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Risk and Return: An Equilibrium Approach

Abstract

Recent empirical evidence suggests, counterintuitively, that expected stock returns are negatively related to the volatility of stock returns at the market level, and that this relation varies substantially over time. This paper investigates this relation in a general equilibrium nominal exchange economy estimated using consumption growth and inflation data. Surprisingly, perhaps, a two regime specification is able to duplicate the salient features of the expected return/volatility relation. Within each regime, the state variables follow a VAR(1), and the probability of a regime shift depends on the level of inflation. In this model, the unconditional correlation between expected returns and volatility is -0.3. Moreover, conditional correlations range from -0.7 to 1.0, depending on the state of the world as described by the levels of consumption growth and inflation. These results highlight the perils of relying on intuition from static models. They also have important implications for the empirical modeling of returns and investment performance evaluation. Finally, the results demonstrate that volatility can be an extremely poor measure of priced risk at the market level.

1 Introduction

Recent empirical studies (e.g., Glosten, Jagannathan, and Runkle (1993), Whitelaw (1994), and Boudoukh, Richardson, and Whitelaw (1996)) document two important stylized facts with regard to the intertemporal relation between equity risk and return at the market level.[1] First, they provide evidence of a negative relation between conditional expected returns and the conditional volatility of returns. Second, they document significant time-variation in this relation. Specifically, in a modified GARCH-M framework, Glosten, Jagannathan, and Runkle (1993) find that the estimated coefficient on volatility in the expected return regression is negative using post WWII monthly data. In a similar dataset, when both conditional moments are estimated as functions of predetermined financial variables, Whitelaw (1994) finds that the unconditional correlation between the fitted moments is negative. Moreover, the correlation varies substantially from approximately -0.8 to 0.8 when measured over 17-month horizons. Finally, in a nonparametric estimation using almost two centuries of annual data, Boudoukh, Richardson, and Whitelaw (1996) find that time-variation in expected returns and the variance of returns, as functions of slope of the term structure, do not coincide.

These empirical results are especially interesting because they run counter to the strong intuition of a positive relation between volatility and expected returns at the market level that comes from such models as the dynamic CAPM (Merton (1980)). Two questions arise naturally. First, are the stylized facts consistent both with general equilibrium and with the time series properties of variables such as consumption and inflation which drive equity returns in these models?[2] Second, what are the features of the model which allow for this counterintuitive behavior of expected returns and volatility?

This paper addresses these two questions in the context of a nominal exchange economy (Lucas (1978)). As such, the exercise is similar in spirit to that of Cecchetti, Lam, and Mark (1990), who attempt to duplicate the serial correlation patterns in equity returns in a relatively simple equilibrium setting. Consumption growth and inflation are modeled as a vector autoregressive process, with potentially multiple regimes in which the parameters differ across regimes (similar to

[1] These papers extend earlier work on the subject by Campbell (1987), and French, Schwert and Stambaugh (1987), among many others.

[2] It is known that equilibrium models can generate a wide variety of relations between the mean and volatility of returns (e.g., Abel (1988), Backus and Gregory (1993)). The question addressed in this paper is whether the more specific intertemporal patterns documented recently are consistent with economic data.

Hamilton (1989)). The parameters of the vector autoregression (VAR) are estimated by matching the means, variances, autocovariances and cross-serial covariances calculated from quarterly data over the period 1953-1994. The stock market is modeled as a claim on aggregate consumption. The quantities of interest are the unconditional correlation between expected equity returns and the volatility of returns and the conditional correlations between these moments of returns in various states of the world. In order to calculate the moments of equity returns, the economy is approximated by a discrete state space economy in which the state variables can take on only a fixed number of discrete values (see Tauchen and Hussey (1991)). The discrete state space (DSS) methodology permits closed form calculations of expected returns and conditional volatility, while generating excellent approximations to the continuous state space.

A single regime model is not able to generate the correlation patterns between expected returns and volatility that are apparent in the data. In fact, these variables are strongly positively correlated both unconditionally and in every state of the world. In contrast, a two regime specification generates both a negative unconditional correlation, and conditional correlations that range from -0.7 to 1.0. The key features of the specification are regime parameters that imply different means, speeds of mean reversion, and variances across the regimes, and regime switching probabilities that depend on the level of inflation. Nevertheless, the aggregate time series properties of consumption growth and inflation are identical to those in the data.

The major contribution of the paper is in establishing the fact that the recent empirical evidence is consistent with reasonable parameterizations of a relatively simple equilibrium model. This finding adds credibility to these empirical stylized facts, and the model provides the economic intuition behind these results. Specifically, the possibility of shifts between regimes which exhibit different risk/return tradeoffs increases volatility while simultaneously reducing the equity risk premium in certain states of the world. The equity risk premium is a function of the covariance between equity returns and the marginal rate of substitution. However, the marginal rate of substitution depends only on next period's consumption growth, while the equity return depends on the infinite future via its dependence on the stock price next period. In states in which a regime shift is likely, this divergence of horizons weakens the link between market returns and the marginal rate of substitution. As a result, the risk premium is low but the volatility of returns is high because of the regime uncertainty. The state dependence of regime shifts is achieved by modeling the probability of a regime shift as a function of level of inflation. This feature also coincides with the importance of nominal interest rates for estimating the moments of equity returns (see, for example, Glosten, Jagannathan, and Runkle (1993) and Whitelaw (1994)).

In addition to the economic intuition that they provide, the results have important implications both for the empirical modeling of equity returns and for investment performance evaluation. Empirical models such as GARCH-M, which impose strong restrictions on the relation between expected returns and volatility, are unable to capture the complexity of the relation demonstrated in this paper. The effects of this misspecification are potentially serious, and investigation of more general specifications is clearly warranted. The results also suggest that performance measures such as Sharpe ratios, which rely on the volatility of returns to measure risk, are potentially misleading. Even at the market level, volatility serves as a poor proxy for priced risk in the two regime specification.

The remainder of the paper is organized as follows. Section 2 develops the asset pricing framework and describes the two regime VAR specification. Section 3 describes the discrete state space pricing procedure for computing the moments of equity returns. The data and estimation technique are described in Section 4, and a single regime model is estimated. In Section 5, we estimate and analyze the two regime model which matches the observed expected return and volatility patterns. Section 6 concludes.

2 The Asset Pricing Framework

From an asset pricing perspective, consider a nominal version of a pure exchange economy with a single consumption good (Lucas (1978)). Assume further the existence of a representative agent whose utility function exhibits constant relative risk aversion. All assets will be priced according to the first order conditions of this agent, giving the standard pricing equation

$$P_t = \beta \, \mathrm{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\alpha} \left(\frac{\pi_t}{\pi_{t+1}} \right) (P_{t+1} + Y_{t+1}) \right], \tag{1}$$

where c_t is real consumption, π_t is the price level, P_t is the nominal, ex-payoff price of the asset, Y_t is the asset's nominal payoff, α is the coefficient of risk aversion, β is the time preference parameter, and $\mathrm{E}_t[\cdot]$ denotes the expectation conditional on information available at time t.[3]

[3]More generally, Harrison and Kreps (1979) show that in the absence of arbitrage there exists a non-negative pricing operator (in this case the nominal marginal rate of substitution) for which this condition holds. This result

Denote the one-period, gross, nominal return on an asset by

$$R_{t+1} \equiv \frac{P_{t+1} + Y_{t+1}}{P_t}, \tag{2}$$

and the one-period, gross, nominal, riskless rate by

$$R_{ft} \equiv \frac{1}{\beta \, \mathrm{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\alpha} \left(\frac{\pi_t}{\pi_{t+1}} \right) \right]}. \tag{3}$$

The expected return on any asset in excess of the riskless rate is proportional to the negative of the covariance of this return with the nominal marginal rate of substitution (NMRS), i.e.,

$$\mathrm{E}_t[R_{t+1} - R_{ft}] = -R_{ft} \, \mathrm{Cov}_t[N_{t+1}, R_{t+1}], \tag{4}$$

where $N_{t+1} \equiv \beta(c_{t+1}/c_t)^{-\alpha}(\pi_t/\pi_{t+1})$ is the NMRS.

