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**Abstract**

We establish a necessary and sufficient condition for the risk aversion of an agent's derived utility function to increase with independent, zero-mean background risk. This condition is weaker than standard risk aversion. For small risks, the condition is that the ratio of the third to the first derivative of the utility function is decreasing in income. In a market with state-contingent marketable claims, an increase in background risk, which raises the agent's derived risk aversion, reduces the slope of the agent's optimal sharing rule. Under a weak aggregation condition, an increase of background risk for many agents in the economy raises the prices of marketable claims in states with a low level of marketable aggregate income relative to the prices in states with a higher level of such income.

# 1 INTRODUCTION

In this paper, we consider an economy in which agents face a non-insurable, independent background risk, with a zero mean. Our main results are concerned with the effect of an increase in this background risk. First, we ask how such an increase would affect an agent's risk attitudes. Second, we derive the effect of an increase in background risk on the agent's demand for risky assets and on the pricing of claims in a pure exchange economy.

It has increasingly been recognized in the literature that an agent's choice between a risky and a riskless asset is complicated by the existence of other unavoidable risks. It is rare for decisions on the purchase of risky assets by a consumer-investor to be taken in the absence of wage income risk, for example. This has led Ross (1981), Kihlstrom et.al. (1981), Nachman (1982), and Pratt and Zeckhauser (1987) to consider the robustness of the Pratt (1964)-Arrow (1965) theory of risk aversion in the presence of such an additional income risk. Essentially, Kihlstrom et.al. (1981) and Nachman (1981) develop the concept of a derived utility function. This is a modified utility function which incorporates the effect of an independent background risk. They show that if the original utility function has the characteristic of positive, decreasing absolute risk aversion, then so does the derived utility function. Here, we show the effect of an increase in the size of such a background risk on the risk aversion of the derived utility function, and hence, on the demand for the marketable risky assets.

Kimball (1990) emphasizes the importance of the marginal utility function of the agent. He shows that the modification of the original utility function in the face of background risk depends on the degree of absolute prudence.[1] In further work, Kimball (1993) shows that positive, decreasing absolute risk aversion and positive, decreasing absolute prudence are necessary and sufficient conditions for standard risk aversion. Standard risk aversion is a property of the utility function such that every risk that raises marginal utility makes another undesirable risk more undesirable.

We establish a necessary and sufficient condition for the risk aversion of an agent's derived utility function to increase with background risk. This condition is weaker than standard risk aversion, which is, therefore, sufficient but not necessary. In the limit, for

[1]The degree of absolute prudence is defined to be the negative of the ratio of the third derivative to the second derivative of the utility function.

small risks, the condition is that the ratio of the third to the first derivative of the original utility function is decreasing in wealth. This result relates closely to Gollier and Pratt (1993), who consider weak properness, a condition on utility functions which is less restrictive than standard risk aversion.[2] They derive necessary and sufficient conditions under which an undesirable background risk increases the risk aversion of the derived utility function. However, Gollier and Pratt consider only the effect of positive, in place of zero, background risk. In contrast, our result concerns the effect of changes in the *size* of a zero-mean background risk on the derived utility function.

In our pure exchange economy, an agent purchases claims on aggregate risky income. We call this *marketable* aggregate income. The agent chooses his/her investment in the marketable aggregate income, given an independent, zero-mean, background risk which changes in size. We assume that the capital market is perfect and complete with respect to claims on the *marketable* aggregate income. The background risk faced by the agent can neither be hedged nor diversified away and, therefore, is referred to as a non-insurable background risk. However, the agent can modify his/her optimal purchases of claims on the marketable aggregate income, in the presence of the background risk. The focus here is on the effect of the background risk on the optimal sharing rule, i.e., the purchase of claims on the marketable aggregate income. We show that, if an increase in background risk raises the agent's derived risk aversion, then he/she reacts to this by adjusting his/her sharing rule so as to receive a higher proportion of the aggregate income in the "low" states and a lower proportion in the "high" states. [A low (high) state is defined as a state in which the marketable aggregate income is low (high).] Further, in the special case where the agent is standard risk averse, the change in the sharing rule can be decomposed into an income effect and a substitution effect, both of which have the same sign.

