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A Model of Optimal Capital Structure with Stochastic Interest Rates

Abstract

This paper develops a model of optimal capital structure with stochastic interest rate which is assumed to follow a mean-reverting process. Closed-form solutions are obtained for both the value of the firm and the value of its risky debt. The paper finds that the current level and the long-run mean of the interest rate process play distinctive roles in our integrated model. The current level of the interest rate is critical in the pricing of risky bonds, while the long-run mean plays a key role in the determination of a firm's optimal capital structure such as the optimal coupon rate and leverage ratio. Our findings demonstrate that a model of optimal capital structure with a constant interest rate cannot price risky bonds and determine the optimal capital structure simultaneously in a satisfactory manner. Furthermore, our numerical results indicate that the correlation between the stochastic interest rate and the asset return of a firm has little impact on the firm's optimal capital structure.

1 Introduction

The problem of optimal capital structure has long been an intriguing one among researchers. Brennan and Schwartz (1978) are perhaps the first to study this problem using the contingent-claims analysis approach of Black and Scholes (1973) and Merton (1974). In an important recent development, Leland (1994) introduces a model of optimal capital structure based on a perpetuity. Leland and Toft (1996) extend the Leland model to examine the effect of debt maturity on bond prices, credit spreads, and optimal leverage. Titman and Tsyplakov (2002) develop a model of a firm that can dynamically adjust both its capital structure and its investment choices. While very insightful, all these models assume that the risk-free interest rate is constant, thus ignoring the impact of the stochastic nature of the interest rate on the firm's optimal capital structure. Empirical evidence has indicated that firms do take into account the slope of the default-free term structure when they issue debt. See, for example, Barclay and Smith (1995), Guedes and Opler (1996), Stohs and Mauer (1996), and Graham and Harvey (2001). In particular, there is evidence that firms prefer short maturity debt when the term structure is steep. Graham and Harvey report that CFOs state that the slope of the term structure is one important consideration when they decide on how to refinance. These empirical evidences call for a model that includes both optimal leverage and stochastic interest rate.

In this paper, we develop a model of optimal capital structure with stochastic interest rate. More specifically, we combine the Leland-Toft optimal capital structure model with the Longstaff-Schwartz (1995) bond valuation model under stochastic interest rate. One advantage of our model is that it has closed-form solutions for both the firm value and the debt value in the spirit of Longstaff and Schwartz (1995). This allows us to perform comparative statics. Modelling the interest rate as a mean-reverting process allows us to examine separately the impact of the long-run mean and the initial interest rate. Numerical results from the implementation of our model indicate that the long-run mean of the interest rate is an important determinant of the optimal capital structure. This is intuitive since the long-run mean plays a key role in the determination of the tax shields and bankruptcy costs resulting from the future debt issues. The initial interest rate level is important in determining the price of current outstanding risky bonds, especially those with short and moderate maturities. The reason is that it takes time for the interest rate to revert to its mean level. Our results also indicate that the correlation between the interest rate and the firm asset return has little impact. Furthermore, we find that the maturity of a bond is also an important determinant in capital structure considerations.

An active and growing body of work has studied the valuation of risky corporate bonds and other derivative instruments in a stochastic interest rate environment. Kim, Ramaswamy, and Sundaresan (1993) calculate various corporate bonds in a series of numerical examples. Longstaff and Schwartz (1995) derive closed-form valuation expressions for fixed and floating rate debt and find that the correlation between the underlying asset return and interest rate has a significant effect on credit spreads.[1] These models, however, assume that the Modigliani-Miller theory holds, i.e., the value of a firm is independent of its capital structure. This implies that the firm does not derive tax benefits from issuing bonds.

The model proposed here combines two strands of the literature, namely, the valuation models with stochastic interest rate in the absence of optimal capital structure, and the optimal capital structure models in the absence of stochastic interest rate.

The remainder of this paper is organized as follows. Section 2 introduces the model and derives various closed-form valuation expressions. Section 3 presents numerical results.

[1] Duffie and Singleton (1999), Jarrow and Turnbull (1995), and many others specify the default outcomes and value credit risk by no arbitrage.

Some concluding remarks regarding possible extensions of the model are given in Section 4. The Appendix reviews the T-forward risk-neutral measure used to derive the valuation expressions in Section 2.

2 The Model

In this section we first set up the valuation problem and then derive the formulas for the firm value and the debt value.

2.1 The setup

Our assumptions and notations are mainly adopted from Leland and Toft (1996) and Longstaff and Schwartz (1995). The main assumptions are summarized as follows.

Assumption 1 *Financial markets are dynamically complete, and trading takes place continuously. Therefore, there exists an equivalent martingale measure (Harrion and Kreps, 1979) or a risk-neutral measure (Cox and Ross, 1976), Q, under which discounted price processes are martingales.*

Below we shall work directly under the risk-neutral measure Q (and the forward measure).

Assumption 2 *The total value of the firm's unlevered assets, V_t, is described by a geometric Brownian motion process given by*

$$\frac{dV_t}{V_t} = (r_t - \delta)\, dt + \sigma_v \, dw_{1t}^Q, \tag{1}$$

where r_t is the interest rate at time t, δ is a constant payout rate, σ_v is a constant, and w_{1t}^Q is a standard Wiener process defined on a complete probability space $(\Omega, \mathcal{P}, \mathcal{F})$.

Assumption 3 *The interest rate r_t follows the Vasicek (1977) model*

$$dr_t = (\alpha - \beta\, r_t)\, dt + \sigma_r\, dw_{2t}^Q, \tag{2}$$

where α, β, and σ_r are constants, and w_{2t}^Q is another standard Wiener process on the same probability space $(\Omega,\ \mathcal{P},\ \mathcal{F})$. The instantaneous correlation between dw_{1t}^Q and dw_{2t}^Q is given by $\rho\, dt$.