In this simple setting it is standard to identify the stock market as the claim on the aggregate consumption stream, i.e., to equate aggregate consumption and the aggregate stock market dividend (see, for example, Cecchetti, Mark, and Lam (1990)). Applying equation (1), the value of equity is

$$S_t = \beta \, \mathrm{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\alpha} \left(\frac{\pi_t}{\pi_{t+1}} \right) (S_{t+1} + C_{t+1}) \right], \tag{5}$$

where $C_{t+1} \equiv c_{t+1}\pi_{t+1}$ is the nominal dividend (and nominal consumption). Substituting for future equity prices, the current value can be written as the sum of discounted future real dividends (consumption) multiplied by the current price level

$$S_t = \pi_t \sum_{s=1}^{\infty} \beta^s \mathrm{E}_t \left[\left(\frac{c_{t+s}}{c_t} \right)^{-\alpha} c_{t+s} \right]. \tag{6}$$

The gross, nominal, 1-period, equity return is

$$\begin{aligned} R_{St+1} &= \frac{S_{t+1} + C_{t+1}}{S_t} & (7)\\ &= \left(\frac{\pi_{t+1}}{\pi_t} \right) \frac{s_{t+1} + c_{t+1}}{s_t} & (8)\\ &= \left(\frac{\pi_{t+1}}{\pi_t} \right) \left(\frac{c_{t+1}}{c_t} \right) \frac{s_{t+1}/c_{t+1} + 1}{s_t/c_t}, & (9) \end{aligned}$$

where s_t is the real equity value, and s_t/c_t is the real (and nominal) price-dividend ratio.

can also be derived in a similar setting with the growth in the money supply as the second fundamental process, instead of inflation. Imposing a cash-in-advance constraint and appropriately specifying the timing of money endowments and consumption decisions produces a unitary velocity of money and the same pricing relation (Labadie (1989)).

This model is not intended to capture all the complexities inherent in equity returns. In fact, similar models have been rejected on the grounds that they cannot match the observed equity premium or other features of the joint time series of equity returns and consumption data.[4] Nevertheless, as will become apparent, this model is both sufficiently complex to produce insight into the time-variation of the mean and volatility of equity returns and sufficiently simple to preserve tractability.

It is not immediately clear how the expected excess equity return and the conditional volatility of this return will be related. The appropriate measure of risk for any asset is the covariation of its return with the NMRS (see equation (4)). Even for the aggregate stock market, this covariance is not necessarily proportional to the variance of returns. Nevertheless, for many specifications the covariance between the market return and the NMRS and the variance of the market return will be closely linked. First, note that the NMRS is positively related to both the inverse of the inflation rate and the inverse of consumption growth (for $\alpha > 0$). Holding the price–dividend ratio constant, the market return is proportional to both inflation and consumption growth. Consequently, it is not unreasonable to expect that the covariance between the NMRS and the return will be negative in this model.[5] Moreover, when inflation and/or consumption growth are volatile, the market return will be volatile, the covariance will be large in magnitude, and the risk premium will be large. It is exactly this intuition that supports a positive relation between the volatility of returns and the risk premium.

In order to overturn this intuition it is necessary to break the link between the NMRS and the market return. This goal can only be achieved by generating variation in the price–dividend ratio that works in the opposite direction, i.e., the price-dividend ratio must covary positively with the NMRS. The NMRS is determined by next period's inflation and consumption growth, while the price–dividend ratio depends on the infinite future of consumption growth (see equation (6)). Therefore, the key is to break the link between what happens next period and the periods thereafter. One potential mechanism for generating this decoupling is regime shifts.

The idea of shifts in regimes has gathered increasing empirical support in the literature (see, for

[4]For early examples, see Hansen and Singleton (1982) and Mehra and Prescott (1985). Numerous attempts have been made to modify the model to better fit the data. These include introducing habit persistence (Constantinides (1990)), consumption adjustment costs (Marshall (1993)), and non-separability of preferences (Epstein and Zin (1989)).

[5]Boudoukh, Richardson, and Whitelaw (1996) illustrate that this is not always the case.

example, the special issue of the *JBES*, July 1994). There exist numerous possible specifications, but for simplicity we consider a two-regime model. In particular, we assume that, at any point in time, the natural logarithms of inflation and consumption growth follow a vector autoregressive process of order 1 (VAR(1)) with multivariate normal errors and a constant covariance matrix. However, we also allow for the possibility of two different VAR regimes. The state processes follow a specified VAR for a number of periods until a regime switch is triggered. These processes then follow a VAR with different parameters until another switch occurs. In particular, the two-regime economy is parameterized as

$$\begin{aligned} X_t &= A_0 + I_{t-1}A_0' + (A_1 + I_{t-1}A_1')X_{t-1} + \epsilon_t + I_{t-1}\epsilon_t' \qquad (10) \\ & \qquad \epsilon_t \sim \text{MVN}(0, \Sigma) \qquad \epsilon_t' \sim \text{MVN}(0, \Sigma') \qquad \text{Cov}(\epsilon_t, \epsilon_t') = 0 \\ X_t &= \begin{bmatrix} \ln \frac{c_t}{c_{t-1}} \\ \ln \frac{\pi_t}{\pi_{t-1}} \end{bmatrix}, \end{aligned}$$

where I_{t-1} takes on the values 0 or 1 and indexes the regime. Extensions to multiple regimes are straightforward. The evolution of the sequence of random variables $\{I_t\}$ is governed by the transition probabilities

$$\Pr[I_t = 0,\ 1|\Phi_{t-1}] = f(I_{t-1}, X_{t-1}), \qquad (11)$$

i.e., the probability of being in a given regime next period depends only on the current regime and the underlying state variables. This parameterization is a generalization of the switching model in Hamilton (1989), which is also studied in the context of stock returns in Cecchetti, Lam, and Mark (1990). In the Hamilton model, the regime affects only the constant term in (10), and the transition between regimes depends only on a history of past regimes.

3 Equity Pricing: The Discrete State Space Methodology

Using the pricing equation (6), and the laws of motion for inflation and consumption (dividend) growth, it is sometimes possible to calculate the conditional moments of the return on equity in closed form. For more complex, multi-regime specifications closed form solutions are no longer available; therefore, we employ a discrete state space methodology that provides approximate numerical solutions.

Consider an n-vector of variables x_t that describes the state of the world at time t and that follows a stationary VAR with a single lag:

$$x_{t+1} = A + Bx_t + \epsilon_{t+1}. \tag{12}$$

The assumption of one lag is made for the convenience of exposition only; longer lags can be handled in the same fashion by simply augmenting the vector of state variables. x_t can be approximated by an n-vector of variables $\widehat{x}_t$ each of which takes on only m discrete values. In this discrete economy there are m^n possible states of the world. The evolution of $\widehat{x}_t$ through time can therefore be described by a $m^n \times m^n$ transition matrix Π whose (i, j) entry is the probability of moving from state i at time t to state j at time $t+1$. The problem, of course, is choosing the discrete values of $\widehat{x}_t$ and the transition probabilities such that $\widehat{x}_t$ best approximates x_t. Tauchen and Hussey (1991) develop such a scheme based on numerical quadrature methods. They choose the discrete points and transition probabilities such that the discretization matches the moments of $\widehat{x}_t$ with those of x_t (see the appendix for details). They also present an extensive discussion of the theoretical convergence of $\widehat{x}_t$ and functions of $\widehat{x}_t$ to their continuous state space analogs as the number of quadrature points goes to infinity.

Given $\widehat{x}_t$ and the transition matrix Π, the solutions to certain expectation equations become relatively easy to calculate. Assume, for example, that log consumption (dividend) growth ($\ln[c_{t+1}/c_t]$) and log inflation ($\ln[\pi_{t+1}/\pi_t]$) follow a VAR(1). Assume further that each will be approximated by an m point discretization (yielding m^2 states of the world). Let $\widehat{l}$ and $\widehat{d}$ denote the m^2 by 1 vectors which contain the values that the real marginal rate of substitution and consumption growth take on in each of the states:[6]

$$\widehat{l} = \left\{ \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\alpha} \right\} \tag{13}$$

$$\widehat{d} = \left\{ \frac{c_{t+1}}{c_t} \right\}. \tag{14}$$

The value of equity in equation (5) can be rewritten in terms of the price–dividend ratio

$$\frac{s_t}{c_t} = \beta \mathrm{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{1-\alpha} \left(\frac{s_{t+1}}{c_{t+1}} + 1 \right) \right]. \tag{15}$$

[6]The time subscript is dropped because the state vectors are the same in every period.

In the discrete state space world the expectation is simply a summation, and this equation can be written as

$$\widehat{v} = \Pi(\widehat{l}. * \widehat{d}. * \widehat{v}) + \Pi(\widehat{l}. * \widehat{d}) \; , \tag{16}$$

which has the solution

$$\widehat{v} = \left[I_{m^2} - (1_{m^2}(\widehat{l}. * \widehat{d})^T). * \Pi\right]^{-1} \left[\Pi(\widehat{l}. * \widehat{d})\right] \; , \tag{17}$$

where $\widehat{v}$ denotes the vector of price–dividend ratios (one entry for each state of the world), I_i is an $i \times i$ identity matrix, 1_i is an i-vector of ones, superscript T denotes transpose, and $.*$ is element-by-element matrix multiplication.