We are then in a position to derive the effects of an increase in an independent, zero-mean, background risk on the relative pricing of claims on marketable aggregate income in different states of the world. We show that in response to an aggregate positive shock to background risk, the prices of claims on low states go up relative to those of claims on high states, provided that the harmonic mean of the agents' derived risk aversion increases in every state. Thus, the reward for risk-bearing increases.

[2]Weak properness is a characteristic of a utility function such that any undesirable risk can never be made desirable by the introduction of any other independent, background risk.

The paper is organized as follows. In section 2 of the paper, we define the problem mathematically and explain our notation. Section 3 examines the properties of risk aversion in the presence of background risk. In the next section, section 4, we derive the effect of an increase in background risk on the optimal sharing rule. In section 5, we consider the effects of changes in the background risk on the relative pricing of claims in different states. We conclude in section 6 with a brief discussion of the applications of our framework to problems in economics and finance.

## 2 STATEMENT OF THE PROBLEM AND DEFINITION OF THE NOTATION

Consider an agent $i$ who faces an independent background risk. In a perfect market, the agent chooses a state-dependent share of marketable aggregate income, subject to the constraint that the cost of acquiring this set of claims is equal to his/her initial wealth. The independent background risk affects his/her choice of the state-dependent share of marketable aggregate income. Thus, the agent's consumption at the end of the single period, $y_i$ is equal to the chosen marketable claim, $x_i$, where $x_i$ is the agent's share of marketable aggregate income, plus an independent background risk $\sigma_i\varepsilon_i$, i.e., $y_i = x_i + \sigma_i\varepsilon_i$. We define $\varepsilon_i$ as a risk with a zero mean and a standard deviation of unity and $\sigma_i$ as a positive scalar denoting the standard deviation of $\sigma_i\ \varepsilon_i$.[3] Hence, an increase in $\sigma_i$ denotes the addition of a mean-preserving spread in the sense of Rothschild and Stiglitz (1970) to the agent's total income $y$,[4] conditional on his/her marketable income $x$. In other words, $\varepsilon_i$ is a zero-mean standard risk, which then permits a comparative statics analysis with respect to the scale of the background risk.[5]

The agent has a utility function, $\nu$(y), which is assumed to be four times differentiable. We first define the Arrow-Pratt equivalent risk premium, for a given risk, conditional on $x$ by the relation

[3] The results that follow do not change even if the mean is constant, but not zero. In a complete market, the agent can separately trade these mean cash flows.

[4] Since we consider only a single period in our analysis, there is no distinction between the income and wealth at the end of the period. We shall refer to $y$ as the total income in the analysis and discussion that follow.

[5] We drop the subscript i when we consider only one agent.

$$E[\nu(y)|x] = \nu[x - \pi(x,\sigma)] \tag{1}$$

where $\pi = \pi(x,\sigma)$ is the equivalent risk premium. $\nu(x - \pi)$ is called the derived utility function as in Nachman (1982). We then define, as in Kimball (1990), the equivalent precautionary risk premium for a given risk, conditional on $x$, by the analogous relation[6]

$$E[\nu'(y)|x] = \nu'[x - \psi(x,\sigma)] \tag{2}$$

where $\psi = \psi(x,\sigma)$ is the equivalent precautionary premium and $\nu'(x - \psi)$ is the derived marginal utility function. Hence, optimal decision rules can be stated in terms of the precautionary premium, $\psi$.

