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The pricing of marked-to-market contingent claims in a no-arbitrage economy

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Abstract

This paper assumes that the underlying asset prices are lognormally distributed, and derives necessary and sufficient conditions for the valuation of options using a Black-Scholes type methodology. It is shown that the price of a futures-style, marked-to-market option is given by Black's formula if the pricing kernel is lognormally distributed. Assuming that this condition is fulfilled, it is then shown that the Black-Scholes formula prices a spot-settled contingent claim, if the interest-rate accumulation factor is lognormally distributed. Otherwise, the Black-Scholes formula holds if the product of the pricing kernel and the interest-rate accumulation factor is lognormally distributed.

1 Introduction

This paper establishes necessary and sufficient conditions under which futures-style options, traded on a marked-to-market basis, can be valued using the Black(1976) model. It contrasts these with the conditions under which the Black-Scholes model can be used to obtain the spot price of an option. Assuming that these conditions obtain, the prices of the two types of contingent claim are compared.

Interest rate options, index options, and individual stock options on the Sydney Futures Exchange (SFE) are traded on a daily marked-to-market basis. In other words, they are futures-style contingent claims. This form of contract is not unique to the SFE, for example on the London International Financial Futures Exchange (LIFFE), options on futures are also traded in this manner. However, most options on other exchanges, options on the spot and options on futures are paid for on a spot cash basis. The pricing of options that are paid for on a spot basis is similar, from a theoretical point of view, to settlement on a forward basis.[1] In this paper we analyse the difference between the pricing of a futures-style claim, i.e. the price of a daily marked-to-market claim, and the forward price of a claim. The contingent claims analyzed are European-style put and call options.[2]

The difference between the marked-to-market futures price and the forward price of a contingent claim is due to the effect of stochastic interest rates on the price of the claim. In the case of the futures and forward prices of the underlying asset itself, the size of this difference depends on the covariance of the asset price and the interest rate accumulation factor.[3] In the case of many assets this effect is probably fairly small, and is often ignored in practice. However, in the case of long term futures and forward contracts on assets, such as bonds, which have high covariances with interest rates, the size of the difference may be substantial. Also, in the case of options, the difference may be far more significant, since in this case, the differnce between the futures and forward prices of the underlying asset is leveraged. As well as the difference of the futures and forward prices being more significant in the case of options, there is a more fundamental consideration. We show here that the conditions for the Black(1976) model are less stringent for futures-style options than for forward-style options. It turns out that the Black model holds for forward-style options if the pricing kernel and the interest rate accumulation factor are lognormally

[1] The spot price of a European-style claim, paying no dividends, is its forward price, discounted at the zero-bond price for the maturity of the option. Hence, any factors that affect the forward price of the claim also affect the spot price.

[2] The general approach used here could be modified for the case of American-style options. In this context, see, for example Ho, Stapleton and Subrahmanyam(1996).

[3] see for example Cox, Ingersoll, and Ross (1981)

distributed.[4] However, in the case of futures-style options, the condition for the Black model is far less stringent, it is sufficient that the pricing kernel is lognormal.

Futures-style contingent claims have been studied by Lieu (1990) and by Duffie and Stanton (1992). Lieu values marked-to-market, European-style, call and put options, assuming that the futures price evolves as a Gauss-Weiner process. He establishes that the Black(1976) model holds under this assumption. Lieu also establishes a put-call-parity relationship for these options. Duffie and Stanton provide a somewhat more general no-arbitrage condition for the pricing of both futures and forward-style claims, again in a continuous time setting.[5] However, their expression for the value of the continuously re-settled call option, is derived under the assumption of constant interest rates.[6] It is, therefore, in their special case, simply the non-discounted Black-Scholes price, and is identical to the forward price of the claim. Lieu, on the other hand, does provide sufficient conditions for the Black model to hold for futures-style options.In comparison, we establish here, by concentrating on the pricing kernel, both necessary and sufficient conditions for the model to hold. We also compare the pricing of futures and forward style claims and show that Lieu's put-call parity relation for futures-style options holds in similar form for forward-style options. Compared to this previous work on the pricing of contingent claims, we start with a more general framework, based directly on the original contribution of Cox, Ingersoll and Ross(1981). This leads us to an analysis of the pricing kernel, and a comparison of the conditions under which the Black model can be used to price the futures-style and forward-style options.

In section 2, we derive no-arbitrage pricing relationships for futures-style options, extending the results of Cox, Ingersoll and Ross (1981). We derive the effect of the marking-to-market interval on the price of the contingent claims, and the implications for put-call parity. In section 3, we assume that the spot price of the asset is lognormally distributed and establish necessary and sufficient conditions for the Black model to price marked-to-market and non-marked-to-market claims respectively. In section 4, we assume that the conditions for the Black model to hold are satisfied and examine the difference between the prices of the futures-style and forward-style claims. Conclusions are presented in section 5.

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[4] The pricing kernel is a utility dependent variable, which prices the option, capturing the effect of risk aversion on the value of the claim. The interest rate accumulation factor is defined precisely interms of the product of the short term bond prices, in equation (3) of this paper.

[5] see Duffie and Stanton (1992) corollary 2 and 3, p.568.