The solution method for pricing finitely-lived securities such as bonds is somewhat different. For example, the price of a 1-period, riskless bond is the expectation of the NMRS

$$Q_{ft} \;\; = \;\; \beta \mathrm{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\alpha} \left(\frac{\pi_t}{\pi_{t+1}} \right) \right], \tag{18}$$

which has the solution (in the discrete state space economy)

$$\widehat{Q} = \Pi \cdot \widehat{N}, \tag{19}$$

where $\widehat{Q}$ is an m^2 vector of bond prices, one for each state of the world, and $\widehat{N}$ is a vector of the NMRS in each state, i.e.,

$$\widehat{N} = \left\{ \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\alpha} \left(\frac{\pi_t}{\pi_{t+1}} \right) \right\} \; . \tag{20}$$

In addition to permitting solutions for asset prices in complex economies, the discrete state space pricing technique allows for relative simple computations of the conditional mean and variance of asset returns. The conditional moments are conditional on the specific state of the world as described by the state variables. The stationary distribution of the states can be found as the solution to $\Pi^T P = P$, where P is the vector of stationary probabilities.

The discussion above focuses on an economy described by a single VAR process on the state variables. The extension to multiple regimes of individual VAR processes is relatively straightforward. Within regime transition probabilities, conditional on remaining within the regime, are derived as before. The probabilities of moving between regimes, conditional on the state, must also be defined. The result is an augmented transition probability matrix and an extended matrix of state variables. Computations then proceed as before.

As an illustration, consider again the problem of estimating the price–dividend ratio, this time in an economy with two regimes. As before, $\widehat{l}$ and $\widehat{d}$ are m^2 by 1 vectors which contain the values of the RMRS and consumption growth in each of the m^2 states. Let Π_1 denote the transition probabilities between these states, using the VAR parameters of regime 1 and assuming there is only a single regime. In other words, $(\widehat{l}, \widehat{d}, \Pi_1)$ defines a single-regime model, and the rows of Π_1 sum to one. Similarly, let $(\widehat{l}, \widehat{d}, \Pi_2)$ define a single-regime model under the VAR parameters of regime 2. The state vectors are identical across regimes, but the transition probabilities depend on the parameters of the two VARs. Assume that the probability of moving to regime 2 next period, conditional on being in regime 1, is state independent and equal to p. Similarly, assume the probability of moving from regime 2 to regime 1 is state independent and equal to q. Construct the augmented matrices $\widehat{l}^*$, $\widehat{d}^*$, and Π^* as follows:

$$\Pi^* = \begin{bmatrix} \Pi_1(1-p) & \Pi_1 p \\ \Pi_2 q & \Pi_2(1-q) \end{bmatrix} \quad \widehat{l}^* = \begin{bmatrix} \widehat{l} \\ \widehat{l} \end{bmatrix} \quad \widehat{d}^* = \begin{bmatrix} \widehat{d} \\ \widehat{d} \end{bmatrix}. \tag{21}$$

There are now $2m^2$ states of the world to reflect the fact that for each value of the RMRS and consumption growth, the necessary information also includes the current regime. Π^* is a valid transition matrix since its rows sum to one. The augmented matrices are used for pricing in (17) and (19).

4 Estimating the Model

4.1 The Data

The model is estimated using quarterly data on per capita, constant dollar (real), total (durable goods, nondurable goods, and services) consumption and the associated implicit price deflator from Citibase. Excluding consumption of durable goods has no qualitative effects on the results. For comparability with earlier studies on both consumption and equity returns, the sample period is 1953-1994.

There are numerous issues with respect to the quality of the data, problems of time aggregation, etc., which are beyond the scope of this paper. Fortunately, the implied intertemporal relation between expected returns and volatility is relatively insensitive to the precise time series properties of the data. The data serve principally to guarantee that the parameterizations of the model are reasonable. As such, the data restrict the potential parameter space and hence mitigate the problem of permitting too many degrees of freedom in the search for parameters which match the time series properties of the mean and volatility of equity returns.

4.2 The Estimation Technique

The discrete state space methodology lends itself naturally to generalized method of moments (GMM, Hansen (1982)) estimation because of the ease of calculating moments of the state variables in the discrete state space world. The difference from a standard GMM estimation is that the moments are not a simple closed form function of the parameters of the model. Nevertheless, it is not necessary to resort to estimation by simulation (Duffie and Singleton (1995)), because, given the parameters and the discretization, the moments are available in closed form.

Consider, for example, the problem of estimating a VAR(1) on real consumption and inflation, i.e., estimating the parameters in

$$\begin{aligned} X_t &= A + BX_{t-1} + \epsilon_t \qquad \epsilon_t \sim \text{MVN}(0, \Sigma) \qquad (22) \\ X_t &= \begin{bmatrix} \ln \frac{c_t}{c_{t-1}} \\ \ln \frac{\pi_t}{\pi_{t-1}} \end{bmatrix}. \end{aligned}$$

A standard GMM estimation would use the moment conditions equivalent to the ordinary least squares normal equations plus an additional three restrictions to identify the elements of Σ. That is, the moment equations would be defined as

$$g(A, B, \Sigma) = \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \epsilon_{1t} \ln \frac{c_{t-1}}{c_{t-2}} \\ \epsilon_{1t} \ln \frac{\pi_{t-1}}{\pi_{t-2}} \\ \epsilon_{2t} \ln \frac{c_{t-1}}{c_{t-2}} \\ \epsilon_{2t} \ln \frac{\pi_{t-1}}{\pi_{t-2}} \\ \epsilon_{1t}^2 - \sigma_{11} \\ \epsilon_{2t}^2 - \sigma_{22} \\ \epsilon_{1t}\epsilon_{2t} - \sigma_{12} \end{bmatrix}, \qquad (23)$$

where

$$\epsilon_{1t} = \ln \frac{c_t}{c_{t-1}} - a_1 - b_{11} \ln \frac{c_{t-1}}{c_{t-2}} - b_{12} \ln \frac{\pi_{t-1}}{\pi_{t-2}} \tag{24}$$

$$\epsilon_{2t} = \ln \frac{\pi_t}{\pi_{t-1}} - a_2 - b_{21} \ln \frac{c_{t-1}}{c_{t-2}} - b_{22} \ln \frac{\pi_{t-1}}{\pi_{t-2}} \tag{25}$$

with the restriction that $\mathrm{E}[g(A, B, \Sigma)] = 0$. The moment conditions are replaced by their sample counterparts, and the parameters are chosen to minimize the quadratic form $g'Wg$ for a suitable choice of the weighting matrix W. This system of equations is exactly identified (i.e., there are nine parameters and nine moment restrictions); therefore, the sample moments can be set to zero. The resulting parameter estimates are identical to those from ordinary least squares estimation, although standard errors may differ.

Estimation using the discrete state space methodology differs slightly from the procedure above. Note that the moment conditions above are essentially setting the variances, covariances, and serial covariances implied by the VAR equal to those in the data. In the discrete state space setting, parameters are chosen to set the moments in the discrete economy equal to those in the data. Given a specific discretization scheme and the number of states of the world, each set of parameters $\{A, B, \Sigma\}$ implies a unique state vector of inflation (denoted i) and consumption growth (denoted d) in each state and a transition matrix (Π). This transition matrix implies, in turn, a stationary distribution P. These vectors and matrices are sufficient to calculate all the moments of the state variables. For example, the mean of consumption growth is $\mathrm{E}[\ln(c_t/c_{t-1})] = P^T d$, the sum of state-by-state consumption growth weighted by the unconditional probability of each state. The cross serial covariance between consumption and inflation is $\mathrm{E}[\ln(\pi_{t-1}/\pi_{t-2})\mathrm{E}_{t-1}[\ln(c_t/c_{t-1})]] =$

$P^T((\Pi d).*i)$. Of course, these moments are functions of the original parameters. Define the vector

$$z_t = \begin{bmatrix} \ln \frac{c_t}{c_{t-1}} \\ \ln \frac{\pi_t}{\pi_{t-1}} \\ \ln \frac{c_t}{c_{t-1}} \ln \frac{c_{t-1}}{c_{t-2}} \\ \ln \frac{c_t}{c_{t-1}} \ln \frac{\pi_{t-1}}{\pi_{t-2}} \\ \ln \frac{\pi_t}{\pi_{t-1}} \ln \frac{c_{t-1}}{c_{t-2}} \\ \ln \frac{\pi_t}{\pi_{t-1}} \ln \frac{\pi_{t-1}}{\pi_{t-2}} \\ (\ln \frac{c_t}{c_{t-1}})^2 \\ (\ln \frac{\pi_t}{\pi_{t-1}})^2 \\ \ln \frac{c_t}{c_{t-1}} \ln \frac{\pi_t}{\pi_{t-1}} \end{bmatrix}. \tag{26}$$

In the discrete state space world, $\mathrm{E}[z_t] = f(A, B, \Sigma)$. Denote the sample counterpart $\mathrm{E}[z_t] = \hat{z}$, i.e., the moments of consumption growth and inflation calculated from the data. The natural analog of the estimation above is to find the parameters that minimize the quadratic form $(f - \hat{z})'W(f - \hat{z})$. These parameter estimates will differ from the standard VAR estimates for two reasons: (1) in finite samples, the slightly different moments conditions will give slightly different results, (2) for a finite number of discrete states, the discrete state space economy will not be exactly equivalent to the continuous state space economy. The number of discrete states necessary to get "good" estimates is an empirical question which is addressed in the following section.