The coefficient of absolute risk aversion is defined as $a(y) = -\nu''(y)/\nu'(y)$ and the coefficient of absolute prudence as $\eta(y) = -\nu'''(y)/\nu''(y)$. The precautionary premium $\psi = \psi(x,\sigma)$ is a positive and a strictly decreasing function of $x$, if the absolute prudence is positive and decreasing, i.e. if $\eta(y) > 0$ and $\eta'(y) < 0$.[7] This is discussed by Kimball (1990) and follows directly by applying the analogous argument of Pratt (1964) for $\pi$, the risk premium, and the coefficient of absolute risk aversion. Finally, it follows from an analogy of the results in Rothschild and Stiglitz (1970) that an increase in background risk raises the precautionary premium, given positive absolute prudence, i.e. $\partial\psi/\partial\sigma > 0$, if $\eta(y) > 0$.[8] Similarly, $\psi = [<]0$ and $\partial\psi/\partial\sigma = [<]0$, if $\eta(y) = [<]0$.

[6]The use of the equivalent precautionary premium rather than just the risk premium simplifies the notion and makes the analysis more intuitive. To see this, differentiate

$$E[\nu(y)|x] = \nu[x - \pi(x,\sigma)]$$

with respect to $x$, which yields

$$E[\nu'(y)|x] = \nu'(x - \pi)(1 - \partial\pi/\partial x) = \nu'(x - \psi)$$

[7]The restriction that prudence is positive is a *necessary* condition for decreasing absolute risk aversion. Positive prudence is not *sufficient* for decreasing absolute risk aversion, which requires additionally that absolute prudence exceeds absolute risk aversion. [see Kimball (1990)].

[8]Rothschild and Stiglitz (1970) show, for utility functions with positive absolute risk aversion, that the addition of a mean preserving spread raises the risk premium. Using the Kimball analogy, it follows that an increase in $\sigma$ raises the background income risk, and hence the precautionary risk premium, if $\eta(y) > 0$.

## 3 PROPERTIES OF RISK AVERSION IN THE PRESENCE OF INCREASING BACKGROUND RISK

In this section, we shall state and prove several properties relating to the risk aversion of the agent's derived utility function in the presence of increasing background risk. To be specific, we derive the necessary and sufficient conditions under which the properties of the original utility function are preserved, in the presence of background risk, in the derived utility function. The absolute risk aversion of the agent's derived utility function is defined as the negative of the ratio of the second derivative to the first derivative of the derived utility function i.e. $\hat{a}(x,\sigma) \equiv -v''(x-\psi)/v'(x-\psi)$. Taking the logarithm of the right side of equation (2), differentiating with respect to $x$, and taking the negative of the resulting function yields

$$\hat{a}(x,\sigma) = -\frac{\partial \log \nu'(x-\psi)}{\partial x} = -\frac{\nu''(x-\psi)}{\nu'(x-\psi)}\left[1-\frac{\partial\psi}{\partial x}\right] = a(x-\psi)\left[1-\frac{\partial\psi}{\partial x}\right]. \quad (3)$$

Note that in the absence of background risk ($\sigma = 0$), $\psi = 0, \partial\psi/\partial x = 0$ and hence $\hat{a}(x,\sigma) = a(x)$, the coefficient of absolute risk aversion of the original utility function. In the theorem that follows, we characterize the behavior of $\hat{a}(x,\sigma)$ in relation to $a(y)$, the coefficient of absolute risk aversion of the original utility function, and explore the properties of derived risk aversion in the presence of background risk.

<u>Theorem 1</u> (Properties of Derived Risk Aversion)

*a) If $\nu'(y) > 0$ and $\nu''(y) < 0$, then*

$$\frac{\partial \hat{a}(x,\sigma)}{\partial x} < [=] 0, \forall (x,\sigma) \Longleftrightarrow \frac{\partial a(y)}{\partial y} < [=] 0, \forall y.$$

*b) If $\nu'(y) > 0$ and $\nu''(y) < 0$, then*

$$\begin{aligned}\frac{\partial \hat{a}(x,\sigma)}{\partial \sigma} &> [=][<] \quad 0, \forall (x,\sigma) \Longleftrightarrow \\ v'''(y_2) - v'''(y_1) &< [=][>] \quad -a(y)[v''(y_2) - v''(y_1)], \\ \forall \quad & (y, y_1, y_2), \quad \textit{where} \quad y_1 \le y \le y_2\end{aligned}$$