Assumption 4 *Assume that bankruptcy occurs when the value of the firm falls below a constant level V_B. If $V_t > V_B$, the firm is solvent and pays the contractual coupon rate to its debt holders. In the event of bankruptcy, bond holders will receive ϕV_B with $\phi \in [0, 1)$ and equity holders get nothing.*

The exogenous flat default boundary assumed here follows from Longstaff and Schwartz (1995) or Leland (1994). As mentioned in Black and Cox (1976), this type of boundary can be considered to model some kind of net asset requirement in the bond covenants (protected debt). This assumption is also made for analytical tractability. An endogenous default boundary can be defined in our setting here. However, with stochastic interest rates, such a boundary could only be obtained by using numerically intensive methods. The martingale technique introduced later on won't be applicable any more.

The sharing rule specified above is similar to the one assumed in Leland and Toft (1996) and is referred to as the absolute priority rule. This assumption can be easily relaxed to allow equity holders to share ϕV_B with the bond holders.

Assumption 5 *We consider a stationary debt structure where a firm continuously sells a constant (principal) amount of new debt with a maturity of m years to replace the same amount of principal of retiring debt. New bond principal and coupon are issued at rates*

$p = P/m$ and $c = C/m$ per year, where P and C are the total principal and total coupon rates of all outstanding bonds, respectively.

This debt structure is the same as the one assumed in Leland and Toft (1996). The advantage of this debt structure is its analytical tractability.

Essentially, we consider the Leland and Toft (1996) debt structure in the Longstaff and Schwartz (1995) setting. Below we shall first derive formulas of risky bond prices and then determine the optimal capital structure.

2.2 Default Probability under the T-forward Measure

Longstaff and Schwartz (1995) derive various debt valuation expressions by solving a partial differential equation (PDE). However, with rollover finite maturity debt, it is more convenient to obtain the corresponding valuation expressions in closed form by the martingale approach. To this end, we first derive the density distribution for the first passage time to be defined below and then derive valuation expressions for finite maturity debt, using the T-forward risk-neutral measure Q^T developed in the Appendix.

Given the interest rate process in Eq. (2), the price of a zero-coupon bond at time t with a maturity of T years is given by (Vasicek (1977))

$$\Lambda(r_t, T-t) = e^{A(T-t) - B(T-t)\, r_t}, \tag{3}$$

where

$$\begin{aligned} A(T-t) &= \left(\frac{\sigma_r^2}{2\beta^2} - \frac{\alpha}{\beta}\right)(T-t) + \left(\frac{\sigma_r^2}{\beta^3} - \frac{\alpha}{\beta^2}\right)(e^{-\beta(T-t)} - 1) - \\ &\quad \left(\frac{\sigma_r^2}{4\beta^3}\right)(e^{-2\beta(T-t)} - 1), \qquad (4) \\ B(T-t) &= \frac{1 - e^{-\beta\,(T-t)}}{\beta}. \qquad (5) \end{aligned}$$

Simple algebra now yields

$$\frac{\sigma_r \, \Lambda_r}{\Lambda} = -\sigma_r \, B(T-t). \tag{6}$$

It follows from the Appendix that under the T-forward risk-neural measure, Q^T, the two processes, $w_{1t}^{Q^T}$ and $w_{2t}^{Q^T}$, defined by

$$\begin{aligned} dw_{1t}^{Q^T} &= dw_{1t}^{Q} + \rho \, \sigma_r \, B(T-t) \, dt, & (7) \\ dw_{2t}^{Q^T} &= dw_{2t}^{Q} + \sigma_r \, B(T-t) \, dt, & (8) \end{aligned}$$

are two standard Wiener processes with correlation coefficient ρ. Under Q^T, the firm value and the interest rate processes are given by

$$\begin{aligned} \frac{dV_t}{V_t} &= (r - \delta - \rho \, \sigma_v \, \sigma_r \, B(T-t)) \, dt + \sigma_v \, dw_{1t}^{Q^T}, & (9) \\ dr &= (\alpha - \beta \, r - \sigma_r^2 \, B(T-t)) \, dt + \sigma_r \, dw_{2t}^{Q^T}. & (10) \end{aligned}$$

Define the first passage time τ as $\tau = \min\{t : V_t \le V_B\}$, which is the first time that the firm value V_t hits V_B in some $\omega \in \Omega$ under Q^T. Denote by $F(T)$ the cumulative distribution function of τ under Q^T. Using a result in Longstaff and Schwartz (1995), we arrive at an expression for $F(T)$:[2]

$$F(T) = \lim_{n \to \infty} \sum_{i=1}^{n} q(t_{i-\frac{1}{2}}), \tag{11}$$

where for $i = 1, 2, \ldots, n$,

$$\begin{aligned} t_i &= i \, \frac{T}{n}, & (12) \\ q(t_{i-\frac{1}{2}}) &= \frac{N(a(t_i)) - \sum_{j=1}^{i-1} q(t_{j-\frac{1}{2}}) \, N(b(t_i; t_{j-\frac{1}{2}}))}{N(b(t_i; t_{i-\frac{1}{2}}))}, & (13) \end{aligned}$$

[2]Note that we have modified the Longstaff and Schwartz's original formulas. For details, see Huang and Huang (2000) and Collin-Dufresne and Goldstein (2001).

