu_{t-\tau}\alpha^{\tau-1} + \varepsilon_t \quad (8)$$

where time is measured in periods of length n years. In (8), c is the rate of mean reversion per period, ν_t and ϵ_t are mutually and intertemporally independent, normally distributed variables.

Equation (8) assumes a spot rate process which is essentially an extension of the Vasicek (1977) process. In its simplest form with $\nu_{t-\tau} \equiv 0$ the spot rate follows the process

$$f(t,t) = \mu(0,t,t) + [f(t-1,t-1) - \mu(0,t-1,t-1)](1-c) + \epsilon_t \quad (9)$$

Here the logarithm of the spot rate is a mean reverting process with a mean reversion coeficient of c per period. The process is 'calibrated' to current expectations of future rates, $\mu(0,t,t)$. The process in equation (9) is not complex enough, however, to mirror actual movements of the term structure. We need to capture changes in expected spot rates that are unrelated to current realisations of the spot rate itself. This is achieved by adding a second dimension to the process. Hence we assume that the short rate follows the two-dimensional $ARMA$ process in equation (8). Equation (8) allows for an independent shift in the conditional expectation of $f(t,t)$. For example, we have with α=0

$$\mu(t-1,t,t) = \mu(0,t,t) + [f(t-1,t-1) - \mu(0,t-1,t-1)](1-c) + \nu_{t-1} \quad (10)$$

Hence, the conditional expectation of the time t spot rate depends on two time $t-1$ measurable stochastic variables, ε_{t-1} which determines $f(t-1,t-1)$ and ν_{t-1} which further shifts the expectation of $f(t,t)$. In general, the parameter α makes the conditional expectation an $ARMA(1, t-1)$ process. If $\alpha > 0$ the effect of a shock to expectations persists to later spot rates. $1-\alpha$ measures the degree of decay in expectations. For example, with $\alpha = 1$, there is no decay at all. In this case the conditional expectation of $f(t,t)$ is affected equally by all realisations of ν between time 0 and time t. Given that the short rate follows the process in equation (8) the expectations of the spot rate $f(t+k, t+k)$ at time t can be found by successive substitution. We find

Lemma 1 *If the logarithm of the spot rate follows the ARMA process*

$$f(t,t)-\mu(0,t,t) = [f(t-1,t-1)-\mu(0,t-1,t-1)](1-c)+\sum_{\tau=1}^{t-1} \nu_{t-\tau}\alpha^{\tau-1}+\varepsilon_t, \quad \forall t, \tag{11}$$

then the expectation of the logarithm of the interest rate i_{t+k} at time t is

$$\begin{aligned} \mu(t, t+k, t+k) &= \mu(0, t+k, t+k) + [f(t,t) - \mu(0,t,t)](1-c)^k \\ &\quad + \sum_{\tau=0}^{t-1} \nu_{t-\tau}\alpha^{\tau} \sum_{\tau=1}^{k} (1-c)^{k-\tau}\alpha^{\tau-1} \end{aligned} \tag{12}$$

Proof. Substitute successively for $f(1,1)$, $f(2,2)$,...,$f(t+k, t+k)$ and take the conditional expectation $\mathrm{E}_t[f(t+k, t+k)]$.

■

To appreciate the meaning of Lemma 1 we will look at various limiting cases. First, if $\alpha = 0, c = 0$, the process for the short rate is a two-dimensional random walk. The expectation at t of $f(t+k, t+k)$ is in this case

$$\mu(t, t+k, t+k) = \mu(0, t+k, t+k) + f(t,t) - \mu(0,t,t) + \nu_t \tag{13}$$

The expectation in (13) is affected both by the degree to which $f(t,t)$ exceeded its expected value, $\mu(0,t,t)$ and by the independent shift factor ν_t. Note that in this case, the shift in the expectation is the same for each k. If $c = 0$, we have

$$\begin{aligned} \mu(t,t+k,t+k) &= \mu(0,t+k,t+k) + f(t,t) - \mu(0,t,t) \\ &\quad + \sum_{\tau=0}^{t-1} \nu_{t-\tau}\alpha^{\tau}\frac{(1-\alpha^k)}{(1-\alpha)} \end{aligned} \tag{14}$$

and with $\alpha = 1$

$$\mu(t,t+k,t+k) = \mu(0,t+k,t+k) + f(t,t) - \mu(0,t,t) + \sum_{\tau=0}^{t-1} \nu_{t-\tau}k \tag{15}$$