[6] see Duffie and Stanton (1992) equation (34), p.571

2 Forward and Futures Contracts

In the financial markets, a *futures* contract normally refers to an agreement to buy or sell an asset at a fixed price, at a future date, where the contract is 'marked-to-market' and settled at a series of specified points in time. Typically, the marking to market is on a trading day basis. On the other hand, a *forward* contract can be thought of as a special type of futures contract, one that is marked-to-market only once, at the maturity of the contract. Since many authors work with continuous time models in the finance literature, it is usually assumed for modelling simplicity that a futures contract is marked-to-market continuously.[7] In this section, we will use the term *futures contract* as a generic term which includes the typical market forward contract and typical market futures contract as special cases.

Consider a point in time T and a contract maturity date s. Then the *futures price* $F_{T,s}$ is the fixed price agreed at time T for delivery of the asset at time s. Suppose that the contract is initially made at time t and is a long contract, i.e., a contract to buy the asset. If the contract is to be marked to market k times at times $t + n_1$, $t + n_1 + n_2$,..., $t + n_1 + n_2 + ... + n_k$ then the holder will, in effect, receive a series of cash flows (or 'dividends')

$$[F_{t+n_1,s} - F_{t,s}, F_{t+n_1+n_2,s} - F_{t+n_1,s}, ..., F_{s,s} - F_{s-n_k,s}]$$

The first 'dividend' in the series, received at the end of the first mark to market interval, n_1, is equal to the change in the futures price over that interval. Subsequent 'dividends' are equal to the futures price change over each time interval.

Given this general definition of a futures contract, we can characterise two important special cases. First, if $n_1 = s - t$, the contract will be marked to market only once, at time s. In this case, there is only one cash flow $(F_{s,s} - F_{t,s})$. Also, since by definition $F_{s,s} = V_s$, where V_s is the spot price of the asset at time s, this cash flow is $(V_s - F_{t,s})$. This contract is normally referred to as a forward contract. In this special case, we will denote the *forward price* at time t as $G_{t,s} \equiv F_{t,s}(n_1)$, since it is the price of a futures contract, marked-to-market only at time $t + n_1$. The second special case is where the futures contract is marked-to-market every trading day, i.e., if trading takes place at time t, $t + 1$, $t + 2$,..., $s - 1$, s then $n_1 = 1$, $n_2 = 1$,..., $n_k = 1$ and $k = s - t$, where $s - t$ is measured in days. This is typically the case for futures contracts traded on organised market futures exchanges. In this special case, we will denote the

[7]This is the case, for example, in Cox, Ingersoll and Ross (1985), and in Duffie and Stanton (1992)

futures price as $H_{t,s} \equiv F_{t,s}(1)$ since the distance between marking to market dates is one trading day.

In the case of other contingent claims such as call or put options, exactly the same principles apply. A futures-style call option is an agreement, made at time t, to buy a call option at time s, which is marked-to-market at intervals between t and s. If it is marked-to-market daily, it is a regular futures-style contract, similar to those traded on the SFE. If it is marked-to-market just once, at time s, it is a forward-style contract.

The Pricing of Futures Contracts

One distinguishing feature of a futures contract is that (normally) no money changes hands on the contract date, t. It is an agreement to pay or receive payment in the future. It follows then that the value of the contract, at time t, must be zero, if arbitrage is to be avoided. In other words, the market must determine the futures price $F_{t,s}$ so as to ensure that the futures contract has zero value at time t. This observation allows us to state the futures price of an asset in the following proposition.

Proposition 1 (The Price of a Futures Contract)

a) Suppose that the futures contract is to be marked-to-market every trading day. Then

$$H_{t,s} = \mathrm{E}_t(V_s \phi_{t,s}) \tag{1}$$

where $\phi_{t,s}$ is the pricing kernel at time t, for claims at time s.

b) Suppose that the futures contract is to be marked to market only once at time s. Then

$$G_{t,s} = \mathrm{E}_t(V_s \psi_{t,s}) \tag{2}$$

where

$$\psi_{t,s} = \phi_{t,s} g_{t,s}$$

where $\psi_{t,s}$ is the 'adjusted' pricing kernel and $g_{t,s}$ is the interest rate accumulation factor, i.e.

$$g_{t,s} = \prod_{\tau=t}^{s-1} B_{\tau,\tau+1}/B_{t,s}, \tag{3}$$

where $B_{t,s}$ is the value at t of a dollar paid at s.

Proof The proof of Proposition 1 follows directly from Cox, Ingersoll and Ross (1981) using the principle of no-arbitrage. For a detailed proof see Satchell, Stapleton, and Subrahmanyam (1989). Also, for a proof in a continuous-time setting, see Duffie and Stanton (1992) □

Note that the futures price for the contract which is marked to market daily, given in equation (1), is the risk adjusted expectation of the value of the firm. Hence, under risk neutrality, where $\phi_{t,s} = 1$ in all states, the futures price is just the expected value of the future value of the firm: $E(V_s)$. Part b) of the proposition shows that $G_{t,s}$: the forward price of the asset, is a more complex variable. Even in the case of risk neutrality we find $G_{t,s} = E_t(V_s g_{t,s})$. Hence the forward price is affected by the period by period stochastic discounting at the prevailing rates of interest, reflected by the term $g_{t,s}$.[8] We can now apply Proposition 1 to the valuation of assets whose payoff is contingent on the price of an underlying asset. We restrict attention to European-style contingent claims that pay off at a terminal date s. These derivative assets pay no dividends from time t (the valuation date) to time s. We consider the futures price, for delivery at s, of such a contingent claim.