$$a(t_i) = -\frac{M(t_i, T|X_0, r_0)}{\sqrt{S(t_i|X_0, r_0)}}, \tag{14}$$

$$b(t_i; t_j) = -\frac{M(t_i, T|X_{t_j})}{\sqrt{S(t_i|X_{t_j})}}, \tag{15}$$

and where $X \equiv V/V_B$, the sum on the RHS of (13) is defined to be zero when $i = 1$, and

$$M(t, T|X_0, r_0) \equiv E_0^{Q^T}\left[\ln X_t\right]; \tag{16}$$

$$S(t|X_0, r_0) \equiv \mathrm{Var}_0^{Q^T}\left[\ln X_t\right]; \tag{17}$$

$$M(t, T|X_u) = M(t, T|X_0, r_0) - M(u, T|X_0, r_0)\,\frac{\mathrm{Cov}_0^{Q^T}\left[\ln X_t, \ln X_u\right]}{S(u|X_0, r_0)} \tag{18}$$

$$S(t|X_u) = S(t|X_0, r_0) - \frac{\left(\mathrm{Cov}_0^{Q^T}\left[\ln X_t, \ln X_u\right]\right)^2}{S(u|X_0, r_0)}, \tag{19}$$

with

$$\begin{aligned} E_0^{Q^T}\left[\ln X_t\right] &= \ln X_0 + \left(-\delta - \frac{\sigma_v^2}{2} + \frac{\alpha - \rho\sigma_v\sigma_r}{\beta} - \frac{\sigma_r^2}{\beta^2}\right) t \\ &\quad + \left(r_0 - \frac{\alpha}{\beta} + \frac{\sigma_r^2}{\beta^2} - \frac{\sigma_r^2}{2\beta^2} e^{-\beta T}\right) B(0,t) \\ &\quad + \left(\frac{\rho\sigma_v\sigma_r}{\beta} + \frac{\sigma_r^2}{2\beta^2}\right) B(0,t) e^{-\beta(T-t)} \end{aligned} \tag{20}$$

$$\begin{aligned} \mathrm{Cov}_0^{Q^T}\left[\ln X_t, \ln X_u\right] &= \left(\sigma_v^2 + \frac{2\rho\sigma_v\sigma_r}{\beta} + \frac{\sigma_r^2}{\beta^2}\right) u - \left(\frac{\rho\sigma_v\sigma_r}{\beta} + \frac{\sigma_r^2}{\beta^2}\right) B(0,u) \\ &\quad - \left(\frac{\rho\sigma_v\sigma_r}{\beta} + \frac{\sigma_r^2}{2\beta} B(0,u)\right) B(0,u) e^{\beta(u-t)} \end{aligned} \tag{21}$$

$$\begin{aligned} \mathrm{Var}_0^{Q^T}\left[\ln X_t\right] &= \left(\sigma_v^2 + \frac{2\rho\sigma_v\sigma_r}{\beta} + \frac{\sigma_r^2}{\beta^2}\right) t - \left(\frac{2\rho\sigma_v\sigma_r}{\beta} + \frac{\sigma_r^2}{\beta^2}\right) B(0,t) \\ &\quad - \frac{\sigma_r^2}{2\beta} B(0,t)^2. \end{aligned} \tag{22}$$

2.3 Valuation Formulas in Closed Form

In this subsection, using the cumulative density function $F(T)$ obtained in the previous subsection, we derive expressions for the bond price, the value of tax benefits, and the value of bankruptcy costs in closed form.

Consider a bond that pays a coupon rate c, has a principal value p, and matures at time t. The payment *rate* $g(s)$ to the debt holders at any time s is equal to

$$g(s) = c\,1(s \le t)\,1(s \le \tau) + p\,\delta(s-t)\,1(s \le \tau) + \phi(t)V_B\delta(s-\tau)\,1(s \le t), \tag{23}$$

where $1(\cdot)$ denotes the indicator function and $\delta(\cdot)$ is the Dirac delta function. Note that $g(s)$ is random because τ, by definition, is random. $\phi(t)$ is the fraction of the asset value, V_B, that the maturity-t bondholders receive in bankruptcy.

Under the risk-neutral measure, Q, the value of the debt at time zero is given by

$$d(V; V_B, t) = \int_0^\infty E^Q[e^{-\int_0^s r(u)du} g(s)]ds = \int_0^\infty \Lambda(s) E^{Q^s}[g(s)]ds. \tag{24}$$

For simplicity, r_0 has been suppressed in $\Lambda = E^Q[e^{-\int_0^s r(u)du}]$. The term inside the square brackets represents the discounted cash flow received during time interval ds. Taking expectation under Q represents the present value of the cash flow, and integrating it gives rise to the present value of the debt. The last step results from the s-forward risk-neutral measure Q^s,[3] defined in the Appendix. Evaluating $E^{Q^s}[g(s)]$, we have

$$\begin{aligned} E^{Q^s}[g(s)] \;=\; & E^{Q^s}[c1(s \le t)1(s \le \tau)] + E^{Q^s}[p\,\delta(s-t)1(s \le \tau)] + \\ & E^{Q^s}[\phi(t)V_B\delta(s-\tau)1(s \le t)] = c1(s \le t)(1-F(s)) + \\ & p\,\delta(s-t)(1-F(s)) + \phi(t)V_B 1(s \le t) f(s). \end{aligned} \tag{25}$$

Note that $F(s) = E^{Q^s}[1(s \le \tau)]$ and $f(s) = E^{Q^s}[\delta(s-\tau)] = \int_0^\infty f(\tau)\delta(s-\tau)d\tau$ are the distribution function and density function of τ under the s-forward risk-neutral measure, respectively.

[3] Note that we have used Q^s to denote the forward risk-neutral measure because the appropriate time here is s.

We can now express $d(V, V_B, t)$ in closed form:

$$\begin{aligned} d(V; V_B, t) &= \int_0^\infty \Lambda(s) c 1(s \le t)(1-F(s))ds + \int_0^\infty \Lambda(s) p\, \delta(s-t)(1-F(s))ds + \\ &\int_0^\infty \Lambda(s)\phi(t) V_B 1(s \le t) f(s) ds = c\int_0^t \Lambda(s)(1-F(s))ds + \Lambda(t)p\,(1-F(t)) \\ &+\phi(t)V_B \int_0^t \Lambda(s) f(s) ds = \frac{C}{m}\int_0^t \Lambda(s)(1-F(s))ds + \frac{P}{m}(\Lambda(t)(1-F(t))) + \\ &\frac{\phi V_B}{m}\left(\Lambda(t)F(t) - \int_0^t \Lambda'(s)F(s)ds\right), \end{aligned} \tag{26}$$

where $\phi(t) = \phi/m$,[4] $c = C/m$, and $p = P/m$ have been substituted.