In this case each of the shift factors affects the expectation. Also, the shift in the expectation depends on k. Also, if $c > 0, \alpha = 0$, we find

$$\mu(t,t+k,t+k) = \mu(0,t+k,t+k) + [f(t,t) - \mu(0,t,t)](1-c)^k + \nu_t(1-c)^{k-1} \tag{16}$$

Finally, with $c > 0, \alpha = 1$

$$\begin{aligned} \mu(t,t+k,t+k) &= \mu(0,t+k,t+k) + [f(t,t) - \mu(0,t,t)](1-c)^k \\ &\quad + \sum_{\tau=0}^{t-1} \nu_{t-\tau}\left[\frac{1-(1-c)^k}{c}\right] \end{aligned} \tag{17}$$

These special cases of the ARMA process are summarized in Table 2.

4 Futures Rates in a No-Arbitrage Economy

We first establish an expression for the futures rate, $F(t, t+k)$. Given the no-arbitrage condition for bond prices and the fact that the interest rate, i_t, is linear in the bond price, we have the following result:

Lemma 2 *In a no-arbitrage economy, if the spot interest rate, defined on a 'banker's discount' basis, is lognormally distributed under the martingale measure, the k period futures rate at time t for an m–year loan is*

$$f(t, t+k) = \mu(t, t+k, t+k) + \frac{kn}{2}\sigma^2(t, t+k, t+k) \tag{18}$$

where n is the length, in years, of the period t to t + 1

Proof. From the no-arbitrage condition, the futures price is equal to the expectation under the equivalent martingale measure,

$$P(t, t+kn, t+kn+m) = \mathrm{E}_t[P(t+kn, t+kn+m)]$$

Hence, using the definition of the interest rates i_{t+k} , the futures price is given by

$$\mathrm{E}_t(1 - mi_{t+k}) = 1 - m\mathrm{E}_t(i_{t+k}), \tag{19}$$

It then follows immediately from the definition of the futures rate that

$$F(t, t+k) = \mathrm{E}_t(i_{t+k}) \tag{20}$$

Since by assumption, i_{t+k} is lognormal, under the martingale measure, with a conditional logarithmic mean and annualised volatility, of $\mu(t, t+k, t+k)$ and $\sigma(t, t+k, t+k)$, we have

$$\begin{aligned} \mathrm{E}_t(i_{t+k}) &= \exp\left(\mu(t, t+k, t+k) + \frac{kn}{2}\sigma^2(t, t+k, t+k)\right) \\ &= F(t, t+k) = \exp f(t, t+k) \end{aligned}$$

■

Lemma 2 states that lognormality of the futures rate follows from lognormality of the spot rate. This is because the conditional logarithmic mean of the spot rate, $\mu(t, t+k, t+k)$ is normal and because the conditional variance, $\sigma(t, t+k, t+k)$ of the spot rate is a constant. Lemma 2 also restricts the correlation of the spot and the futures rates. Combining the results of Lemmas 1 and 2 we can write the logarithm of the kth futures rate as

$$\begin{aligned} f(t, t+k) &= \mu(0, t+k, t+k) + [f(t,t) - \mu(0,t,t)](1-c)^k \\ &\quad + \sum_{\tau=0}^{t-1} \nu_{t-\tau}\alpha^\tau \sum_{\tau=1}^{k} (1-c)^{k-\tau}\alpha^{\tau-1} \\ &\quad + \frac{kn}{2}\sigma^2(t, t+k, t+k) \end{aligned} \tag{21}$$

The conditional variance of the futures rate is

$$\begin{aligned} \sigma^2(t-1, t, t+k)/n &= (1-c)^{2k}\mathrm{var}_{t-1}[f(t,t)] \\ &\quad + \left[\sum_{\tau=1}^{k}(1-c)^{k-\tau}\alpha^{\tau-1}\right]^2 \mathrm{var}_{t-1}(\nu_t) \end{aligned} \tag{22}$$