As we have seen above, forward prices of assets are more complex, in general, than futures prices. While the day-to-day marked-to-market futures price of an asset is the "risk-adjusted" expectation of the spot price at time s, the forward price includes an interest rate term. The European-style contingent claim has a payoff denoted $c(V_s)$ at time s which is a function of V_s, the price of the underlying asset. In the particular case of a European call option with a strike price K, for example, we can write

$$c(V_s) = \max(V_s - K, 0)$$

[8]Since $E_t(\phi_{t,s}) = 1$, it follows that there exists an equivalent probability measure * under which $H_{t,s} = E_t^*(V_s)$, i.e. under which the futures price follows a martingale. This measure is often termed the 'risk neutral' equivalent martingale measure. It is less often recognised that since $E_t(\psi_{t,s}) = 1$, also, there must exist a second equivalent martingale measure ** under which the forward price follows a martingale. In the special case of non-stochastic interest rates these measures are of course the same. [The existence of the equivalent measures * and ** follows from the proof given by Duffie (1992), p 82-3.]

We shall denote the spot price at t of the general contingent claim as $V_t[c(V_s)]$. Similarly, its futures price at t for delivery at time s will be denoted in general by $F_{t,s}[c(V_s)]$. In the case of the usual market futures contract, where the contract is marked-to-market every trading day we denote the futures price of the contingent claim as $H_{t,s}[c(V_s)]$. The forward price of the contingent claim at t, also for delivery at time s, is denoted by $G_{t,s}[c(V_s)]$.

The Futures Price of a Contingent Claim

Applying Proposition 1 to the evaluation of the futures price of a contingent claim we have the following:

Proposition 2 (The futures price of a contingent claim)

a) Suppose that the futures contract to buy a European-style contingent claim with payoff function $c(V_s)$ is marked-to-market every trading day. Then

$$H_{t,s}[c(V_s)] = \mathrm{E}_t[c(V_s)\phi_{t,s}] \tag{4}$$

b) Suppose that the futures contract on the contingent claim is marked-to-market only once at time s. Then

$$G_{t,s}[c(V_s)] = \mathrm{E}_t[c(V_s)\psi_{t,s}]$$

where

$$\psi_{t,s} = \phi_{t,s} g_{t,s}$$

Proposition 2 is quite general, resulting solely from the assumption of a no-arbitrage economy. In fact, the results shown in Proposition 2 are too general to be implementable in practice. In the case of options, for example, we normally try to derive, following Black and Scholes (1973), pricing relationships which are preference free. Proposition 2 is not sufficient to derive such a valuation result, since the $\phi_{t,s}$ and $\psi_{t,s}$ variables are preference-dependent, and hence potentially unobservable. In the next section, therefore, we specify a scenario where we can establish preference-free pricing relationships for contingent claims. However, the proposition does allow us to establish the following put-call parity relationship:-

Corollary 1 (Put-call parity for futures options (Lieu) *Suppose that the futures contract to buy a European-style contingent claim with payoff function $c(V_s)$ is marked-to-market every trading day. Then, if the claim is a call option with strike price K, and has a futures price $H_t[C]$ then the price of a put option is*

$$H_t[P] = H_t[C] - H_{t,s} + K$$

Alternatively, if the claim is marked-to-market just once at time s, and the call has a forward price of $G_t[C]$ then the forward price of the put option is

$$G_t[P] = G_t[C] - G_{t,s} + K$$

The proof of the corollary follows directly from substituting the payoff function of the put and the call option in Proposition 2. Note that the second part of the corollary is actually a generalisation of Lieu's result and itself follows directly also from the usual put-call-spot parity.

3 Contingent Claims Pricing Given Lognormally Distributed Asset Prices

Proposition 2 highlights the importance of the pricing kernel $\phi_{t,s}$ for the pricing of contingent claims and of the compound pricing function $\psi_{t,s} = \phi_{t,s} g_{t,s}$. In this section, we assume that the asset price V_s is lognormally distributed and investigate the conditions under which the preference-dependent variables, $\psi_{t,s}$ and $\phi_{t,s}$, can be eliminated from the valuation equations. Essentially, this amounts to looking for conditions under which the Black (1976) model holds for the forward and the futures prices of contingent claims[9], since in the Black model, no preference-dependent variables occur in the valuation formula.

We adopt the following notation. Since V_s is lognormally distributed, we denote the logarithmic mean and variance as follows:-

$$\begin{aligned} \mathrm{E}_t\left[\ln\left(\frac{V_s}{V_t}\right)\right] &= \mu_\nu \\ \mathrm{var}_t\left[\ln\left(\frac{V_s}{V_t}\right)\right] &= \sigma_{\nu\nu} \end{aligned}$$

Also $\sigma_\nu = \sigma_{\nu\nu}^{1/2}$. Note that, in these definitions, the mean and the standard deviation: σ_ν, are *not* annualized. Thus, they are likely to be functions of the time to maturity $s - t$. Similarly, we denote

$$\begin{aligned} \mathrm{E}_t[\ln(\phi_{t,s})] &= \mu_\phi \\ \mathrm{E}_t[\ln(\psi_{t,s})] &= \mu_\psi \end{aligned}$$

and the variances $\sigma_{\phi\phi}$, $\sigma_{\psi\psi}$, and the covariances $\sigma_{\phi\nu}$, $\sigma_{\psi\nu}$ and $\sigma_{\phi\psi}$ are defined analogously.