Assuming that the newly issued debt (at time 0) is priced at par, i.e., $d(V; V_B, m) = P/m$, the coupon rate C can then be solved in terms of P by

$$C = \frac{P\,(1 - \Lambda(m)(1-F(m))) - \phi V_B(\Lambda(m)F(m) - \int_0^m \Lambda'(s)F(s)ds)}{\int_0^m \Lambda(s)(1-F(s))ds}. \tag{27}$$

Integrating $d(V, V_B, t)$ from 0 to m, we obtain the total value of all outstanding debts:

$$\begin{aligned} D(V) &= \frac{C}{m}\int_0^m \left(\int_0^t \Lambda(s)(1-F(s))ds\right) dt + \frac{P}{m}\int_0^m \Lambda(st)(1-F(t))dt + \\ &\frac{\phi V_B}{m}\int_0^m \Lambda(t)F(t)dt - \frac{\phi V_B}{m}\int_0^m \left(\int_0^t \Lambda'(s)F(s)ds\right) dt, \end{aligned} \tag{28}$$

which can be simplified to

$$\begin{aligned} D(V) &= \frac{C}{m}\int_0^m \Lambda(t)(1-F(t))(m-t)dt + \frac{P}{m}\int_0^m \Lambda(t)(1-F(t))dt + \\ &\frac{\phi V_B}{m}\int_0^m \left(\Lambda(t)F(t) - \Lambda'(t)F(t)(m-t)\right) dt. \end{aligned} \tag{29}$$

The tax shield accumulation rate is given by $\theta\, C\, 1(s < \tau)$, with θ being the corporate tax rate. As in Brennan and Schwartz (1978) and Leland (1994), it is assumed here that the firm loses its tax benefits forever after bankruptcy has occurred. Consequently, the value of

[4] This assumes that all the debts with remaining time to maturity within $[0, m]$ have the same seniority. ϕ is the total fraction of the assets that bondholders receive in bankruptcy.

the tax shields is given by

$$
\begin{aligned}
TB(V) &= \int_0^\infty E^Q[e^{-\int_0^s r(u)du}\theta\, C\, 1(s<\tau)]ds = \theta C \int_0^\infty \Lambda(s) E^{Q^s}[1(s<\tau)]ds \\
&= \theta C \int_0^\infty \Lambda(s)(1-F(s))ds. \qquad (30)
\end{aligned}
$$

Similarly, the rate of bankruptcy costs is given by $(1-\phi)V_B\delta(s-\tau)$, and the value of bankruptcy costs is equal to

$$
\begin{aligned}
BC(V) &= \int_0^\infty E^Q[e^{-\int_0^s r(u)du}(1-\phi)V_B\delta(s-\tau)]ds = (1-\phi)V_B \int_0^\infty \Lambda(s)f(s)ds \\
&= -(1-\phi)V_B \int_0^\infty \Lambda'(s)F(s)ds. \qquad (31)
\end{aligned}
$$

Finally, the total firm value consists of three terms: the firms unlevered asset value, plus the value of tax shields, less the value of bankruptcy costs:

$$
\begin{aligned}
v(V) &= V + TB(V) - BC(V) = V + \theta C \int_0^\infty \Lambda(s)(1-F(s))ds + \\
& \quad (1-\phi)V_B \int_0^\infty \Lambda'(s)F(s)ds. \qquad (32)
\end{aligned}
$$

The total principal P shall be determined by maximizing $v(V)$. Substituting C in terms of P from (27) into the above expression for $v(V)$, we arrive at an unconstrained, univariate maximization problem.

3 Numerical Results

In this section, we implement the optimal capital structure model developed in the previous section. We consider first the benchmark case where the interest rate is assumed to be constant. We then examine our proposed model with stochastic interest rate.

In the numerical calculations, the following base parameters are fixed: the asset return volatility $\sigma_v = 0.2$; the corporate tax rate $\theta = 0.35$; the bankruptcy cost parameter $\phi = 0.5$;

the asset payout ratio $\delta = 0.02$; the interest rate process parameters are taken from Longstaff and Schwartz (1995) and are given by $\sigma_r^2 = 0.001$, $\alpha = 0.06$, and $\beta = 1.0$. The base value for the correlation coefficient between the interest rate and the asset return is assumed to be zero.

3.1 Optimally levered firms with constant interest rates

Though a lot of work has been performed pricing defaultable bonds in a stochastic interest rate environment, almost all work on optimal structure has assumed constant interest rate. To see the impact of the interest rate variability on capital structure, we first consider the benchmark case in which the interest rate is a constant.

Table 1 reports results with different interest levels (constant) and different bankruptcy constraints. One observation from the table is that in a low interest rate environment, the optimal coupon rate and leverage ratio are lower. One reason behind this is as follows. The coupon payments contribute less to the total market value of a bond if the interest rate is low, but only the coupon part is tax deductible. Therefore, with the same market value, a bond in a low interest environment offers less tax shield. Another reason is that, in a low interest rate environment, the risk-neutral drift of the asset return is lower, and thus, the (risk-neutral) probability of bankruptcy is higher, and the expected bankruptcy cost is higher. Consequently, it follows that a firm optimally levers less in a low interest rate environment.

The result that a firm optimally levers less in a low interest rate environment may seem odd at first glance. Intuition suggests that firms may find it more attractive to issue debt in a low interest rate environment because the coupons they pay will be lower. However, implicitly behind this reasoning is the assumption that interest rate will likely increase in the future. Ours is a comparative static result. It is assumed that the interest rate will be at that level forever. Therefore, in a sense, it makes no sense to talk about high or low interest rate. Each level represents a different state of the world.

Another observation from the table is that with less restrictive bankruptcy trigger level ($V_B = 0.9P$), a firm optimally levers more. The reason is that with a lower bankruptcy level, the probability and expected cost of bankruptcy are both lower. We can think of the level of V_B as the strength of the bond covenants to force bankruptcy. As the rights of debt holders to force default increase (higher V_B), firms find it optimal to use less leverage.