Also, since the variance of the spot rate is

$$\sigma^2(t-1, t, t)/n = \mathrm{var}_{t-1}[f(t,t)] \tag{23}$$

it follows that the covariance of the spot and the k th futures rate is

$$\begin{aligned}\text{cov}_{t-1}[f(t,t), f(t,t+k)] &= (1-c)^k \text{var}_{t-1}[f(t,t)] \\ &= (1-c)^k \sigma^2(t-1,t,t)/n \end{aligned} \tag{24}$$

and the correlation of the spot and futures rates is therefore

$$\rho(t-1,t,t+k) = \frac{(1-c)^k \sigma(t-1,t,t)}{\sigma(t-1,t,t+k)} \tag{25}$$

This expression for the correlation of the short rate and the kth futures rate illustrates an important implication of the no-arbitrage model. Given the volatilities of the spot and futures rates, we are not able to independently choose both the correlation and the degree of mean reversion. The no-arbitrage model restricts the correlation between the two factors to be a function of the degree of mean reversion of the short rate.

We can now establish an important property of the kth futures rate that allows us to solve for the cross-sectional term structure of interest rates. We have:

Lemma 3 *Given the conditions of Lemma 2, the logarithmic mean of the* kth *futures rate is related to the conditional logarithmic mean of the spot interest rate at* $t+k$ *by*

$$\mu(0,t,t+k) = \mu(0,t+k,t+k) + \frac{kn}{2}\sigma^2(t,t+k,t+k) \tag{26}$$

The lemma relates the means of the futures and corresponding spot rates. The extra term reflects the fact that from Lemma 2 the futures rate itself is lognormal with volatility $\sigma(0,t,t+k)$.

Proof. See Appendix 1.

We can now derive the main result of the paper. This is a two-factor cross-sectional relationship between interest rates at time t. The following proposition follows from Lemmas 1, 2, and 3. We show now that the *ARMA* time-series process assumed in the statement of Lemma 1 is necessary and sufficient to generate a two-factor term structure.

Proposition 1 *In a no-arbitrage economy in which the short rate of interest follows a lognormal process of the form*

$$f(t,t) = \mu(0,t,t) + [f(t-1,t-1) - \mu(0,t-1,t-1)](1-c) + \sum_{\tau=0}^{t-1} \nu_{t-\tau}\alpha^{\tau} + \epsilon_t$$

the term structure of futures rates at time t is generated by a two-factor model. The kth futures rate is given by

$$\begin{aligned} f(t,t+k) &= \mu(0,t,t+k) + a_k[f(t,t) - \mu(0,t,t)] \\ &\quad + b_k[f(t,t+1) - \mu(0,t,t+1)] \end{aligned} \tag{27}$$

where

$$b_k = [(1-c)^{k-1} + \ldots + \alpha^{k-1}]$$

and

$$a_k = (1-c)^k - (1-c)b_k.$$

Also, a short rate process in the form of (9) is necessary for the two-factor model in equation (27).

Proof. **Sufficiency** Solving the model by successive substitutions and taking the conditional expectation yields

$$\mu(t, t+k, t+k) = \mu(0, t+k, t+k) + [f(t,t) - \mu(0,t,t)](1-c)^k + V_t \sum_{\tau=1}^{k} (1-c)^{k-\tau} \alpha^{\tau-1}$$

where

$$V_t = \sum_{\tau=0}^{t-1} \nu_{t-\tau} \alpha^{\tau}$$

Substituting a similar expression for $\mu(t, t+1, t+1)$ and using Lemmas 2 and 3 yields the two–factor model (27).

Necessity See appendix 3.