We now define precisely what we mean by the Black (1976) model for the futures price of a contingent claim.

[9]Note that we are concerned here with the futures price of an option, not an option on a futures contract which is valued in Brenner, Courtadon and Subrahmanyam (1985) and in Ramaswamy and Sundaresan (1985).

Definition *(The Black model)*

Suppose that the price of the underlying asset, V_s, is lognormally distributed with logarithmic standard deviation σ_ν. Then, the Black model holds for the futures price of a European-style contingent claim on the asset, if the futures price of the claim can be computed using the 'risk-neutral' distribution of the underlying asset. In this context, the risk-neutral distribution of the asset is a probability distribution which is lognormal, with a mean equal to the futures price of the asset, $F_{t,s}$, and a logarithmic standard deviation equal to σ_ν.

In other words, if the Black model holds, the futures price of the claim is given by

$$F_{t,s}[c(V_s)] = \int_{-\infty}^{\infty} c(V_s)\hat{f}(V_s)\mathrm{d}V_s, \tag{5}$$

where $\hat{f}(V_s)$ is lognormal with a mean equal to the futures price of the underlying asset and a logarithmic standard deviation σ_ν.

The definition of the Black model given here represents a slight generalization of the model introduced by Black (1976) for the purpose of valuing options on futures for two reasons. First, it applies to general European-style contingent claims rather than just to put and call options. Second, our definition applies, in special cases, to the forward price of a claim and to the daily marked-to-market futures price of the claim. Since the distribution $\hat{f}(V_s)$ is lognormal, we find that the Black-Scholes formula, with an interest rate of zero, holds for the futures price of an option, *if the Black model holds.* Hence, for the case of a call option on a stock, with exercise price K, we can write

$$F_{t,s}[c(V_s)] = BS[F_{t,s}, K, \sigma_\nu, 0, s-t]$$

where the function

$$BS[V_t, K, \sigma_\nu, r, s-t]$$

denotes the Black-Scholes formula for the value of a call option on a stock, given the current stock price V_t, an exercise price K, a volatility σ_ν (non-annualized), a continuously compounded interest rate r, at time t, for loans with maturity s, and a time to maturity $s-t$.

The Black model is a preference-free valuation relationship for the contingent claim. Hence, the implications of the Black model holding for a contingent claim are quite strong. For example, if the Black model holds, the forward price of a contingent claim, $G_{t,s}[c(V_s)]$, can be computed knowing only the forward price of the underlying asset, $G_{t,s}$, the volatility σ_ν, and the form of the payoff function $c(V_s)$. Similarly, computation of the daily marked-to-market futures price of the contingent claim, $H_{t,s}[c(V_s)]$, requires only the futures price of the underlying asset, $H_{t,s}$, the volatility σ_ν, and the payoff function $c(V_s)$. In the special case where $c(V_s)$ is the payoff function for a call option with strike price K, the computations require either $G_{t,s}$ or $H_{t,s}$, σ_ν and K. The forward price of the call option, if the Black model holds, is

$$G_{t,s}[c(V_s)] = BS[G_{t,s}, K, \sigma_\nu, 0, s-t]$$

and the daily marked-to-market futures price, if the Black model holds, is

$$H_{t,s}[c(V_s)] = BS[H_{t,s}, K, \sigma_\nu, 0, s-t]$$

It also follows that, if the Black model holds for the forward price of the claim, then the Black-Scholes model holds for the spot price of the option in the stochastic interest rate economy. The condition specified in the definition is thus a strong one.

The following proposition establishes necessary and sufficient conditions for the Black model to hold. It involves the lognormality of the pricing kernel, $\phi_{t,s}$, and of the compound variable $\psi_{t,s} = \phi_{t,s} g_{t,s}$.

Proposition 3 (Conditions for the Black Model to hold)

a) Suppose that the futures contract to buy a European-style contingent claim with payoff function $c(V_s)$ is marked-to-market every trading day and that V_s is lognormally distributed. The Black model holds for the futures price of the claim if $\phi_{t,s}$ is joint lognormally distributed with V_s. The Black model also holds for the futures price only if $\mathrm{E}_t(\phi_{t,s}|V_s)$ is lognormally distributed.

b) Suppose that the futures contract on the above claim is marked-to-market only once, at time s, and that V_s is lognormally distributed. The Black model holds for the forward price of the claim if $\psi_{t,s} = \phi_{t,s} g_{t,s}$ is joint lognormally distributed with V_s. The Black model also holds for the forward price only if $\mathrm{E}_t(\psi_{t,s}|V_s)$ is lognormally distributed.

Proof See Appendix. □

We now discuss the significance of proposition 3 in theory and practice, and the parts of the proposition that are already estabished in the literature. Most of the literature on options pricing, until recently, has been in the context of non-stochastic interest rates. In this case, parts a) and b) of Proposition 3 are the same. Rubinstein (1976) shows in this context the sufficiency part of the proposition. Brennan (1979) establishes the necessary condition, in the context of a representative investor economy. When interest rates are stochastic, parts a) and b) of the proposition have to be distinguished. Merton (1973) shows in this case that lognormal zero-bond prices (together with a lognormal pricing function $\phi_{t,s}$) is sufficient for the Black-Scholes model to hold. Part b) extends Merton's sufficient conditions and establishes a necessary condition on the compound pricing kernel $\psi_{t,s}$.