We can also see from the table that the coupon rate and credit spread of optimally levered firms seem to be related to the maturity of the bonds in a non-uniform fashion. This is due to the interplay of the tax shields, which favors long maturity debt, and the bankruptcy cost, which favors short maturity debt. On the one hand, for the same market value of the bond, the tax shield for short maturity debt is less than that of a longer maturity debt because the coupon payments contribute less to the market value of the short maturity debts and only the coupon part is tax deductible. Therefore, tax shield favors long maturity debt. On the other hand, for the same coupon and principal, the expected bankruptcy cost is higher for a longer maturity debt. Therefore, debt maturity is an important consideration in capital structure considerations.

A somewhat surprising result is that at high interest rates, for given coupon and principal of the bond, its yield curve is inverted. Initially the yield increases with maturity and then declines. Since the interest rate is constant, the yield spread curve has the same property. This implies that initially the market thinks the bond is risky but if the firm survives the first few years, then its default probability will be considered to decrease. Another implication of an inverted (risky) yield curve is that the yield of the total debt is higher than that of the newly issued debt, whose maturity is twice of the average maturity of the total debt.

3.2 Optimally levered firms with stochastic interest rate with base values

We now examine the impact of stochastic interest rate. More specifically, we look at how the characteristics of an optimally levered firm–such as the leverage ratio and credit spreads–change when we vary a particular parameter in the interest rate process.

In Table 2, we consider the impact of the initial interest rate. As can be seen from the table, the initial interest rate level is not as crucial as in the constant interest rate case. By comparing with table 1, it is clear that a firm levers a lot more when the initial interest rate is only 3%. In the stochastic case, the long-run mean of the interest rate is 6%. Therefore, when a firm determines the debt amount initially, it considers not only the impact of the current issue on the firm value but also the impact of all future issues on the firm value. Therefore, the long-run mean is important. Since the interest rate in the real world is stochastic, it is important to take into account the long-run mean of the interest rate process in deciding the leverage ratio. For this reason, the characteristics of the optimally levered firm with different initial interest rates are quite similar to the one with an initial interest rate of 6%, which happens to be the long-run mean of the interest rate. Our results indicate that when the current interest rate is different from the long-run mean, a stochastic interest rate has a significant impact on the characteristics of the optimally levered firms. It is also clear that bonds with longer maturities are even closer to their counterparts with an initial interest rate 6% because the mean-reverting property of the interest rate has more time to affect the initial bonds. The credit spread is lower for the total debt than the newly issued ones. The reason is quite simple. The average maturity of the total debt is only half of the maturity of the newly issued debt and generally shorter maturity debts have lower credit spread because the probability of bankruptcy is lower for shorter maturity debts. Notice that, however, even though this result is intuitive, in the high constant rate case, the yield curve may be inverted.

Note that even though in the stochastic interest rate case, the initial interest rate level is less important than the constant interest rate case, this initial level of the stochastic interest rate is still important in determining the price of a corporate bond. Note that if a constant interest rate is to be used, the long-run mean appears to be the appropriate rate to use. From table 2, we note that the characteristics of optimally levered firms change only moderately when the initial interest rate changes from far below the long-run mean to far above the long-run mean. However, using the long-run mean as the constant interest rate level will grossly misprice the current outstanding debt. Suppose, for example, the initial (current) interest is 3%. To price current outstanding debt, especially short-term ones, the long-run mean, say 6%, is not appropriate. On the other hand, to price the future debts, or more precisely, the tax shields and expected bankruptcy costs resulting from future debts, the long-run mean is more appropriate than the current interest rate level, especially when it is far from the long-run mean. Therefore, the assumption of a constant interest rate is incapable of correctly pricing the current outstanding debt and determining the optimal capital structure, especially when the current interest rate is different from the long-run mean. In a stochastic interest rate model like ours, both the current outstanding debt and future tax shields and expected bankruptcy costs are determined appropriately.

3.3 Optimally levered firms with stochastic interest rate with different correlation coefficients

Now we consider the effects of the correlation between the interest rate and the return of the firm value on the characteristics of optimally levered firms. We keep the other parameters at their base values and the initial interest rate is the long-run mean 6%. We note that for a given maturity, the characteristics of optimally levered firms with different correlation between the interest rate and the return of the firms' assets are similar. The reason is due to the small impact of the correlation on the drift of the unlevered firm value dynamics under the T-forward risk-neutral process. Note that in (9), under Q^T, the drift of V_t is changed by $\rho\sigma_r\sigma_v B(T-t)$, which is less than $0.0063\rho B(T-t)$ since we used $\sigma_v = 0.2$ and $\sigma_r = 0.0316$. Because $\beta = 1.0$, $B(T-t)$ is smaller than 1. Therefore, $|\rho\sigma_r\sigma_v B(T-t)|$ is less than 0.0063 and thus the correlation's impact is small. However, we do note that the drift increases if the correlation ρ is negative and decreases if ρ is positive. For this reason, the optimal coupon and leverage ratio are higher for negative correlations.

3.4 Optimally levered firms with different correlations and initial interest rates

In the previous subsection, we considered the impact of correlation with the same initial interest rate. Here we consider the effects with different correlations and different initial interest rates. Table 4 demonstrates clearly that the initial interest rate is more important than the correlation between the interest rate and the return of the firm's assets, especially for bonds with shorter maturities. The reason is quite simple: for short maturity debts, there is less time for the interest rate to revert to its long-run mean before the initial debts mature. Again, with the same maturity and initial interest rate, firms with negative correlation between the interest rate and the return of the firm's asset optimally lever more because negative correlation implies the risk-neutral bankruptcy probability is smaller.

4 Concluding Remarks

Existing models of the optimal capital structure of a firm do not consider stochastic interest rates. This paper considers stochastic interest rate, a firm's capital structure, and the valuation of the firm's debt in a unified framework. Expressions for the total value of the firm and for the firm's risky debt are obtained in closed form.