Proposition 1 relates the kth futures rate to the spot rate $f(t,t)$ and the first futures rate, $f(t+1, t+1)$. If $m = 91/365$, for example, this means that the kth three–month futures rate is related to the spot three–month rate and the one period futures, three–month rate. In a recent contribution, Duffie and Kan (1993) have pointed out that if the model is linear in two such rates, it can always be expressed in terms of any two forward rates. In our context, it may be more practical to express the kth futures rate as a function of the spot rate and the nth futures rate. Hence, we derive the following implication of Proposition 1:

Corollary 1 *Suppose we choose any two futures rates as factors, where N_1 and N_2 are the maturities of the factors then the following linear model holds:*

$$f(t, t+k) = \mu(0, t, t+k) + A_k(N_1, N_2)[f(t, t+N_1) - \mu(0, t, t+N_1)] + B_k(N_1, N_2)[f(t, t+N_2) - \mu(0, t, t+N_2)] \quad (28)$$

where

$$B_k(N_1, N_2) = (a_k b_{N_1} - b_k a_{N_1})/(a_{N_2} b_{N_1} - b_{N_2} a_{N_1}),$$

$$A_k(N_1, N_2) = (-a_k b_{N_1} + b_k a_{N_1})/(a_{N_2} b_{N_1} - b_{N_2} a_{N_1}),$$

and

$$b_k = [(1-c)^{k-1} + \ldots + \alpha^{k-1}],$$

and

$$a_k = (1-c)^k - (1-c)b_k.$$

Corollary 1 follows by solving equation (27) for $k = N_1$, and $k = N_2$ and then substituting back into equation (27).

Corollary 2 The Random Walk Case

Suppose that $c = 0$ and the logarithm of the interest rate follows a random walk. In this case, the kth futures is

$$f(t,t+k) = \mu(0,t,t+k) + \left(\frac{N-k}{N}\right)[f(t,t) - \mu(0,t,t)] + \left(\frac{k}{N}\right)[f(t,t+N) - \mu(0,t,t+N)]. \tag{29}$$

Proof

Corollary 2 follows directly from Corollary 1 with

$$b_{k,N} = \frac{k}{N},$$

and hence,

$$a_{k,n} = \frac{N-k}{N}.$$

Here, the kth futures is affected by changes in the Nth futures according to how close k is to N. Equation (29) is a simple two–factor 'duration' type model.

Corollary 3 The Stochastic Process for the Futures Rates

Given that the spot rate follows the process assumed in Proposition 1 (sufficiency) then the kth futures rate follows the process

$$\begin{aligned} f(t,t+k) - \mu(0,t,t+k) &= (1-c)[f(t-1,t+k-1) - \mu(0,t-1,t+k-1)] \\ &\quad -(1-c)V_{t-1}[K] + (V_{t-1}+\varepsilon_t)(1-c)^k + V_t[K] \end{aligned}$$

where

$$\begin{aligned} V_t &= \sum_{\tau=0}^{t-1} \nu_{t-\tau}\alpha^{\tau} \\ K &= \sum_{\tau=1}^{k} (1-c)^{k-\tau}\alpha^{\tau-1} \end{aligned}$$

Proof See Appendix 4.

5 Forward rates and zero-coupon bond yields

We have derived the distribution of futures prices and futures rates, at time t, using an assumption about the process for short rates and the no-arbitrage condition. The term structure of futures rates at a point in time is closely related to the term structure of forward rates. The latter are required to completely describe the yields on zero-coupon bonds and to value an arbitrary set of cash flows at time t. The general relationship between futures prices and forward prices is well known from Cox, Ingersoll, and Ross (1981)(CIR). The CIR result states that the difference between the forward and the futures price of an asset, depends on the covariance, under the equivalent martingale measure, of the asset futures price and the money market accumulation factor. We can apply the CIR result to find an equivalent relationship, in our model, between the forward and futures rates of interest. in the case of the model of futures rates here, we have:

Proposition 2 *The forward rate at time t for delivery at $t+k$ of an n-year zero-coupon bond is*

$$G(t, t+k) = F(t, t+k) - cov[F(t, t+k), \psi], \tag{30}$$

where

$$\psi = B_{0,1}B_{1,2}...B_{t-1,t}/B_{0,t}$$

and where cov refers to the covariance of the variables under the martingale measure.