The significance of Proposition 3 stems from the widespread use of the Black model in practice. The Black model, and its close relative, the Black-Scholes model, are used extensively to price options, including interest rate and bond options, when interest rates are stochastic. The necessary and sufficient conditions established in Proposition 3b, especially, give some idea of the validity of their use when interest rates are stochastic and, in particular, correlated with the underlying asset price.

Much of the proof of Proposition 3 concerns the details of the lognormal distribution. The proof in the appendix shows the main steps in the argument and relies heavily on the method of proof used by Brennan (1979), page 60. The difference here is that our condition is a restriction on the pricing kernel, $\phi_{t,s}$ rather than on the utility function of the representative investor. Otherwise, our proof for the futures price of the contingent claim follows the same steps as Brennan's for the spot price of the claim.

In Proposition 3a, the necessary condition is somewhat weaker than the sufficient condition . The relevant pricing function is the conditional expectation $\mathrm{E}_t(\phi_{t,s} \mid V_s)$ rather than $\phi_{t,s}$ itself. However, in order to relate our results to those of Brennan (1979) and Rubinstein (1976), note that if we were to assume, as they did, that V_s is joint lognormally distributed with aggregate wealth and that there exists a representative investor, then Proposition 3a would imply that the representative investor had constant proportional risk aversion (CPRA) preferences.

Proposition 3b shows that joint lognormality of $\psi_{t,s} = \phi_{t,s}g_{t,s}$ with V_s is a sufficient condition for the Black model to hold for the forward price of the contingent claim. Hence, if $\phi_{t,s}$ is joint lognormal with V_s, a further sufficient condition in this case is that the stochastic bond price factor $g_{t,s}$ is also joint-lognormally distributed. It is significant that many of the analytical models, in the literature, which capture the effect of stochastic interest rates on contingent claims prices, do assume Gaussian interest rates.[10] The necessary condition in this case is that $\mathrm{E}_t(\psi_{t,s} \mid V_s)$ is lognormal. If $\phi_{t,s}$ is itself lognormal, this is close to stating that a necessary condition for the Black model to hold for the forward price of the claim is that the bond price factor $g_{t,s}$ is lognormal.

Finally, we should observe that Proposition 3 holds for ***any*** no-arbitrage economy in which V_s is lognormally distributed. In particular, the conditions hold *regardless of the trading environment that exists between dates t and s.* In order to establish propositions 1 and 2, we need only assume that trading takes place at points in time $t, t+1, t+2, ..., s$ where the number of trading dates is arbitrary. Thus, Proposition 3 is not dependent on the number of trading dates between t and s. The number of dates could be any element in the set $[0,\infty]$. The two cases that have been emphasized in the literature are:

1. continuous trading (number of dates $\rightarrow \infty$),
2. discrete trading at t and s (number of intermediate dates $= 0$).

In the latter instance, where no intermediate trading is possible, we have the world assumed by Brennan (1979) and Rubinstein(1976). These two cases, $(0, \infty)$, constitute special cases of Proposition 3. The results hold in these two cases, but also apply to economies with an arbitrary number of trading dates. To this extent, the theorems in this section represent an extension of previous applications of the Black model to be found in the literature. In addition of course, the results here apply to an economy with stochastic interest rates.

Another important implication of Proposition 3 relates to continuous-time economies. Proposition 3 shows that the assumption of joint lognormality of V_s and $\phi_{t,s}$ is an alternative sufficient condition for the Black model to hold. More significantly, despite the fact that no explicit assumption is made regarding risk preferences and the pricing function $\phi_{t,s}$, in deriving the Black model in a continuous time economy, Proposition 3 establishes that lognormality of $\mathrm{E}(\phi_{t,s} \mid V_s)$ is an implicit assumption. Furthermore, if there exists a representative investor, and if V_s and aggregate wealth are joint lognormally distributed, the implicit assumption is that the representative investor has CPRA preferences.

We now state some of these implications of Proposition 3 in the form of corollaries. We have:

Corollary 2 (The Spot Price of a Contingent Claim in the Black Model)

[10] Usually, as in Jamshidian (1989) or Turnbull and Milne (1991), a one-factor Gaussian model of the term structure is assumed.

If $\psi_{t,s} = \phi_{t,s} g_{t,s}$ is joint lognormal with V_s

a)

$$V_t[c(V_s)] = B_{t,s} \int_{-\infty}^{\infty} c(V_s) f^{\#}(V_s) \mathrm{d}V_s,$$

with $\mathrm{E}_t^{\#}(V_s) = G_{t,s}$.

b) and if further, V_s is uncorrelated with $g_{t,s}$

$$V_t[c(V_s)] = B_{t,s} \int_{-\infty}^{\infty} c(V_s) \hat{f}(V_s) \mathrm{d}V_s,$$

with $\hat{\mathrm{E}}_t(V_s) = H_{t,s}$.

c) also, if the contingent claim is a European-style call option with

$$c(V_s) = \max[V_s - K, 0]$$

where K is the exercise price of the option, the effect of stochastic discounting on the price of the option is $< (=)(>) 0$ if and only if $cov(V_s, g_{t,s}) < (=)(>) 0$.