*c) If $\nu'(y) > 0$ , $\nu''(y) < 0$, and $\nu'''(y) > 0$, then*

$$\frac{\partial[-\frac{\partial\nu'(x-\psi)/\partial\sigma}{\partial\nu'(x-\psi)/\partial x}]}{\partial x} < [=][>]0, \forall(x,\sigma) \Longleftrightarrow \frac{\partial\eta(y)}{\partial y} < [=][>] \ 0 \ \ \forall y.$$

Proof of 1a):

Sufficiency

First, Kihlstrom et.al. (1981) and Nachman (1982) have shown that $\partial a(y)/\partial y < 0$ implies $\partial\hat{a}(x,\sigma)/\partial x < 0$. For a simple proof, we refer to Kimball (1993, p 598). In the case of exponential utility $\partial a/\partial y = 0$ and $\psi(x,\sigma)$ is independent of $x$ so that $\hat{a}(x,\sigma) = a(y)$, a constant. This establishes sufficiency.

Necessity

Consider the case where $\hat{a}(x,\sigma) = C_2(\sigma)$, i.e., independent of $x$. Then the derived marginal utility function must be exponential, i.e.,

$$\nu'(x-\psi(x,\sigma)) = -C_1(\sigma)\exp(-C_2(\sigma)x)$$

where $C_1(\sigma)$ is also independent of $x$. Hence, for $\sigma \to 0, y \to x$ and $\hat{a}(x,\sigma) \to a(y) = C_2(0)$, a constant.

Now, suppose that $\hat{a}(x,\sigma)$ decreasing everywhere for every $\sigma > 0$. Consider a small risk such that $|\psi|$ and $|\partial\psi/\partial x|$ are small. Then, from equation (3) it follows that $\hat{a}(x,\sigma) \approx a(x)$. Hence, $\hat{a}(x)$ is decreasing everywhere only if $a(x)$ is.

Proof of 1b):

From the definition of $\hat{a}(x,\sigma)$,

$$\hat{a}(x,\sigma) = \frac{\mathrm{E}[-v''(x+\sigma\epsilon) \mid x]}{\mathrm{E}[v'(x+\sigma\epsilon) \mid x]} \tag{4}$$

Differentiating with respect to $\sigma$ and dropping (for notational convenience) the conditioning on $x$, i.e., "$\mid x$", we have the following condition:

$$\frac{\partial\hat{a}(x,\sigma)}{\partial\sigma} > [=][<]0 \Longleftrightarrow f(x,\sigma) > [=][<]0 \tag{5}$$

for *any* distribution of $\varepsilon$, where $f(x,\sigma)$ is defined as

$$f(x,\sigma) \equiv .\mathrm{E}\left[\epsilon\left\{-v'''(x+\sigma\epsilon) - v''(x+\sigma\epsilon)\hat{a}(x,\sigma)\right\}\right] \tag{6}$$

Necessity

We now show that

$$\begin{aligned} f(x,\sigma) &> [=][<] \ 0 \Longrightarrow \\ v'''(y_2) - v'''(y_1) &< [=][>] \ -a(y)\,[v''(y_2) - v''(y_1)] \,, \forall \ \ y_1 \le y \le y_2 \end{aligned} \tag{7}$$