Proof. Applying the general result in CIR we have the time $t+k$ forward price of the zero-coupon bond for delivery at time t

$$P'(t, t+k, t+k+m) = P(t, t+k, t+k+m) + cov[P(t, t+k, t+k+m), \psi].$$

Now, defining the forward rate by the relation

$$G(t, t+k) = (1 - P'[t, t+k, t+k+m)]/m$$

and given the futures rate

$$F(t, t+k) = [1 - P(t, t+k, t+k+m)]/m,$$

then, substituting in the CRR relationship we find

$$G(t, t+k) = F(t, t+k) - cov[F(t, t+k), \psi].$$

Proposition 2 allows us to compute forward rates, futures rates, and any zero-coupon bond price given the term structure of futures rates.

6 The Two-Factor Model:A Binomial Discrete Example

In Ho, Stapleton and Subrahmanyam (1995)(HSS), a method is described for approximating a multivariate lognormal distribution with a multivariate binomial distribution. To approximate the joint distribution at time t of two variables X and Y with logarithmic mean and standard deviation $\mu_x, \sigma_x, \mu_y, \sigma_y$ and correlation ρ_{xy} is particularly simple. As described in HSS it can be achieved either by choosing the conditional probability parameters in an appropriate manner, or by orthogonalizing the variables. The latter method may be more efficient for highly correlated variables hence we choose that method here. The objective is to replicate the joint distribution of the short rate, $f(t,t)$ and the Nth futures rate $f(t,t+N)$. The required inputs are the means of the two rates $\mu(0,t,t)$ and $\mu(0,t,t+N)$, their volatilities $\sigma(0,t,t)$ and $\sigma(0,t,t+N)$, and the correlation between the two rates, ρ. The volatilities and correlation are given in equations (22), (23) and (25) being determined by the variances of the factors V_t and ε_t. The means can be determined from the current term structure in the following manner.

We assume that the term structure of zero bond prices,

$$\{P(0,km), k = 1,2,\ldots,K\},$$

is observable at time 0, where k represents the index of m-year periods. The bond price for any maturity t is approximated by

$$P(0,t) = P(0,km)^x P(0,(k+1)m)^{1-x}, \tag{31}$$

where

$$km < t < (k+1)m,$$

and

$$x = (k+1)m - t.$$

In equation (31), the bond price $P(0,t)$ is obtained by a geometric interpolation of the given zero bond prices. The futures rate at time 0 for delivery at time t, using the relationship in equation (2) between futures prices and values is

$$F(0,t) = [1 - P(0,t,t+m)]/m. \qquad (32)$$

The expected m–year rate at time 0, given the no-arbitrage condition is

$$\mathrm{E}_0(i_t) = F(0,t). \qquad (33)$$

Hence, we must have

$$\mu(0,t,t) + \frac{1}{2}\sigma^2(0,t,t)t = \ln\left[[1 - P(0,t,t+1)]/m\right] \qquad (34)$$

which, given an estimate of $\sigma(0,t,t)$, determines the mean $\mu(0,t,t)$.

The mean of the futures rate $f(t, t+N)$ can be computed in a similar manner. A futures contract made at time 0 to enter a futures contract at time t to buy an asset at time $t+N$ is simply a futures contract to buy the asset at time $t+N$. Hence the futures rate at time 0 is also the rate for the futures, futures contract. Thus, $F(0,t,t+N)$ is the futures rate at 0 for entering an N–period futures at t. The expected futures rate is then

$$\mathrm{E}(F(t,N)) = F(0,t+N) \qquad (35)$$

by no-arbitrage, and

$$F(0,t+N) = [1 - P(0,t+N,t+N+m)]/m, \qquad (36)$$

which implies

$$\ln(F(0,t+N)) = \mu(0,t,t+N) + \frac{1}{2}\sigma^2(0,t,t+N)t, \tag{37}$$

which determines $\mu(0,t,t+N)$, given an estimate of $\sigma(0,t,t+N)$.

Having generated binomial approximations for the short rate $f(t,t)$ and the Nth futures rate $f(t,t+N)$ we can then apply the linear equation (28) to derive the term structure of futures rates. The remaining input parameter is the vector of means of the futures rates for $k = 1, \ldots K$. Again, we have

$$\mu(0,t,t+k) = \ln[1 - P(0,t+k,t+k+m)]/m - \frac{1}{2}\sigma^2(0,t,t+k)t, \tag{38}$$

using a similar argument to that used for determining $\mu(0,t,t+N)$.