Proof Statements 2*a* and 2*b* follow from the spot-forward relationship that holds for zero-dividend paying assets. To prove 2c) note that, for a call option

$$\frac{\partial V_t[c(V_s)]}{\partial F_{t,s}} > 0$$

If $\mathrm{cov}(V_s, g_{t,s}) < (=)(>) 0, G_{t,s} < (=)(>) H_{t,s}$ and hence

$$G_t[c(V_s)] < (=)(>) H_t[c(V_s)]$$

□

Proposition 3 and Corollary 2 establish the conditions for the preference-free Black model to hold for the futures prices of a contingent claim. We now explore some economic scenarios under which these conditions will hold.

Corollary 3 (Sufficient conditions for the Black Model)

a) In a representative agent economy, if the representative agent has CPRA preferences and aggregate wealth is lognormally distributed, then the Black Model holds for the futures price of a contingent claim on V_s.

b) In a representative agent economy, if the representative agent has CPRA preferences and $g_{t,s}$ is lognormal then the Black Model holds for the forward price of a contingent claim on V_s.

c) If $\phi_{t,s} = 1$ (risk neutrality) and $g_{t,s}$ is lognormal, then the Black Model holds for the forward price of a contingent claim on V_s.

d) If the futures price of the stock evolves as a lognormal diffusion process, then the Black Model holds for the futures price of the contingent claim.

e) If the price of a bond follows a mean reverting lognormal diffusion process as in Vasicek (1977) then if the futures price of the bond evolves as a lognormal diffusion process then the Black Model holds for the forward price of a contingent claim on the bond.

Proof In a representative agent economy, it can be shown that $\phi_{t,s}$ is the relative marginal utility of the agent. If aggregate wealth is lognormal and if the agent's utility for wealth exhibits constant proportional risk aversion, then $\phi_{t,s}$ and V_s are joint lognormal and the sufficient conditions of Proposition 3a are satisfied.[11] If, in addition, $g_{t,s}$ is lognormal then $\phi_{t,s}g_{t,s}$ is also joint lognormal with V_s. Corollary 3b) is important in the context of interest rate options. A sufficient set of conditions for the use of the Black model in the pricing of options on bonds is risk neutrality and lognormal bond prices (i.e., normally distributed interest rates). Corollary 3c) is simply a special case of 3b). Corollary 3d) provides a link with the traditional analysis of options in diffusion models. If the futures price of an asset follows a multiplicative binomial process, the variable $\psi_{t,s}$ must also evolve as a multiplicative binomial process. Hence in the limit, as the trading interval tends to zero, the distribution of $\psi_{t,s}$ limits to the lognormal distribution. Finally, in the context of bond options, 3e) applies the theorem to the mean-reverting Vasicek model. Note, howeve, that Proposition 3 requires only that the variables are joint lognormal at time s and does not specify the degree of mean reversion. □

[11] See Rubinstein (1976) or Brennan (1979).

Finally, we present a corollary that highlights the economic significance of the necessary condition in Proposition 3.

Corollary 4 (Necessary conditions for the Black Model)

a) In a representative agent economy, if the Black Model holds for the futures price of all contingent claims in the economy, then the representative agent has CPRA preferences.

b) If part a) of this corollary holds and the Black Model holds for the forward price of all contingent claims in the economy, then the interest rate accumulation factor $g_{t,s}$ has a conditional expectation which is lognormally distributed.

Proof

Corollary 4a) follows from the proof of Brennan (1979). Corollary 4b) is significant in that it restricts the range of possible assumptions which allows the use of the Black model in the context of interest rate options. □

4 The effect of marking-to-market on the valuation of contingent claims

In Sections 2 and 3, we have applied general results on futures pricing to determine the futures prices of contingent claims. We assumed in section 3 that the spot price at time s is lognormally distributed, and derived necessary and sufficient conditions for the Black Model to hold for the futures price of a contingent claim. In the special case of a forward contract, this establishes the necessary and sufficient conditions for the Black and Scholes option pricing formula to hold in stochastic-interest-rate economies.

Further implications can now be drawn, from these relationships, to calculate the effect of the daily marking-to-market convention on the valuation of European-style options. We will assume here that the conditions of Proposition 3a) and b) are fulfilled, i.e. that the spot price of the asset is lognormally distributed, and that the Black model holds for the daily marked-to-market futures price of the contingent claim and for its forward price. Given these relationships, it is straightforward to compute the effect of the marking-to-market convention on the pricing of contingent claims. The forward price of the contingent claim is

$$G_{t,s}[c(V_s)] = \int_{-\infty}^{\infty} c(V_s) f^{\#}(V_s) \mathrm{d}V_s, \tag{6}$$

where

$$\mathrm{E}_t^{\#}(V_s) = G_{t,s}$$

We now take the special case of a European-style call option on V_s with a strike price K. In this example equation (6) becomes

$$G_{t,s}[c(V_s)] = \int_{K}^{\infty} (V_s - K) f^{\#}(V_s) \mathrm{d}V_s \tag{7}$$

which, given the lognormality of V_s, is simply the Black equation. The futures price of the claim is also given by the Black equation, but with the futures price of the asset substituted for the forward price. For a general contingent claim, we would have in this case

$$H_{t,s}[c(V_s)] = \int_{-\infty}^{\infty} c(V_s) \hat{f}(V_s) \mathrm{d}V_s \tag{8}$$

and for the call option, in particular, we have

$$H_{t,s}[c(V_s)] = \int_K^\infty (V_s - K)\hat{f}(V_s)\mathrm{d}V_s \tag{9}$$

Equation (9) reduces to a Black equation with the futures price substituted for the forward price. The difference between equation (7) and equation (9)

$$\int_K^\infty (V_s - K)f^{\#}(V_s)\mathrm{d}V_s - \int_K^\infty (V_s - K)\hat{f}(V_s)\mathrm{d}V_s \tag{10}$$

measures the precise effect of the marking-to market convention on the call option price, given stochastic interest rates .