4.3 The Base Case

In order to get an idea of how well the discrete state space economy and the corresponding estimation technique works, we first estimate the VAR(1) in equation (22) using both the methodology of the previous section and standard GMM. The GMM results are reported in Table 1, with heteroscedasticity consistent standard errors in parentheses.

The results are consistent with previous results in the literature (e.g., Boudoukh (1993)). Consumption growth is positively related to lagged consumption growth and negatively related to lagged inflation. Inflation is more highly autoregressive, and is weakly related to lagged consumption growth. The innovations in consumption growth and inflation are weakly negatively related. There is some evidence that one lag is not sufficient to capture the full dynamics at a quarterly frequency. Specifically, additional coefficients exhibit statistical significance in extended specifica-

	Constant	$\ln(c_{t-1}/c_{t-2})$	$\ln(\pi_{t-1}/\pi_{t-2})$	Σ	
		GMM			
$\ln(c_t/c_{t-1})$	0.610	0.200	-0.205	0.422	-0.026
	(0.116)	(0.075)	(0.101)	(0.052)	(0.020)
$\ln(\pi_t/\pi_{t-1})$	0.153	0.065	0.818		0.149
	(0.067)	(0.043)	(0.050)		(0.019)
		DSS			
$\ln(c_t/c_{t-1})$	0.621	0.192	-0.213	0.423	-0.027
	(0.222)	(0.099)	(0.105)	(0.055)	(0.035)
$\ln(\pi_t/\pi_{t-1})$	0.116	0.066	0.853		0.162
	(0.222)	(0.215)	(0.144)		(0.058)

Table 1: Estimation of VAR(1) on Consumption Growth and Inflation

Results from estimation of a VAR(1) on quarterly consumption growth and inflation for the period 1953-1994. The GMM estimation uses the moments conditions given in equation (23). The DSS estimation uses the moment conditions in equation (26). Heteroscedasticity consistent standard errors are in parentheses.

tions. Nevertheless, explained variation increases little with the addition of more lags. For ease of exposition, the remainder of the paper considers only the VAR(1), but the results are qualitatively similar for other specifications.

The same VAR(1) parameters are also estimated using the discrete state space economy. The parameters are chosen to set the nine moments of consumption growth and inflation given in equation (26) equal to their sample counterparts. Five discrete states are used for both consumption growth and inflation, giving twenty-five states in total. The results are again given in Table 1, with heteroscedasticity consistent standard errors in parentheses.

The parameter estimates are very similar to those from the standard GMM estimation. In every case the standard errors are somewhat larger, more so for the inflation equation than the consumption growth equation. The differences in the estimates and the increase in standard errors can potentially be attributed to two factors. First, the GMM estimation uses the OLS moment conditions, which, under certain assumptions, will be more efficient than other sets of moment conditions that identify the same parameters. Second, the discrete approximation may make it harder to accurately identify the parameters.

The role of the discretization can be illustrated by examining the parameter estimates and standard errors as the number of discrete states increases. Rather than looking at all nine parameters, we focus on the coefficient on lagged inflation in the inflation equation. The results for the other

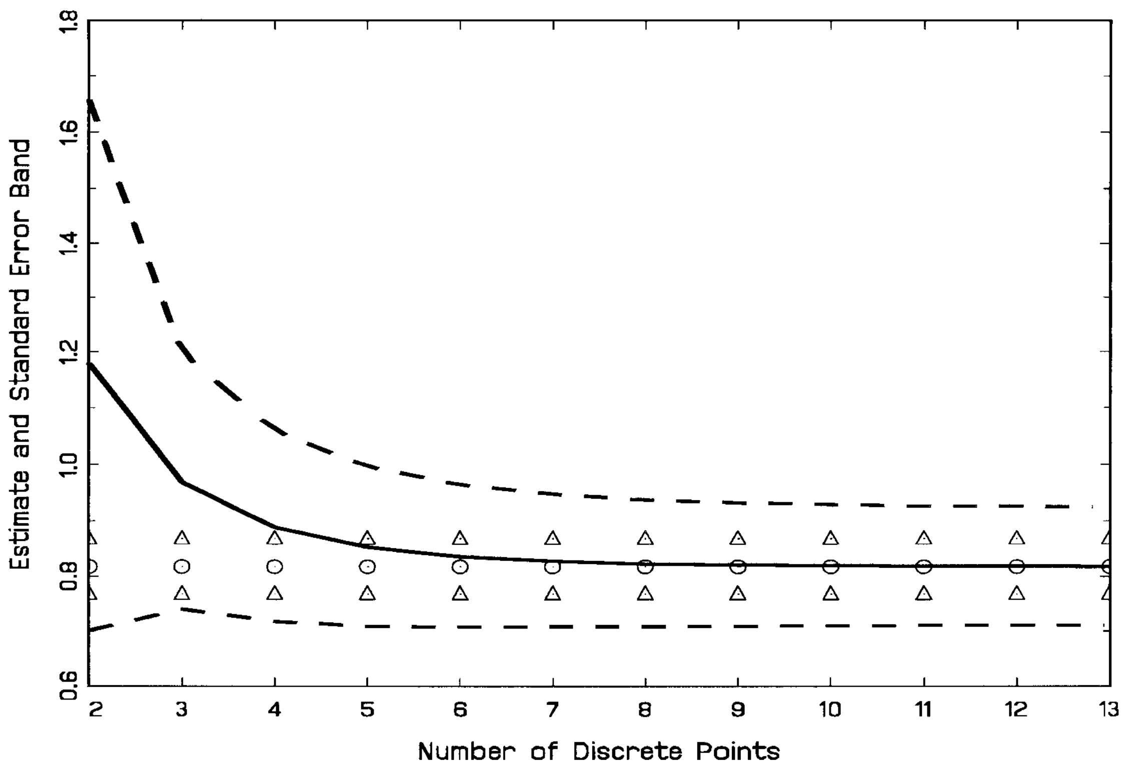


Figure 1: Parameter Estimates for Different Discrete Approximations
DSS estimates of the coefficient on lagged inflation in the inflation regression as the number of discrete states increases. Estimates are shown by the solid line, and the one standard error band is given by the dashed lines. The standard GMM estimate and the corresponding one standard error band are shown by circles and triangles respectively.

parameters are similar. Figure 1 shows the DSS parameter estimate (solid line) and a one standard error band (dashed lines) as the number of discrete points increases from two to thirteen. The GMM parameter estimate and one standard error band are shown as circles and triangles respectively. It is clear from the graph that the DSS estimate converges to the GMM estimate as the number of discrete points increases. The standard errors fail to achieve the level of the GMM standard errors because of the less efficient moments conditions. For the purposes of this paper, however, we are primarily interested in the point estimates and the implied behavior of equity returns, therefore the magnitude of the standard errors is not an issue.

Given the parameter estimates, it is straightforward to see what the VAR(1) implies for the properties of the DSS economy. For this analysis we use nine discrete states for both consumption growth and inflation, at which point convergence to the standard GMM parameter estimates has been obtained. Figure 2 shows consumption growth and inflation in the eighty-one (9^2) states of the world. Quarterly inflation takes on nine different values varying from just more than -1% to just less than 3%. On the other hand, quarterly consumption growth takes on eighty-one different values between -3% and 4%. The number of different consumption growth states is an artifact of the method of discretization. The discretization is actually done for two conditionally independent processes. The negative conditional correlation between the two state variables causes the non-orthogonalized second state variables (in this case consumption growth) to take on more than nine values (see the appendix for details).