When the interest rate is assumed to be a constant, the level of the interest rate has a significant impact on both the optimal coupon and the leverage ratio. When the interest rate is assumed to follow a mean-reverting stochastic process, however, the long-run mean as well as the current level of the interest rate process are required to price the risky bond and determine the optimal capital structure of the firm. On the one hand, the current interest rate level is crucial in the pricing of the risky bond. On the other hand, the long-run mean plays a key role in the determination of the tax shields and bankruptcy costs resulting from the future debt. Therefore, a model of optimal capital structure with a constant interest rate cannot simultaneously price risky corporate debts and determine the optimal capital structure appropriately. A stochastic interest rate process is needed to account for the evolution of the interest rate. While the long-run mean is shown to be important in determining the optimal capital structure, numerical results indicate that the correlation between the stochastic interest rate and the return of the firm's assets has little impact. Finally, besides the long-run mean, the maturity of the bond is also an important determinant in capital structure considerations.

For tractability of the model, we have assumed that the default boundary V_B is an exogenously specified constant. Extending our model to allow for an endogenous default boundary, in the sense of Leland (1994) and Leland and Toft (1996), is an important but challenging topic for future research.

A The Forward Risk-Neutral Measure

In this appendix, we use the Girsanov theorem to derive the T-forward risk-neutral measure in a multi-dimensional setting. Without loss of generality, we assume a probability space Q generated by two standard Wiener processes

$$\tilde{w}_t^Q = \begin{bmatrix} w_{1t}^Q \\ w_{2t}^Q \end{bmatrix}, \tag{33}$$

with correlation matrix

$$\tilde{\rho}(t) = \begin{bmatrix} 1 & \rho(t) \\ \rho(t) & 1 \end{bmatrix}. \tag{34}$$

In the following, Q should be interpreted as the risk-neutral probability measure and r_t the riskless interest rate and is given by

$$dr_t = \mu(r,t)dt + \sigma(r,t)dw_{2t}^Q. \tag{35}$$

We leave other random variables generated by w_{1t}^Q and w_{2t}^Q unspecified.

Suppose we want to compute the following expectation

$$h = E^Q[e^{-\int_0^T r(u)du} H(\{\cdots\}, T)], \tag{36}$$

where $\{\cdots\}$ indicates that $H(\{\cdots\}, T)$ may depend on the sample path in space Q from 0 to T. Let $\Lambda(r_0, T)$ be the discount bond price at $t = 0$ with maturity T. Define

$$\xi_T = \frac{e^{-\int_0^T r(u)du}}{\Lambda(r_0, T)}. \tag{37}$$

Then

$$h = \Lambda(r_0, T)E^Q[\xi_T H(\{\cdots\}, T)]. \tag{38}$$

It is clear that ξ_T is strictly positive and $E^Q[\xi_T] = 1$. Therefore it can be used as a Radon-Nikodym derivative to define a new probability measure Q^T equivalent to the original measure Q such that

$$E^{Q^T}[1_{\{A\}}] = E^Q[\xi_T 1_{\{A\}}] \tag{39}$$

for any event A. Under the new measure Q^T,

$$h = \Lambda(r_0, T) E^{Q^T}[H(\{\cdots\}, T)]. \tag{40}$$

To find the Wiener processes under Q^T, define the likelihood ratio

$$\xi_t = E_t^Q[\xi_T] = \frac{e^{-\int_0^t r(s)ds}\Lambda(r_t, T-t)}{\Lambda(r_0, T)}. \tag{41}$$

It follows that

$$\log \xi_t = -\int_0^t r(s)ds + \log \Lambda(r_t, T-t) - \log \Lambda(r_0, T). \tag{42}$$

Ito's lemma implies

$$d\log \xi_t = -rdt + \frac{d\Lambda(r_t, T-t)}{\Lambda(r_t, T-t)} - \frac{1}{2}\left(\frac{d\Lambda(r_t, T-t)}{\Lambda(r_t, T-t)}\right)^2 = \tag{43}$$

$$\left[-r\Lambda + \Lambda_t + u(r,t)\Lambda_r + \frac{1}{2}\sigma^2(r,t)\Lambda_{rr}\right] dt/\Lambda + \frac{\sigma(r,t)\Lambda_r}{\Lambda} dw_2^Q(t) - \frac{1}{2}\left(\frac{\sigma(r)\Lambda_r}{\Lambda}\right)^2 dt. \tag{44}$$

The term inside the square bracket is the fundamental PDE satisfied by the discount bond price Λ, therefore

$$d\log \xi_t = -\frac{1}{2}\left(\frac{\sigma(r,t)\Lambda_r}{\Lambda}\right)^2 dt + \frac{\sigma(r,t)\Lambda_r}{\Lambda} dw_2^Q(t). \tag{45}$$

Another application of Ito's lemma yields

$$d\xi_t = \xi_t \frac{\sigma(r,t)\Lambda_r}{\Lambda} dw_2^Q(t) = \xi_t \tilde{\beta}(t)^{\mathrm{T}} d\tilde{w}_t^Q, \tag{46}$$

where

$$\tilde{\beta}(t) = \begin{bmatrix} 0 \\ \frac{\sigma(r,t)\Lambda_r}{\Lambda} \end{bmatrix}, \qquad d\tilde{w}_t^Q = \begin{bmatrix} dw_{1t}^Q \\ dw_{2t}^Q \end{bmatrix}. \tag{47}$$

Now the multi-dimensional Girsanov's theorem implies that under the new probability measure Q^T,

$$\tilde{w}_t^{Q^T} = \begin{bmatrix} w_{1t}^{Q^T} \\ w_{2t}^{Q^T} \end{bmatrix} = \tilde{w}_t^Q - \int_0^t \tilde{\rho}(s)\tilde{\beta}(s)ds \tag{48}$$

are two standard Winner processes with correlation matrix $\tilde{\rho}(t)$. In differential form,

$$d\tilde{w}_t^{Q^T} = d\tilde{w}_t^Q - \tilde{\rho}(t)\tilde{\beta}(t)dt. \tag{49}$$