In the binomial version of the model, we choose a parameter u representing the number of up and down movements of the binomial process, for each factor. Since there are two factors, the output of the model is a set of $(u+1)^2$ term structures at time t. For the case of $u = 2$, the nine term structures generated are illustrated in Figure 1.

7 Conclusions

We have derived a stochastic model of the term structure of futures rates in which the two factors are any two futures rates. The remaining futures rates are then log-linear functions of those two factor rates. The factors are themselves derived from an assumed two-dimensional ARMA process for the short rate. The model is simple to compute and can be used either to value options or to generate interest-rate scenarios, which can then be used to evaluate the risk of interest-rate dependent portfolios.

Appendix 1: Proof of Lemma 3

Proof that

$$\mu(0,t,t+k) = \mu(0,t+k,t+k) + \frac{kn}{2}\sigma^2(t,t+k,t+k).$$

Proof. In the proof of Lemma 2, we have the no-arbitrage condition

$$F(t,t+k) = \mathrm{E}_t(i_{t+k}). \tag{39}$$

Hence, the expectation of the futures rate is given by

$$\mathrm{E}_0[F(t,t+k)] = \mathrm{E}_0(i_{t+k}), \tag{40}$$

by the law of iterated expectations.

Taking the logarithm of equation (40) and using the lognormal property, we have

$$\mu(0,t,t+k) + \frac{tn}{2}\sigma^2(0,t,t+k) = \mu(0,t+k,t+k) + \frac{(t+k)n}{2}\sigma^2(0,t+k,t+k). \tag{41}$$

From the lognormality of i_{t+k},

$$(t+k)n\sigma^2(0,t+k,t+k) = \mathrm{var}_0[\mu(t,t+k,t+k)] + kn\sigma^2(t,t+k,t+k). \tag{42}$$

But, using Lemma 2,

$$\mathrm{var}_0[\mu(t,t+k,t+k)] = nt\sigma^2(0,t,t+k). \tag{43}$$

Substituting equations (43) into (42), and then (42) into (41), yields

$$\mu(0,t,t+k) = \mu(0,t+k,t+k) + \frac{kn}{2}\sigma^2(t,t+k,t+k).$$

■

Appendix 2: Interpolation Methods for the Simulations

The current term structure consists of 16 zero coupon bond prices of maturities varying from 1 day to 30 years. [1] Given $P(0,\tau_j), j = 1, 2, \ldots, 16$, we assume $m = \frac{91}{365}$ and generate $P(0, km), k = 1, 2, \ldots, 120$ by geometric interpolation. In this method

$$P(0, km) = P(0, \tau_j)^x P(0, \tau_{j+1})^{1-x}, \tag{44}$$

where

$$\tau_j \leq km \leq \tau_{j+1}, \tag{45}$$

and

$$x = \frac{\tau_{j+1} - km}{\tau_{j+1} - \tau_j}. \tag{46}$$

Given this input we have sufficient data to generate the required term structures of futures rates. We use the HSS method first to generate the short rate $f(t,t)$ and the Nth futures rate $f(t, t+N)$. We then use the linear model (28), to generate the futures rates $f(t, t+k)$, for $k = 2, 3, \ldots, 120$ for each scenario. Given a binomial lattice with u stages for each rate, there are $(u+1)^2$ scenarios and hence $(u+1)^2$ term structures.

[1] Here we input zero coupon rates for maturities 1 day, 7 days, 1 month, 3 months, 6 months, 9 months, 1 year, 2 years, 3 years, 4 years, 5 years, 7 years, 10 years, 15 years, 20 years, and 30 years. In practice, data availability and the number of grid points depends on the currency used.