For further analysis we need to specify the degree of risk aversion and the time preference parameter. Throughout this paper we use $\alpha = 2$ and $\beta = .997$. These values are theoretically reasonable, and other values produce qualitatively similar results. The implied riskless rate of interest varies from 1.9% to 16.9% on an annualized basis. Of greater interest are the values of the expected equity risk premium and the volatility of equity returns. The DSS economy suffers from the same equity risk premium puzzle as its continuous state space counterpart. For these particular parameter values, the risk premium is too small by a factor of approximately one hundred. Addressing this anomaly is beyond the scope of this paper, and the analysis focuses on the relative values of the risk premium in different states of the world. Figure 3 plots the conditional expected risk premium (multiplied by one hundred) against the conditional volatility of the risk premium.

The graph shows a strong positive relation between the risk premium and the volatility of

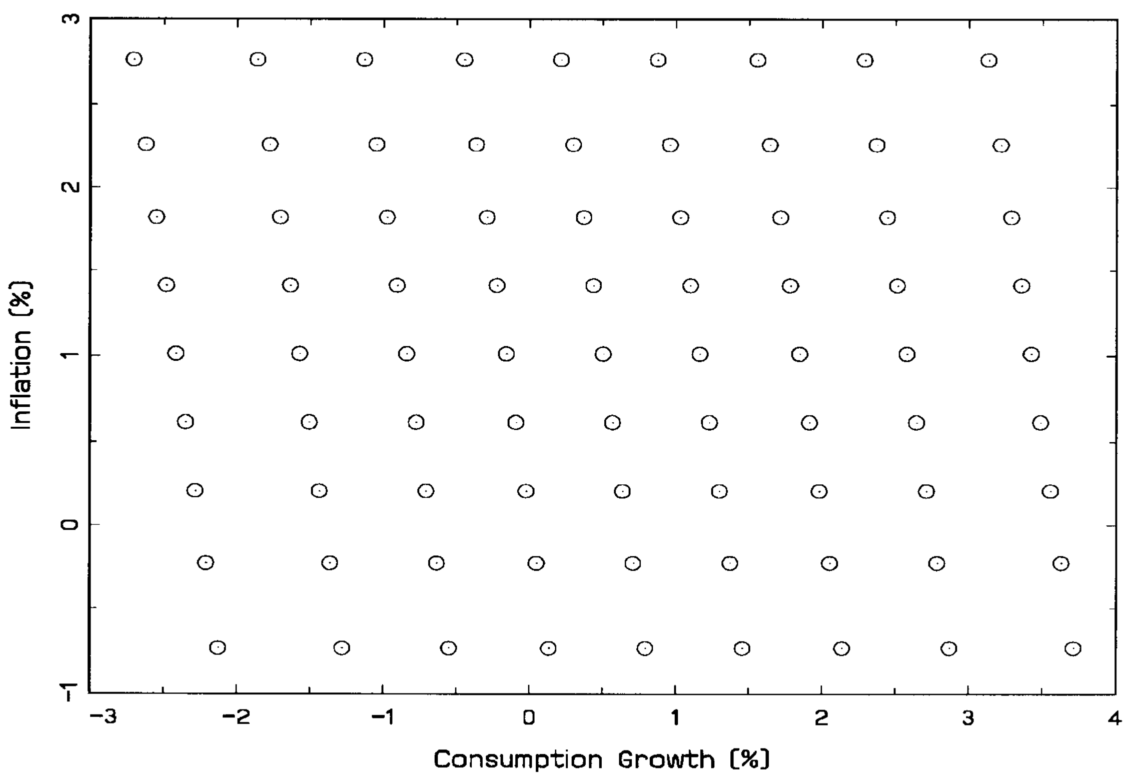


Figure 2: Consumption Growth and Inflation in the DSS Economy
State by state consumption growth and inflation in the DSS economy where each state variable is approximated by a nine point discretization. Parameter estimates come from a VAR(1) estimation using quarterly data.

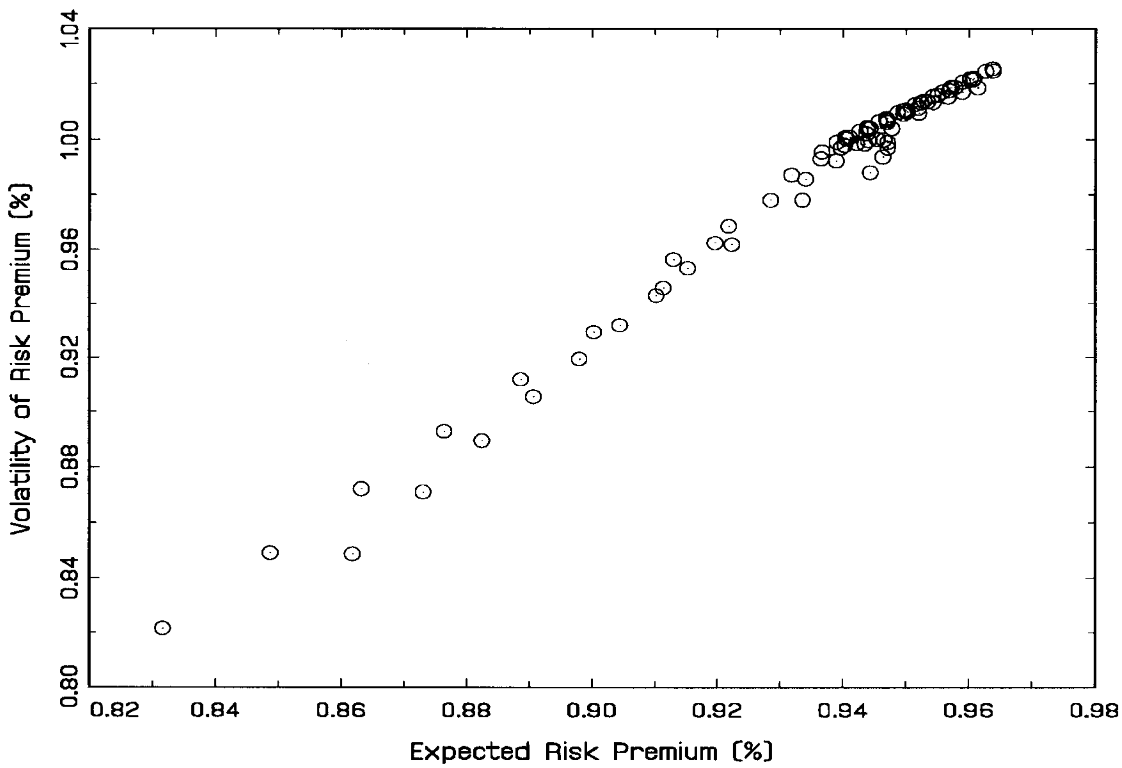


Figure 3: The Risk Premium and the Volatility of Returns
State by state values of the conditional equity risk premium (times 100) and the conditional volatility of returns. Parameter estimates come from a VAR(1) estimation using quarterly data.

returns. This result coincides with the intuition of a risk/return tradeoff at the market level in a dynamic CAPM setting (Merton (1980)). However, there is also some evidence that the relation is not monotonic, especially at lower levels of the risk premium. Unfortunately, the plot is somewhat misleading for judging the importance of this effect. The risk premium and volatility are plotted for each of the eighty-one states of the world, but the probability of each of these states occurring varies a great deal. For example, the state with the lowest risk premium corresponds to extreme values for both inflation and consumption growth. The unconditional probability of observing this particular state is less than 0.00002%. In other words, this risk premium will be observed, on average, less than once every one million years.

A more accurate picture of the relation between the expected risk premium and the volatility is the correlation between these conditional moments of returns. The unconditional correlation is 0.990, suggesting a very strong positive relation. We are also interested in potential time-variation in this relation. In the context of the DSS economy, time-variation is equivalent to variation across different states of the world. The most natural state-by-state measure is the conditional correlation between the conditional expected risk premium and the conditional volatility. Of course, at time $t+1$, the conditional moments based on time $t+1$ information are known. Therefore, we consider the correlation at time t between the conditional expected risk premium and conditional volatility at time $t+1$. For example, suppose the economy is in a particular state (out of the possible 81 states) at time t. Next period (time $t+1$) the economy can be in any of the eighty-one states (with different probabilities), with corresponding conditional risk premiums and volatilities. The question we want to answer is whether high risk premiums are associated with high volatilities in these subsequent states? Conditional correlations will vary across states because conditional transition probabilities vary across states.