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Table 1: Characteristics of Optimally Levered Firms with Constant Interest Rate: $\sigma_v = 0.2$; $\theta = 0.35$; $\phi = 0.5$; $\delta = 2\%$

Maturity m (Years)	Coupon C (Dollars)	Principal P (Dollars)	Optimal Leverage Ratio	Credit Spread Total Debt (Basis Points)	Credit Spread Newly Issued (Basis Points)	Firm Value $v(V)$ (Dollars)
Panel A: Constant Interest Rate $r_0 = 3\%$ and $V_B = P$						
1.0	0.6176	20.5882	0.1987	0.0000	0.0000	103.6029
5.0	0.6283	20.8999	0.2017	0.6043	0.6451	103.6146
10.0	0.7877	24.7836	0.2397	16.1291	17.8134	103.8407
20.0	0.9580	27.4572	0.2665	43.4026	48.8942	104.3381
Panel B: Constant Interest Rate $r_0 = 6\%$ and $V_B = P$						
1.0	2.4301	40.5001	0.3686	0.0186	0.0192	109.8807
5.0	3.3803	49.7279	0.4517	73.7005	79.7677	110.7958
10.0	3.2781	47.9478	0.4339	77.4542	83.6904	111.1916
20.0	3.0897	46.0659	0.4154	68.5268	70.7147	111.1333
Panel C: Constant Interest Rate $r_0 = 9\%$ and $V_B = P$						
1.0	4.7490	52.6036	0.4601	2.6483	2.7821	114.3440
5.0	6.1185	59.8206	0.5186	115.8387	122.8091	115.8210
10.0	5.6558	57.1965	0.4948	88.2098	88.8410	115.6358
20.0	5.3945	55.7980	0.4818	72.7368	66.7967	115.3747
Panel D: Constant Interest Rate $r_0 = 3\%$ and $V_B = 0.9P$						
1.0	0.7292	24.3056	0.2331	0.0000	0.0000	104.2535
5.0	0.7528	24.9875	0.2397	1.1949	1.2740	104.2789
10.0	1.1689	34.0796	0.3283	39.3084	42.9928	104.7507
20.0	1.3142	35.1986	0.3389	66.2648	73.3636	105.4574
Panel E: Constant Interest Rate $r_0 = 6\%$ and $V_B = 0.9P$						
1.0	2.7731	46.2150	0.4153	0.0385	0.0398	111.2719
5.0	4.9811	63.7446	0.5692	171.2418	181.4226	112.9948
10.0	4.1046	57.0966	0.5074	112.1969	118.8833	113.2291
20.0	3.7298	53.9606	0.4778	90.2216	91.2042	113.0315
Panel F: Constant Interest Rate $r_0 = 9\%$ and $V_B = 0.9P$						
1.0	5.4269	59.9717	0.5161	4.6805	4.9122	116.2305
5.0	7.6570	70.4273	0.5966	180.5942	187.2129	118.4640
10.0	6.7047	65.9760	0.5585	117.7985	116.2332	118.0199
20.0	6.2902	63.9390	0.5407	92.5907	83.7871	117.6183

Table 2: Characteristics of Optimally Levered Firms with Stochastic Interest Rate with Base Parameters: $\sigma_v = 0.2$; $\alpha = 0.06$; $\beta = 1.0$; $\sigma_r^2 = 0.001$; $\rho = 0$; $\theta = 0.35$; $\phi = 0.5$

Maturity m (Years)	Coupon C (Dollars)	Principal P (Dollars)	Optimal Leverage Ratio	Credit Spread Total Debt (Basis Points)	Credit Spread Newly Issued (Basis Points)	Firm Value $v(V)$ (Dollars)
Panel A: Initial Interest Rate $r_0 = 3\%$ and $V_B = P$						
1.0	1.7123	41.8706	0.3894	0.0002	0.0408	107.6783
5.0	3.6721	56.0164	0.5071	23.8926	124.5056	111.9693
10.0	3.3144	51.6853	0.4646	33.2413	83.2486	112.6234
20.0	3.1245	49.5142	0.4426	46.2934	59.8242	112.6577
Panel B: Initial Interest Rate $r_0 = 6\%$ and $V_B = P$						
1.0	2.7869	46.4925	0.4147	0.0030	0.2712	112.1220
5.0	4.3414	58.9209	0.5206	32.8860	140.1551	113.8083
10.0	3.7114	54.0841	0.4767	40.6638	90.2966	113.6959
20.0	3.4082	51.6394	0.4542	53.1024	64.4086	113.3671
Panel C: Initial Interest Rate $r_0 = 9\%$ and $V_B = P$						
1.0	3.9664	50.1499	0.4291	0.0161	0.8782	116.7087
5.0	5.0483	61.7168	0.5322	43.4948	154.4233	115.7025
10.0	4.1414	56.5406	0.4888	49.3796	97.6445	114.8094
20.0	3.7148	53.8405	0.4660	60.5302	69.2923	114.1049
Panel D: Initial Interest Rate $r_0 = 3\%$ and $V_B = 0.9P$						
1.0	1.9503	47.6877	0.4392	0.0005	0.0793	108.7401
5.0	5.7367	71.1636	0.6270	78.8985	276.1910	114.6721
10.0	4.1122	60.9477	0.5365	51.4572	116.9725	114.8322
20.0	3.7234	57.4568	0.5035	61.7316	76.9479	114.6856
Panel E: Initial Interest Rate $r_0 = 6\%$ and $V_B = 0.9P$						
1.0	3.1702	52.8664	0.4648	0.0056	0.4913	113.7572
5.0	6.5094	73.6921	0.6324	94.8493	286.6396	116.7986
10.0	4.5970	63.6789	0.5490	62.2818	125.9666	116.0744
20.0	4.0622	59.8997	0.5161	70.4812	82.5317	115.5039
Panel F: Initial Interest Rate $r_0 = 9\%$ and $V_B = 0.9P$						
1.0	4.5135	57.0181	0.4788	0.0293	1.5408	118.9232
5.0	7.3212	76.2254	0.6376	112.3469	295.8597	118.9732
10.0	5.1205	66.4680	0.5612	74.7832	135.2299	117.3616
20.0	4.4276	62.4259	0.5288	79.9280	88.4518	116.3546