Appendix 3: Proof of Proposition 1, Necessity

Assume that the kth futures rate is given by

$$f(t,t+k)-\mu(0,t,t+k)=a_k[f(t,t)-\mu(0,t,t)]+b_k[f(t,t+1)-\mu(0,t,t+1)] \quad (47)$$

For compactness write this as

$$f'(t,t+k)=a_k f'(t,t)+b_k f'(t,t+1) \quad (48)$$

Consider the orthogonal component z_t from

$$f'(t,t+1)=\alpha+\beta f'(t,t)+z_t \quad (49)$$

and

$$\begin{aligned} f'(t,t+k) &= a_k f'(t,t)+b_k[\alpha+\beta f'(t,t)+z_t] \\ &= (a_k+b_k\beta)f'(t,t)+b_k\alpha+b_k z_t \end{aligned} \quad (50)$$

To establish necessity, assume that the kth futures rate is given by (47), (48), (49) and (50) above. We also have, given lognormality of the rate

$$f(t,t+k)=\mu(t,t+k,t+k)+\frac{kn}{2}\sigma(t,t+k,t+k)^2$$

hence

$$f(t,t+k)-\mu(0,t,t+k)=\mu(t,t+k,t+k)+\frac{kn}{2}\sigma(t,t+k,t+k)^2-\mu(0,t,t+k)$$

and substituting (50)

$$\mu(t,t+k,t+k)+\frac{kn}{2}\sigma(t,t+k,t+k)^2-\mu(0,t,t+k) = (a_k+b_k\beta)f'(t,t)+b_k\alpha+b_k z_t$$

$$\begin{aligned} \mathrm{E}_t[f(t+k,t+k)] &+ \frac{kn}{2}\sigma(t,t+k,t+k)^2 - \mu(0,t,t+k) \\ &= (a_k+\beta b_k)[f(t,t)-\mu(0,t,t)] + b_k\alpha + b_k z_t \end{aligned}$$

For the first futures:

$$\begin{aligned} \mathrm{E}_t[f(t+1,t+1)] &+ \frac{n}{2}\sigma(t,t+1,t+1)^2 - \mu(0,t,t+1) \\ &= (a_1+b_1)[f(t,t)-\mu(0,t,t)] + b_1\alpha + b_1 z_t \end{aligned}$$

$$\begin{aligned} \Rightarrow f(t+1,t+1) &+ [\frac{n}{2}\sigma(t,t+1,t+1)^2 - \mu(0,t,t+1)] \\ &= (a_1+b_1)[f(t,t)-\mu(0,t,t)] + b_1\alpha + b_1 z_t + \varepsilon_{t+1} \end{aligned}$$

where $\mathrm{E}_t(\varepsilon_{t+1}) = 0$.

Hence the spot rate follows a two-dimensional process with innovations z_t, ε_{t+1}.

■

Appendix 4: Derivation of the Process for the kth Futures Rate

Using Lemmas 2 and 3, the deviation of the kth futures rate from its expectation is related to that of the spot rate by the equation [substitue (26) in (21)]

$$f(t, t+k) - \mu(0, t, t+k) = (1-c)^k[f(t,t) - \mu(0,t,t)] + V_t[K] \tag{51}$$

where

$$\begin{aligned} V_t &= \sum_{\tau=0}^{t-1} \nu_{t-\tau}\alpha^{\tau} \\ K &= \sum_{\tau=1}^{k} (1-c)^{k-\tau}\alpha^{\tau-1} \end{aligned}$$

Also, by assumption, the spot rate is

$$f(t,t) - \mu(0,t,t) = (1-c)[f(t-1,t-1) - \mu(0,t-1,t-1)] + V_{t-1} + \varepsilon_t \tag{52}$$

The kth forward at time $t-1$ is similarly given by

$$\begin{aligned} f(t-1, t+k-1) \quad &- \quad \mu(0, t-1, t+k-1) \\ &= \quad (1-c)^k[f(t-1,t-1) - \mu(0,t-1,t-1)] \\ &\quad\; + V_{t-1}[K] \end{aligned} \tag{53}$$

Substituting (53) in (52) and (52) in (51) yields the corollary.