Table 2 presents the eighty-one state by state conditional correlations. States are indexed by their level of inflation and consumption growth from lowest (1) to highest (9). The conditional correlations range from a low of 0.937 to a high of 1.000. In every state of the world, the conditional risk premium and conditional volatility are expected to be strongly positively related. There is, however, a systematic pattern with lower correlations for medium levels of both inflation and consumption growth. The correlations are, in general, more sensitive to the level of inflation. The results, in conjunction with those in Figure 3, represent a base case with which to compare more

$\ln(c_t/c_{t-1})$	$\ln(\pi_t/\pi_{t-1})$								
	1	2	3	4	5	6	7	8	9
1	1.000	0.999	0.998	0.992	0.984	0.971	0.944	0.989	0.998
2	1.000	0.999	0.997	0.990	0.984	0.961	0.954	0.992	0.998
3	0.999	0.999	0.997	0.988	0.983	0.950	0.964	0.994	0.999
4	0.999	0.999	0.996	0.987	0.982	0.942	0.972	0.995	0.999
5	0.999	0.999	0.995	0.985	0.979	0.937	0.978	0.996	0.999
6	0.999	0.999	0.994	0.985	0.975	0.937	0.983	0.997	0.999
7	0.999	0.998	0.992	0.984	0.968	0.942	0.987	0.998	0.999
8	0.999	0.998	0.990	0.984	0.958	0.951	0.990	0.998	0.999
9	0.999	0.997	0.988	0.983	0.945	0.963	0.993	0.998	0.999

Table 2: State by State Conditional Correlations
State by state conditional correlations between the conditional risk premium and the conditional volatility. States are indexed by their level of consumption growth and inflation from low (labeled 1) to high (labeled 9).

complex economies.

5 Matching the Moments of Returns

5.1 Estimating a Two Regime Model

The nominal consumption-based model with state variables that follow a single regime VAR(1) calibrated to inflation and consumption growth data is clearly not able to match two salient features of the data: (i) a negative unconditional relation between the risk premium and volatility, and (ii) significant time-variation in this relation. There are numerous generalizations of this model, including non-time separable preferences, heterogeneous agents, transactions costs, etc., which could conceivably alter these results. One simple extension, however, is to look at more general stochastic processes for the state variables. In particular, we consider a two regime economy estimated from the same data used in the base case analysis above.

Recall the specification in equations (10) and (11) specialized to a single lag

$$X_t = A_0 + I_{t-1}A_0' + (A_1 + I_{t-1}A_1')X_{t-1} + \epsilon_t + I_{t-1}\epsilon_t' \tag{27}$$

$$\epsilon_t \sim \text{MVN}(0,\Sigma) \qquad \epsilon_t' \sim \text{MVN}(0,\Sigma') \qquad \text{Cov}(\epsilon_t,\epsilon_t') = 0$$

$$X_t = \begin{bmatrix} \ln \frac{c_t}{c_{t-1}} \\ \ln \frac{\pi_t}{\pi_{t-1}} \end{bmatrix}$$

$$\Pr[I_t = 0,\ 1|\Phi_{t-1}] = f(I_{t-1}, X_{t-1}), \tag{28}$$

wherein consumption growth and inflation can follow one of two VAR(1) processes, depending upon the current regime. Ignoring for a moment the regime switching process, the specification has eighteen free parameters. Clearly, the nine moment conditions in equation (26) are insufficient to identify all of these parameters. In a GMM framework, there are two alternatives – add additional moment restrictions or place restrictions on the parameters to reduce the degrees of freedom. There are also alternative methodologies for estimating regime switching models (e.g., Gray (1995)). Given the quality of the data and the simplicity of the model, there is little to be gained from linking the specification more closely to the data. The major focus of the paper is to explore volatility and expected return patterns in an equilibrium setting. Consequently, we impose the following restrictions on the parameters

$$A_0' = A_0 \tag{29}$$

$$A_1' = A_1 + \begin{bmatrix} 0.1 & -0.02 \\ 0.0 & 0.05 \end{bmatrix} \tag{30}$$

$$\Sigma' = \Sigma + \begin{bmatrix} 0.05 & 0.0 \\ 0.0 & 0.01 \end{bmatrix}, \tag{31}$$

which reduces the degrees of freedom to nine. The two regimes are relatively close in terms of parameter values, with changes in the speed of mean reversion, and the means and variances of inflation and consumption growth across regimes. One might consider the two regimes to be the expansionary and contractionary phases of the economy, or alternatively different Federal Reserve policy regimes. In either case, the magnitudes of the parameter shifts are plausible. It should also be noted that there is nothing unique about this particular set of restrictions. The potential parameter space is huge, and different sets of restrictions produce different results. These restrictions are chosen because they provide an interesting set of results which provides insights into risk and return in this equilibrium setting.

The remaining issue is the transition probabilities between regimes. For simplicity, these probabilities are set exogenously. In both regimes the probability of moving to the other regime is independent of the level of consumption growth. In regime 2, the probability of a regime shift increases in the level of inflation. In particular, for the nine levels of inflation, from lowest to highest, the probabilities of switching into regime 1 the following period are (0%,0%,0%,0%,0%,6%,12%,18%,24%) respectively. In regime 1, the probability of a regime shift decreases in the level of inflation, with probabilities of (24%,18%,12%,6%,0%,0%,0%,0%,0%). One interpretation of this structure is that regime switches are triggered by Federal Reserve policy changes, which, in turn, are influenced by the level of inflation. Alternatively, one might believe that the inflation level signals transitions between phases of the business cycle for other unspecified reasons. The role of these transition probabilities, and, more generally, of regime switches is addressed in the following section.

Given the structure above, estimation proceeds as before. The procedure is to search for the parameters which generate a DSS economy whose moments match the moments of consumption growth and inflation in the data. Table 3 gives the estimated parameters for the two regimes using the moment conditions in equation (26).

	Constant	$\ln(c_{t-1}/c_{t-2})$	$\ln(\pi_{t-1}/\pi_{t-2})$	Σ	
		Regime 1			
$\ln(c_t/c_{t-1})$	0.619	0.170	-0.197	0.411	-0.025
$\ln(\pi_t/\pi_{t-1})$	0.140	0.056	0.780		0.140
		Regime 2			
$\ln(c_t/c_{t-1})$	0.619	0.270	-0.217	0.461	-0.025
$\ln(\pi_t/\pi_{t-1})$	0.140	0.056	0.830		0.150

Table 3: Estimation of Two Regime VAR System
Results from estimation of a regime switching VAR(1) on quarterly consumption growth and inflation for the period 1953-1994. The DSS estimation uses the moment conditions in equation (26).

For the parameters that are allowed to vary across regimes, the estimates straddle those of the single regime model reported in Table 1. Although regime switches affect the autocovariances, cross-serial covariances, and variances of consumption growth and inflation, the probability of switching is small enough that these unconditional moments are close to the average of the moments across the regimes, assuming no regime switches. The next section examines the implications of this regime switching specification for the comovements of the mean and volatility of equity returns.

5.2 Correlation Results

The two regime specification has a total of 162 states of the world, 81 within each regime. The 81 states in each regime are identical in terms of their levels of consumption growth and inflation, but they differ with respect to their transition probabilities, both because of the differing VAR parameters and the differing switching probabilities. Consequently, the conditional expected risk premium and the conditional volatility of returns can take on 162 different values.

Figure 4 graphs these state by state conditional moments. Regime 1 states are marked by circles and regime 2 states are marked by squares. Note that the points tend to form "lines" of nine states. These groups correspond to a single level of inflation and nine different levels of consumption growth. For example, the "line" in the upper left corner of the graph (low expected risk premium, high volatility) corresponds to an inflation rate of 2.5% and consumption growth varying from -2.6% (highest risk premium and volatility) to 3.1% (lowest risk premium and volatility). In general, the risk premium and the volatility are less sensitive to the level of consumption growth than to the level of inflation. This result is to be expected because inflation is the much more autoregressive of the two variables. Consequently, current inflation has a large effect on expected inflation. Consumption growth varies substantially, but the effect of current consumption growth on expectations is much smaller.

The most notable feature of Figure 4, relative to the single regime results in Figure 3, is the weakening of the relation between volatility and the risk premium. In general, the points suggest a negative relation between the risk premium and volatility. In fact, the unconditional correlation between the first two conditional moments of returns is -0.328. This contrasts markedly with the 0.990 correlation in the single regime model. Of greater importance, it coincides with the empirical results in Glosten, Jagannathan and Runkle (1993) and Whitelaw (1994). Both of these papers report a negative relation between conditional expected returns and conditional volatility. The analysis here shows that this negative relation is consistent both with general equilibrium and with the fundamental time series properties of consumption growth and inflation.