Table 3: Characteristics of Optimally Levered Firms with Stochastic Interest Rate with Different Correlation: $\sigma_v = 0.2$; $\alpha = 0.06$; $\beta = 1.0$; $\sigma_r^2 = 0.001$; $V_B = P$; $\theta = 0.35$; $\phi = 0.5$

Maturity m (Years)	Coupon C (Dollars)	Principal P (Dollars)	Optimal Leverage Ratio	Credit Spread Total Debt (Basis Points)	Credit Spread Newly Issued (Basis Points)	Firm Value $v(V)$ (Dollars)
Panel A: Initial Interest Rate $r_0 = 6\%$ and $\rho = 0.75$						
1.0	2.5319	42.2514	0.3827	0.0007	0.0684	110.4136
5.0	3.9521	54.5552	0.4913	26.7767	127.8191	111.8328
10.0	3.5018	50.4945	0.4530	41.8276	97.5690	111.9598
20.0	3.2188	48.0706	0.4298	58.7829	74.0058	111.7186
Panel B: Initial Interest Rate $r_0 = 6\%$ and $\rho = 0.5$						
1.0	2.6126	43.5969	0.3930	0.0011	0.1065	110.9471
5.0	4.0777	55.9849	0.5012	28.6001	131.7372	112.4497
10.0	3.5662	51.6129	0.4604	41.2936	95.0002	112.5059
20.0	3.2809	49.2213	0.4378	56.9776	70.9347	112.2375
Panel C: Initial Interest Rate $r_0 = 6\%$ and $\rho = 0.25$						
1.0	2.6976	45.0081	0.4036	0.0018	0.1687	111.5156
5.0	4.2072	57.4403	0.5110	30.6236	135.8377	113.1073
10.0	3.6407	52.8540	0.4688	41.0847	92.8850	113.0847
20.0	3.3439	50.4103	0.4459	55.0852	67.7370	112.7865
Panel D: Initial Interest Rate $r_0 = 6\%$ and $\rho = -0.25$						
1.0	2.8819	48.0635	0.4262	0.0051	0.4433	112.7693
5.0	4.4793	60.4234	0.5300	35.4245	144.7009	114.5553
10.0	3.7828	55.3481	0.4847	40.2062	87.5224	114.3417
20.0	3.4739	52.9116	0.4626	51.0320	60.9521	113.9804
Panel E: Initial Interest Rate $r_0 = 6\%$ and $\rho = -0.5$						
1.0	2.9839	49.7385	0.4384	0.0090	0.7394	113.4608
5.0	4.6220	61.9459	0.5391	38.2941	149.5160	115.3508
10.0	3.8546	56.6459	0.4927	39.7094	84.5462	115.0230
20.0	3.5401	54.2194	0.4711	48.8333	57.3285	114.6290
Panel F: Initial Interest Rate $r_0 = 6\%$ and $\rho = -0.75$						
1.0	3.0953	51.5495	0.4514	0.0166	1.2698	114.2006
5.0	4.7695	63.4860	0.5479	41.5640	154.6523	116.1964
10.0	3.9265	57.9751	0.5007	39.1639	81.3396	115.7400
20.0	3.6074	55.5700	0.4797	46.5333	53.5681	115.3122

Table 4: Characteristics of Optimally Levered Firms with Stochastic Interest Rate with Different Correlation and Initial Interest Rate: $\sigma_v = 0.2$; $\alpha = 0.06$; $\beta = 1.0$; $\sigma_r^2 = 0.001$; $V_B = P$; $\theta = 0.35$; $\phi = 0.5$

Maturity m (Years)	Coupon C (Dollars)	Principal P (Dollars)	Optimal Leverage Ratio	Credit Spread Total Debt (Basis Points)	Credit Spread Newly Issued (Basis Points)	Firm Value $v(V)$ (Dollars)
Panel A: Initial Interest Rate $r_0 = 3\%$ and $\rho = -0.75$						
1.0	1.9275	47.1071	0.4322	0.0018	0.2750	109.1616
5.0	4.0973	60.8780	0.5392	31.8391	142.0945	114.1619
10.0	3.5213	55.6016	0.4901	32.2114	75.2066	114.5599
20.0	3.3184	53.4298	0.4690	40.6354	49.7744	114.5377
Panel B: Initial Interest Rate $r_0 = 3\%$ and $\rho = 0.75$						
1.0	1.5281	37.3717	0.3515	0.0	0.0080	106.4706
5.0	3.2919	51.3580	0.4732	18.4873	109.7921	110.1700
10.0	3.1137	48.0764	0.4394	33.9977	89.6750	110.9819
20.0	2.9410	45.9640	0.4174	51.1678	68.6957	111.0676
Panel C: Initial Interest Rate $r_0 = 9\%$ and $\rho = 0.75$						
1.0	3.6391	46.0467	0.4015	0.0046	0.2763	114.5271
5.0	4.6539	57.5939	0.5069	36.8653	144.5609	113.5549
10.0	3.9238	52.9717	0.4664	51.0588	105.8202	112.9773
20.0	3.5201	50.2559	0.4424	67.1008	79.6749	112.3973
Panel D: Initial Interest Rate $r_0 = 9\%$ and $\rho = -0.75$						
1.0	4.3825	55.2303	0.4621	0.0751	3.4501	119.3566
5.0	5.4796	66.0473	0.5555	52.7467	165.9900	118.2829
10.0	4.3639	60.4041	0.5112	47.2834	87.7010	116.9617
20.0	3.9188	57.7829	0.4906	52.9556	57.5973	116.1156