Two important points should be noted. First, simply because the Black equation (8) holds, it does not mean that stochastic interest rates do not have an impact on the prices of forward-style contingent claims. Contingent claim prices are, in general, affected; but, the impact is captured *precisely* by the effect that stochastic interest rates have on the asset forward price. The effect of stochastic interest rates is reflected in call option prices in a manner similar to the way in which risk aversion is reflected through the spot price of the asset on which the option is written. The second point is that although the difference between the forward and the futures price of an asset may be small, the effect on option prices may be much larger. The size of the expression in equation (10) depends, not only on the difference in the means of the $\hat{f}$ and $f^{\#}$ distributions (the futures and forward price of the asset), but also upon the strike price of the option. For instance, if the strike price K is relatively large (i.e. the call option is an out-of-the-money option), a small difference between the means of $\hat{f}$ and $f^{\#}$ will have a large percentage effect on the option price. The difference is also likely to be particularly significant in the case of options on interest rates and bonds, where the covariance between interest rates and the underlying asset price is relatively large, in absolute magnitude, and the resulting difference between the asset forward and futures prices is significant.

5 Conclusions

The martingale property of asset prices in a no-arbitrage economy is a fundamental result in financial economies. However, the valuation models for contingent claims that follow from this property are often too general to be directly implementable. Many option valuation models in the literature, therefore, have restricted the stochastic process followed by the underlying asset prices and assumed a lognormal diffusion or a square-root process, in order to obtain more specific results, that are useful in practice. Examples of such models for European-style options are the Black and Scholes (1973) model for the lognormal diffusion case and the Cox Ingersoll and Ross (1985) model for the square-root case.

In this paper (in particular in Proposition 3), we have presented necessary and sufficient conditions for the valuation of contingent claims using a commonly used, preference-free valuation relationship, referred to as the Black model. A sufficient condition for the Black model to hold in the case of a daily marked-to-market futures contract on the contingent claim is that the pricing kernel is joint lognormally distributed with the asset price. On the other hand, a set of sufficient conditions for the Black model to hold for the forward price of the claim are that discount factors as well as the pricing kernel are joint-lognormal distributed. Necessary conditions derived suggest that for these pricing relationships to hold in a representative agent, stochastic-interest-rate economy, the agent must have constant-proportional-risk-averse preferences and the conditionally expected zero-bond price must be lognormal.

Supposing that the Black model holds for both the futures price and the forward price of an option, we provide a precise formula for the effect of the mark-to market convention on an option price. If the forward price of the asset is less than the futures price then the effect of stochastic interest rates on the forward price of a call option is unambiguously negative. The effect is captured by replacing the futures price of the asset by its forward price in the Black formula. While the difference between the forward and the futures price of the underlying asset is generally small, the effect on the option price is levered up.[12] The more out of the money is the option, the greater is the relative effect of stochastic interest rates on its value.

Proposition 3 has a direct application when options are traded on a forward or a futures basis. In this regard, it is worth noting that many over-the-counter options on foreign exchange are traded on a forward basis. Also, the Sydney Futures Exchange and the London International Financial Futures Exchange

[12]For empirical estimates of the difference between forwards and futures prices see French (1983) and Hodrick and Strivastava (1987).

trade futures-style options that are settled on a daily marked-to-market basis.

A number of questions remain to be answered. First, to what extent do our results extend to the valuation of American-style and other path-dependent options? Second, is it possible in some cases to derive preference-free valuation models, different from the Black model, when the conditions of Proposition 3 are not met? Finally, exactly how restrictive are the necessary conditions derived in this paper for the zero-bond prices? Do they invalidate use of the Black model in practice? We leave the answers to these questions to await subsequent research.

Appendix: Proof of proposition 3

3a) Sufficiency

First, we use the assumption of joint lognormality of V_s and $\phi_{t,s}$ to find an explicit expression for the futures price of the asset. Using Proposition 1 this futures price is

$$H_{t,s} = \mathrm{E}(V_s \phi_{t,s})$$

Given the lognormality of V_s and $\phi_{t,s}$ this can be specified as

$$H_{t,s} = V_t e^{\mu_\nu + \frac{1}{2}\sigma_{\nu\nu} + \sigma_{\phi\nu}}$$

using the fact that $\mathrm{E}_t(\phi_{t,s}) = 1$. This asset pricing model relates the mean and the covariance of the asset, as follows:

$$\mu_\nu + \sigma_{\phi\nu} = ln[H_{t,s}/V_t] - \frac{1}{2}\sigma_{\nu\nu}$$

Second, we evaluate the conditional pricing kernel. Again using the properties of lognormality

$$\mathrm{E}_t(\phi_{t,s} \mid V_s) = e^{-\mu_\nu \sigma_{\phi\nu}/\sigma_{\nu\nu} - \frac{1}{2}(\sigma_{\phi\nu}^2/\sigma_{\nu\nu})}(V_s/V_t)^{\sigma_{\phi\nu}/\sigma_{\nu\nu}} \tag{11}$$