The role of regime switches can also be seen in the graph. Consider regime 1. From the five lowest inflation states there is zero probability of switching to regime 2 in the following period. These states correspond to the 45 points marked by circles in the lower right hand corner of the graph. When there is no chance of an immediate regime shift, the relation between the risk premium and volatility looks much like it does in the single regime world. However, as the probability of a regime shift increases, so does the conditional volatility of returns. The uncertainty about next period's regime increases volatility, but there is no corresponding increase in the risk premium. In fact, the risk premium decreases because regime switches reduce the link between the marginal rate of substitution and the return on the market. The real return on the market consists of two components, the dividend and the capital gain. The capital gain is a function not only of current

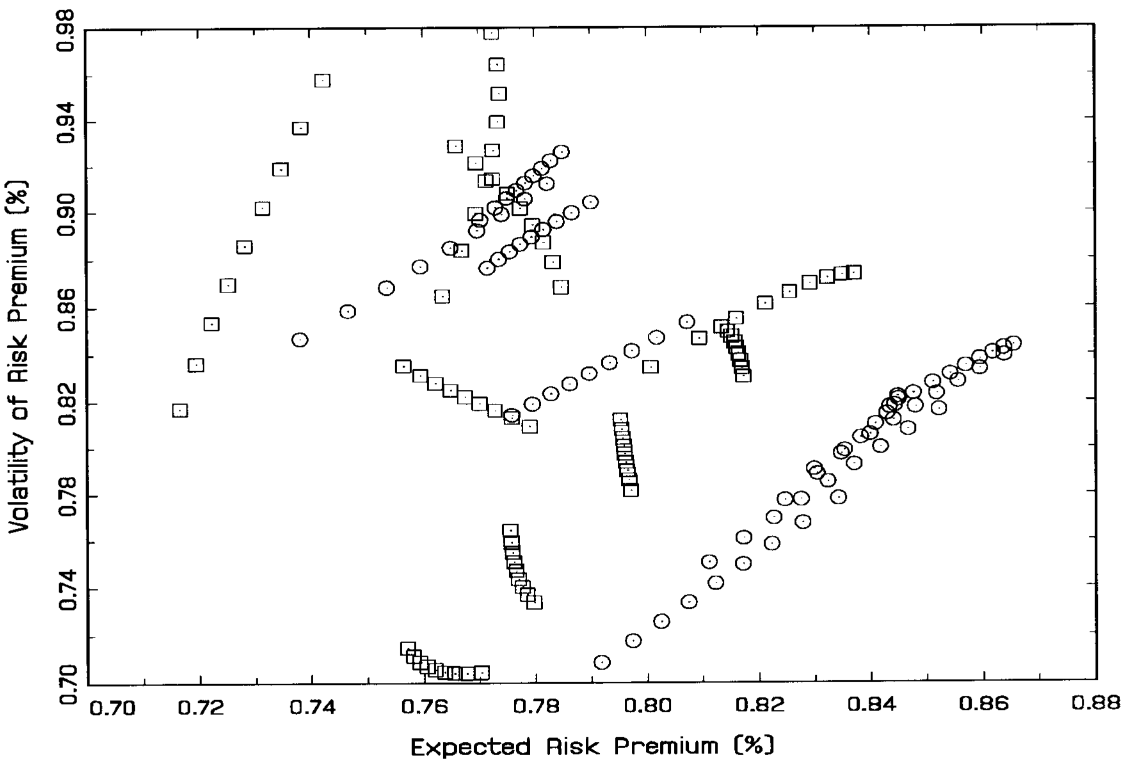


Figure 4: The Risk Premium and the Volatility of Returns
State by state values of the conditional equity risk premium (times 100) and the conditional volatility of returns. Parameter estimates come from a regime switching VAR(1) estimation using quarterly data. States in regimes 1 and 2 are marked by circles and squares respectively.

consumption/dividend growth, but also of the infinite future of consumption/dividend growth. In contrast, the RMRS depends only on current consumption growth. In the presence of regime shifts, the future and present are less closely linked and the marginal rate of substitution and the return are less negatively correlated. A similar pattern is exhibited by the states in regime 2.

A slightly different perspective on the role of regime shifts can be gained by looking at the two regimes individually, as if they were each single regime economies. Figure 5 graphs the state by state levels of the risk premium and the volatility for these two economies. Regimes 1 states are marked by circles and regime 2 states are marked by squares. As expected, this graph bears a strong resemblance to the single regime economy picture in Figure 3. If there are no regime switches there is a strong positive relation between the risk premium and the volatility in both regimes. Regime 2 exhibits higher risk premiums and volatilities partly because of the higher conditional volatilities. Note that the states in the two regimes correspond to exactly the same levels of consumption growth and inflation. However, as the parameters change so do the volatility and the risk premium. Allowing the possibility of a regime shift is all that is necessary to create the complex dynamics inherent in Figure 4.

The contrast between the single regime model and the two regime model is equally apparent when considering the conditional correlation between the risk premium and the volatility. Table 4 reports these state by state conditional correlations. As before, the correlations are ordered by the levels of consumption growth and inflation in each state of the world. They are also separated by regime. Figure 6 presents the same information in a graphical form. The levels of consumption growth and inflation are shown on the x-axis and y-axis, and the conditional correlation is plotted on the z-axis.

Conditional correlations range from a low of -0.696 to a high of 0.994. This contrasts with a range of 0.937 to 1.000 for the single regime specification. The same parameters which generate a negative unconditional correlation, generate substantial time-variation in this relation. This time-variation is again consistent with results in the empirical literature (Boudoukh, Richardson, and Whitelaw (1996), Whitelaw (1994)). Note that the principal determinant of the conditional correlation is the level of inflation. In regime 1, low inflation states have low correlations and high inflation states have high correlations, while in regime 2 the reverse is true. Correlations vary less across levels of consumption growth. This relative insensitivity to consumption growth is

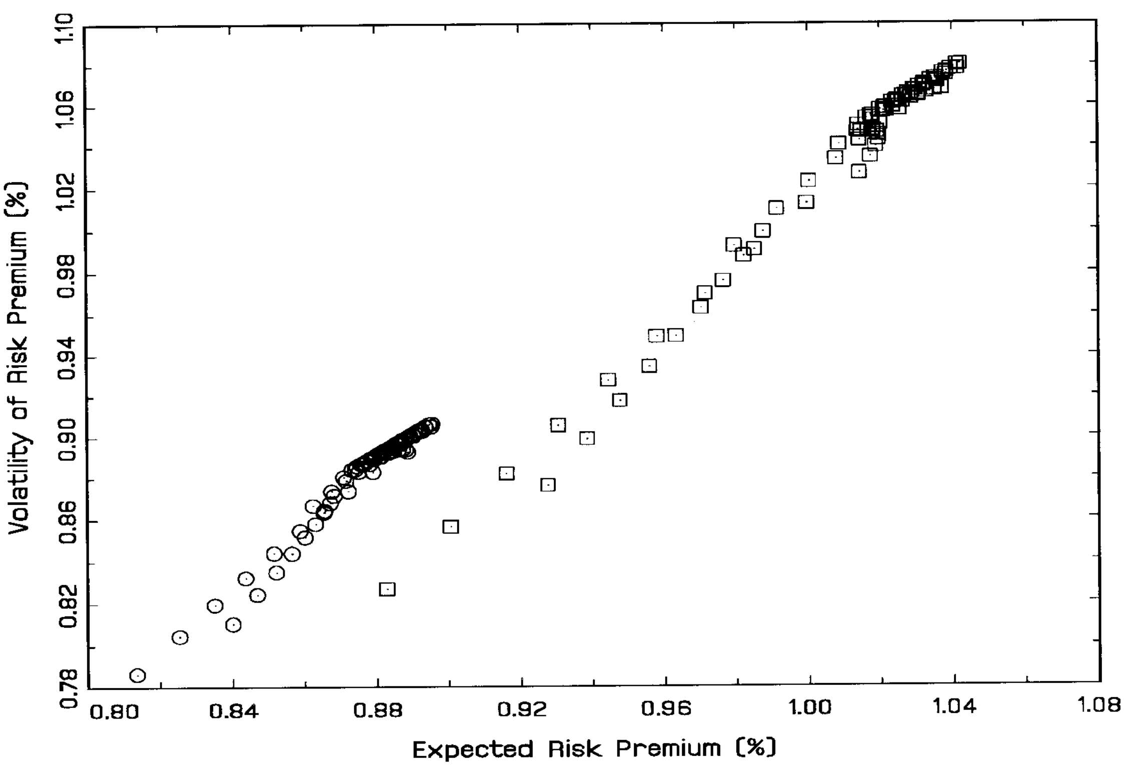
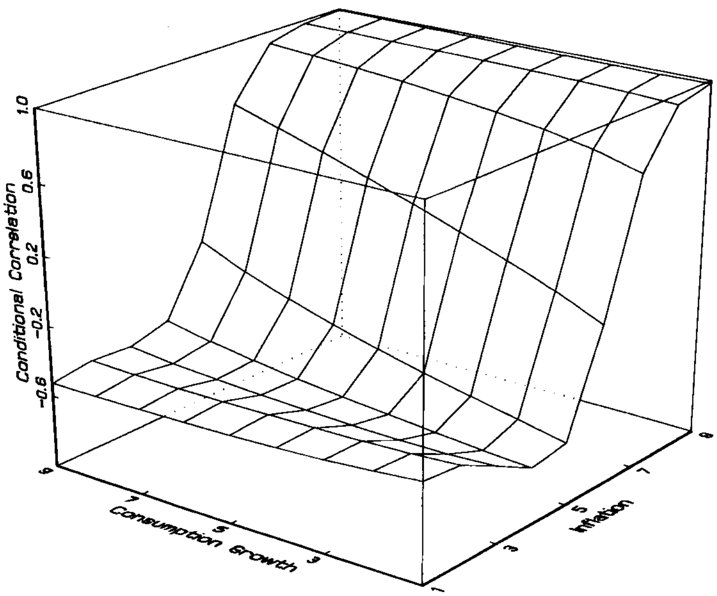