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Figure 1
Yield Curves Generated by Binomial Version of Two-Factor Model

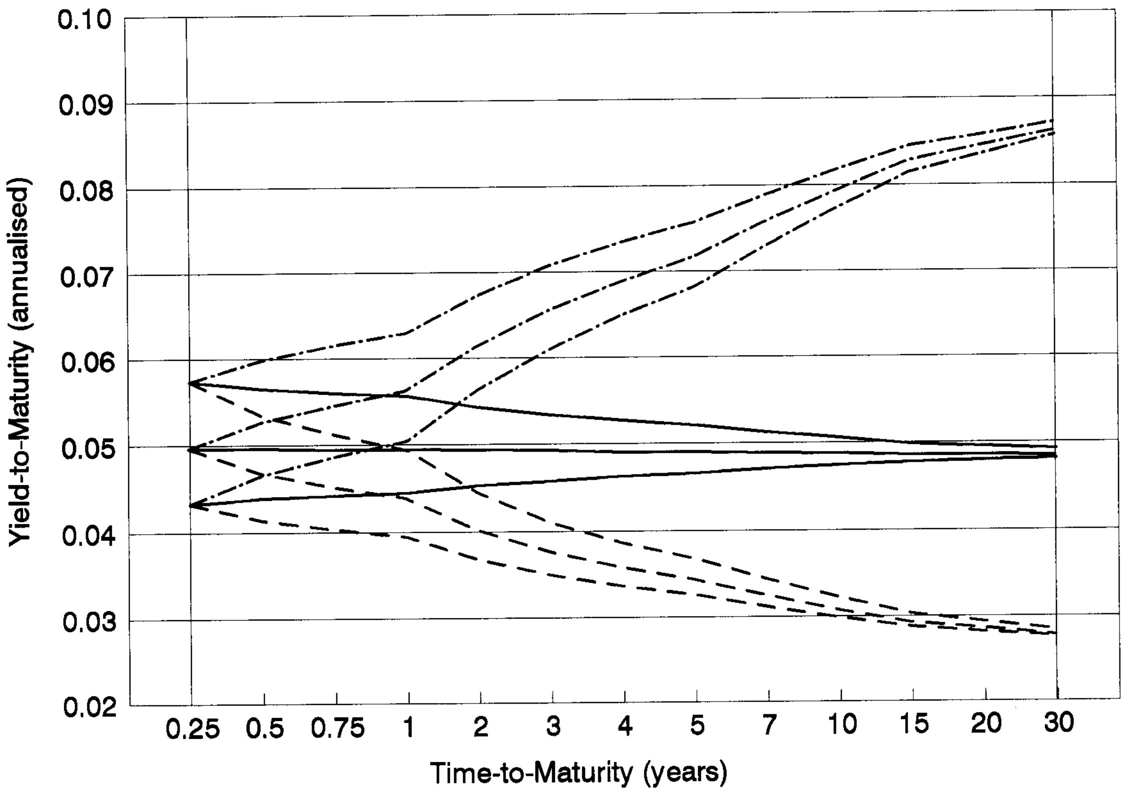
Figure 1 plots the yield curves generated by a binomial discrete version of the no-arbitrage two-factor model of the term structure, using the methodology described in Ho, Stapleton and Subrahmanyam (1993). The parameter N, representing the number of up and down movements of the binomial process for each factor, is set at $N = 2$. This gives $(N + 1)^2 = 9$ yield curves. The nine term structures depicted show the shift and tilt characteristics implied by the two-factor model. The parameters used in the simulation is a flat current term structure with sixteen maturities of 5 per cent per annum. The two factors are a short rate and a futures rate, with volatilities of 0.20 per annum, and maturities of 3 and 6 months, respectively. The mean reversion parameter is set at 0.10 per annum.

Table 2

Special Cases of the ARMA Process

Effect on Conditional Expectation of $t + k$th spot rate

Shock memory	mean reversion $c = 0$	mean reversion $c > 0$
$\alpha = 0$	Two-dimensional random walk. Shift in expectation same for each i_{t+k}	Mean reverting process. Shift in expectation dampened by mean reversion.
$\alpha = 1$	Shift in expectation affected by past shocks but proportional to k	Shift in expectation affected by past shocks, but dampened by mean reversion.


0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
Yield-to-Maturity (annualised)
0.25
0.5
0.75
1
2
3
4
5
7
10
15
20
30
Time-to-Maturity (years)