Now we can find the futures price of the contingent claim. From Proposition 2a this is, in general

$$H_{t,s}[c(V_s)] = \mathrm{E}_t[c(V_s)\phi_{t,s}]$$

Using the law of iterated expectations, this can be written

$$H_{t,s}[c(V_s)] = \mathrm{E}_t[c(V_s)\mathrm{E}_t(\phi_{t,s} \mid V_s)]$$

Defining $z = \ln(V_s/V_t)$ this can be written as

$$H_{t,s}[c(V_s)] = \int_{-\infty}^{\infty} c(V_s)\mathrm{E}_t(\phi_{t,s} \mid V_s)(1/\sigma_\nu\sqrt{2}\pi)e^{-\frac{1}{2\sigma_{\nu\nu}}[z-\mu_\nu]^2}\mathrm{d}z, \tag{12}$$

Substituting from equation (11) into (12) and completing the square, we find

$$H_{t,s}[c(V_s)] = \int_{-\infty}^{\infty} c(V_s)(1/\sigma_\nu\sqrt{2}\pi)e^{-\frac{1}{2\sigma_{\nu\nu}}[z-(\mu_\nu+\sigma_{\phi\nu})]^2}\mathrm{d}z, \tag{13}$$

Now, taking the logarithm of the asset futures price we have

$$H_{t,s}[c(V_s)] = \int_{-\infty}^{\infty} c(V_s)(1/\sigma_\nu\sqrt{2}\pi)e^{-\frac{1}{2\sigma_{\nu\nu}}[z-(\ln H_{t,s}/V_t-\frac{1}{2}\sigma_{\nu\nu})]^2}\mathrm{d}z, \tag{14}$$

Finally, note that (14) can be written as

$$H_{t,s}[c(V_s)] = \int_{-\infty}^{\infty} c(V_s)\hat{f}(V_s)\mathrm{d}V_s, \tag{15}$$

where the mean of V_s under the $\hat{}$ distribution is

$$\hat{\mathrm{E}}_t(V_s) = H_{t,s}$$

and its logarithmic variance is σ_ν^2. Hence the Black model holds for the price of the contingent claim. □

3a) Necessity

The necessity part of Proposition 3a follows directly from Brennan (1979) Theorem I. We simply apply that theorem to the futures price rather than the spot price. We can rewrite equation (4) in the form

$$H_{t,s}[c(V_s)] = \int_{-\infty}^{\infty} [c(V_s)\mathrm{E}_t(\phi_{t,s} \mid V_s)]f(V_s)\mathrm{d}V_s \tag{16}$$

and we know that if the Black Model holds, it implies that

$$H_{t,s}[c(V_s)] = \int_{-\infty}^{\infty} c(V_s)\hat{f}(V_s)\mathrm{d}V_s \tag{17}$$

Since equation (16) and equation(17) must be equivalent for all contingent claims, $c(V_s)$, it follows that

$$\mathrm{E}_t(\phi_{t,s} \mid V_s) f(V_s) = \hat{f}(V_s). \tag{18}$$

Since V_s is lognormally distributed, we can write

$$\begin{aligned} &\mathrm{E}_t(\phi_{t,s} \mid V_s)(1/\sigma_v\sqrt{2}\pi)e^{-\frac{1}{2\sigma_{vv}}[\ln(V_s/V_t)-\mu_v]^2} \\ &\qquad = (1/\sigma_v\sqrt{2}\pi)e^{-\frac{1}{2\sigma_{vv}}[\ln(V_s/V_t)-\mu_v^*]^2} \end{aligned} \tag{19}$$

and

$$\mathrm{E}_t(\phi_{t,s} \mid V_s) = \exp\{-(\mu_v^2 - \mu_{v*}^2)/2\sigma_{vv}\}(V_s/V_t)^{-(\mu_v^* - \mu_v)/\sigma_{vv}} \tag{20}$$

which is lognormal. □

3b) Sufficiency

Part b) of Proposition 3 concerns the special case of the forward price of the contingent claim. Since, from Proposition 2, this forward price is

$$G_{t,s}[c(V_s)] = \mathrm{E}_t(V_s\psi_{t,s})$$

where the stochastic interest rate adjusted pricing function $\psi_{t,s} = \phi_{t,s}g_{t,s}$ again has the property $\mathrm{E}_t(\psi_{t,s}) = 1$, a similar argument can be used to establish Proposition 3b. Note that the explicit expression for the asset forward price is

$$G_{t,s} = V_t e^{\mu_\nu + \frac{1}{2}\sigma_{\nu\nu} + \sigma_{\psi\nu}}$$

The forward price of the contingent claim in this case is

$$G_{t,s}[c(V_s)] = \int_{-\infty}^{\infty} c(V_s) f^{\#}(V_s)\mathrm{d}s$$

where the mean of V_s under the $\#$ distribution is

$$\mathrm{E}_t^{\#}(V_s) = G_{t,s}$$

and its logarithmic variance is again σ_ν^2. □

3b) Necessity

The necessary condition for the forward price of the contingent claim to be given by the Black model in Proposition 3b follows by the same argument as that used in Proposition 3a in the case of the futures price. □

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