Figure 5: The Risk Premium and the Volatility of Returns
State by state values of the conditional equity risk premium (times 100) and the conditional volatility of returns. Parameter estimates come from the individual regimes within the regime switching VAR(1) estimation using quarterly data.

$\ln(c_t/c_{t-1})$	$\ln(\pi_t/\pi_{t-1})$								
	1	2	3	4	5	6	7	8	9
					Regime 1				
1	-0.434	-0.458	-0.569	-0.686	-0.654	-0.094	0.659	0.944	0.982
2	-0.462	-0.476	-0.582	-0.668	-0.600	0.030	0.729	0.956	0.981
3	-0.483	-0.487	-0.586	-0.648	-0.545	0.139	0.779	0.964	0.981
4	-0.498	-0.496	-0.584	-0.625	-0.486	0.240	0.817	0.969	0.982
5	-0.510	-0.501	-0.577	-0.600	-0.421	0.334	0.849	0.973	0.984
6	-0.518	-0.504	-0.566	-0.572	-0.348	0.423	0.875	0.975	0.987
7	-0.524	-0.505	-0.551	-0.538	-0.265	0.506	0.897	0.977	0.989
8	-0.526	-0.501	-0.532	-0.497	-0.169	0.585	0.916	0.979	0.992
9	-0.525	-0.491	-0.505	-0.442	-0.049	0.664	0.934	0.981	0.994
					Regime 2				
1	0.984	0.987	0.910	0.537	0.369	0.148	-0.214	-0.581	-0.696
2	0.988	0.985	0.883	0.487	0.427	0.140	-0.253	-0.577	-0.674
3	0.990	0.982	0.855	0.453	0.484	0.125	-0.284	-0.564	-0.655
4	0.991	0.979	0.823	0.433	0.538	0.107	-0.310	-0.543	-0.638
5	0.992	0.975	0.788	0.425	0.585	0.084	-0.329	-0.517	-0.622
6	0.992	0.970	0.749	0.429	0.623	0.059	-0.342	-0.485	-0.607
7	0.992	0.962	0.707	0.445	0.649	0.029	-0.346	-0.448	-0.593
8	0.992	0.953	0.661	0.474	0.660	-0.003	-0.338	-0.405	-0.579
9	0.992	0.938	0.610	0.518	0.647	-0.038	-0.313	-0.354	-0.565

Table 4: State by State Conditional Correlations

State by state conditional correlations between the conditional risk premium and the conditional volatility. States are indexed by their level of consumption growth and inflation from low (labeled 1) to high (labeled 9).



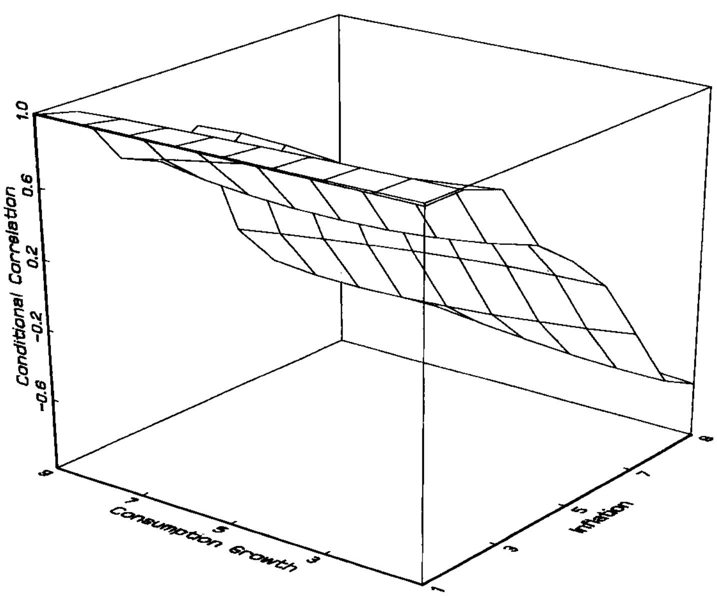


Figure 6: Conditional Correlations
State by state values of the conditional correlation between the equity risk premium and the volatility of returns in the regime switching VAR(1). Parameter estimates come from the DSS estimation using quarterly data.

attributable to the smaller effect of current consumption growth on future economic conditions. In both regimes the negative correlations are generally associated with those states in which a regime shift is possible. This effect spills over to an adjacent intermediate state in some cases because from this state there is a significant probability of entering a regime shifting state in the following period.

5.3 Sensitivity Analysis

One interesting question is what features of the regime switching model make negative unconditional and conditional correlations between the risk premium and volatility possible? There are only two basic requirements. First, the two regimes are sufficiently different to generate different volatility/expected return tradeoffs. In the example above these differences are in the speed of mean reversion, the long-run mean, and the volatility of both inflation and consumption growth. Similar results can be generated by focusing on only one of the processes, or fewer of the parameters, but the differences across the regimes on these dimensions must be larger. Second, the transition probabilities depend on the level of the state variables. In other words, the probability of a regime shift is related to either consumption growth or inflation. When linked to inflation, the effect is stronger because of the higher degree of autocorrelation in this variable. As an illustration, when the probability of a regime switch is set to 6% in every state of the world, the unconditional correlation rises to 0.6 and the conditional correlation in negative in only 10 of the 162 states.

6 Conclusion

A two regime nominal exchange economy, calibrated to the time series behavior of consumption growth and inflation data, is able to duplicate two interesting features of the empirical relation between expected returns and volatility. Specifically, the model generates a negative unconditional correlation between these moments of returns and substantial time-variation in this relation. The results are driven by the different risk/return tradeoffs in the two regimes and a regime switching process that depends on the level of inflation. This paper demonstrates not only that a negative and time-varying relation between expected returns and volatility is consistent with rational expectations, but also that such a relation is consistent with aggregate consumption and inflation data in a representative agent framework.

Given the importance of regime shifts to the results, this paper indicates that further research in this area is warranted, both for economic data such as industrial production, consumption and, inflation and for financial data such as interest rates and stock returns. Initial interest rate evidence (Bekaert, Hodrick and Marshall (1995), Gray (1995)), provides support for multi-regime models.

One important implication of the results is that empirical models that impose a strong, often linear, relation between expected returns and volatility, such as GARCH-M, need to be employed with caution. The time series behavior implied by the model in this paper is inconsistent with these empirical specifications. A potentially promising approach is to model both expected returns and volatility as functions of predetermined financial variables nonparametrically, thus allowing the data to tell the story (e.g., Boudoukh, Richardson, and Whitelaw (1996)).

A second implication of the results is that volatility is not a good proxy for priced risk at the market level. The extent of this breakdown in a relatively simple setting is surprising. As a result, performance evaluation using traditional measures such as Sharpe ratios may be misleading. By exploiting periods of time when volatility and expected returns are unrelated, or even negatively related, investment managers may be able to inflate the Sharpe ratios of their portfolios without any improvement in the true risk/return characteristics.

Appendix: Discretization Methodology

Consider the problem of determining a discrete approximation to the VAR(1)

$$x_{t+1} = A + Bx_t + \epsilon_{t+1} \qquad \epsilon_t \sim \text{MVN}(0, \Sigma)\,, \tag{32}$$

where x_t is an n-vector of state variables. Denote the conditional mean of x_{t+1} as μ_t, and the Cholesky decomposition of the conditional covariance matrix as Ω, i.e.,

$$\text{E}_t[x_{t+1}] \equiv \mu_t = A + Bx_t \tag{33}$$

$$\Sigma \equiv \Omega^T\Omega\,. \tag{34}$$

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