Consider a background risk with three possible outcomes, $\epsilon_0$, $\epsilon_1$, and $\epsilon_2$, such that $\epsilon_0 = 0$, $\epsilon_1 < 0$, and $\epsilon_2 > 0$. Define

$$\begin{aligned} y_i &= x + \sigma\epsilon_i, \quad i = 1, 2 \\ y_0 &= x. \end{aligned}$$

and let $p_i$ denote the probability of the outcome, $\varepsilon_i$. For the special case of such a risk, equation (6) can be written as

$$f(x,\sigma) = p_1|\epsilon_1| \left\{ -v'''(y_2) + v'''(y_1) - [v''(y_2) - v''(y_1)]\hat{a}(x,\sigma) \right\} \tag{8}$$

since

$$E[\varepsilon] = \sum_{i=0}^{2} p_i \varepsilon_i = 0$$

so that

$$p_1|\varepsilon_1| = p_2\varepsilon_2$$

Now $\hat{a}(x,\sigma)$ can be rewritten from (4) as

$$\begin{aligned} \hat{a}(x,\sigma) &= E\left\{ \frac{v'(x+\sigma\epsilon)}{\mathrm{E}[v'(x+\sigma\epsilon)]} \frac{-v''(x+\sigma\epsilon)}{v'(x+\sigma\epsilon)} \right\} \\ &= E\left\{ \frac{v'(x+\sigma\epsilon)}{\mathrm{E}[v'(x+\sigma\epsilon)]} a(x+\sigma\epsilon) \right\} \end{aligned} \tag{9}$$

Hence, $\hat{a}(x,\sigma)$ is the expected value of the coefficient of absolute risk aversion, using the risk-neutral probabilities given by the respective probabilities multiplied by the ratio of the marginal utility to the expected marginal utility. Thus, $\hat{a}(x,\sigma)$ is a convex combination of the coefficients of absolute risk aversion at the different values of $y$. For the three-point distribution being considered, $\hat{a}(x,\sigma)$ is a convex combination of $a(y_0)$, $a(y_1)$, and $a(y_2)$. Hence, as $p_0 \to 1$, $\hat{a}(x,\sigma) \to a(y_0)$. Since $y_0$ can take any value in the range $[y_1, y_2]$, $f(x,\sigma)$ must have the required sign for *every* value of $a(y_0)$, where $y_1 \le y_0 \le y_2$. Thus, since

$p_1|\varepsilon_1| > 0$, this is true only if the condition in (7) holds. This is the same condition as stated in Theorem 1b.

Sufficiency

See Appendix. □

Proof of 1c):

The inequality on the left hand side in theorem 1c) is equivalent to

$$\frac{\partial[\frac{\partial\nu'(\cdot)/\partial\sigma}{\partial\nu'(\cdot)/\partial x}]}{\partial x} > [=][<]0$$

Using the definition of the precautionary premium in $\nu'(x - \psi(x,\sigma)) = E[\nu'(x + \sigma\varepsilon)]$, the term within square brackets in the inequality is equal to

$$\frac{E[\nu''(x + \sigma\varepsilon)\varepsilon]}{E[\nu''(x + \sigma\varepsilon)]}.$$

We need to show that this term increases [is constant] [decreases] in $x$ if and only if the coefficient of absolute prudence decreases [is constant] [decreases]. However, showing that this term increases [is constant] [decreases] with the coefficient of absolute prudence is the same as showing that

$$\frac{E[\nu'(x + \sigma\varepsilon)\varepsilon]}{E[\nu'(x + \sigma\varepsilon)]}. \tag{10}$$

increases [is constant] [decreases] in x if and only if the coefficient of absolute risk aversion decreases [is constant] [increases].

To establish this latter statement consider an agent with no background risk, facing the choice between a riskless and a risky asset, where the excess return on the risky asset is equal to $\mu + \varepsilon$ , and $\mu$ is the expected excess return of the risky asset over the riskless rate. Then, if $\sigma$ denotes the optimal dollar investment in the risky asset, the optimality condition is

$$E[\nu'(z + \sigma(\mu + \varepsilon))\varepsilon)] = -\mu E[\nu'(z + \sigma(\mu + \varepsilon))]$$

where $z$ is the risk-free income (excluding the excess return).

Defining $x = z + \sigma\mu$ we have

$$\frac{E[\nu'(x+\sigma\varepsilon)\varepsilon]}{E[\nu'(x+\sigma\varepsilon)]} = -\mu.$$

Now, we know from Pratt (1964), Theorem 7, that if the agent's wealth increases, there will be a higher [constant] [lower] optimal dollar investment in the risky asset if and only if the agent has decreasing [constant] [increasing] risk aversion. Hence the term in (10) evaluated at the higher level of $x$, for given $\sigma$, is higher [constant] [lower] if and only if the agent has decreasing [constant] [increasing] absolute risk aversion. □

Theorem 1a) argues that there is an equivalence between the risk aversion of the original utility function and that of the derived utility function. Hence, every result in the literature that is based on decreasing or constant absolute risk aversion also holds in the presence of an independent background risk. The implication runs in both directions in the case of constant or decreasing risk aversion. This is not true in the case of increasing risk aversion.

It is quite difficult to establish and interpret the necessary and sufficient condition under which an increase in an independent, zero-mean, background risk will raise the risk aversion of the derived utility function. This is because we have to cover all possible distributions of $\varepsilon$. The condition may be interpreted as a type of "smoothness" condition on the first, second and third derivatives of the utility function. Perhaps it is more easily understood by analyzing the special case of small risks. In this case, we have

Corollary 1b.1: *In the case of small risks, the condition in Theorem 1b becomes*

$$\frac{\partial \hat{a}(x,\sigma)}{\partial \sigma} > [=][<]0 \quad \textit{iff} \quad \frac{\partial \theta}{\partial y} < [=][>]0, \forall y$$

*where* $\theta(y) = v'''(y)/v'(y)$.

Proof: Let $y_2 - y_1 \to 0$. In this case, $v'''(y_2) - v'''(y_1) \to v''''(y)$. Similarly $v''(y_2) - v''(y_1) \to v'''(y)$. Hence, the condition in theorem 1b yields, in this case, $v''''(y) < [=][>] - a(y)v'''(y)$. This is equivalent to $\partial\theta/\partial y < [=][>]0, \forall y$. □

In Corollary 1b.1, we define an additional characteristic of the utility function $\theta(y) = \frac{\nu'''(y)}{\nu'(y)}$ as a *combined* prudence/risk aversion measure. This measure is defined by the product of the coefficient of absolute prudence and the coefficient of absolute risk aversion. The corollary says that derived risk aversion increases [stays constant] [decreases] with background risk if and only if $\theta(y)$ decreases [stays constant] [increases] with increasing income.

Hence, it is significant that *neither* decreasing prudence *nor* decreasing absolute risk aversion is necessary for an increase in derived risk aversion. However, the combination of these conditions is sufficient for the result to hold, since the requirement is that the product of the two must be decreasing. The condition is thus weaker than standard risk aversion, which requires that *both* absolute risk aversion and absolute prudence should be positive and decreasing.

We now apply the necessary and sufficient condition in Theorem 1b to show that standard risk aversion is a sufficient condition for an increase in background risk to cause an increase in the derived risk aversion [see also Kimball (1993)]. We state this as

Corollary 1b.2: *Standard risk aversion is a sufficient, but not necessary, condition for derived risk aversion to increase with an increase in background risk.*

Proof: Standard risk aversion requires both positive, decreasing absolute risk aversion and positive decreasing prudence. Further, $a'(y) < 0 \rightarrow \eta(y) > a(y)$. Also, standard risk aversion requires $v'''(y) > 0$. It follows that the condition in Theorem 1b) for an *increase* in the derived risk aversion can be written as[9]

$$\frac{v'''(y_2) - v'''(y_1)}{v''(y_2) - v''(y_1)} < -a(y_1)$$

or alternatively

$$\eta(y_1) \quad \left[\frac{1 - \frac{v'''(y_2)}{v'''(y_1)}}{1 - \frac{v''(y_2)}{v''(y_1)}}\right] > a(y_1)$$

Since $\eta(y_1) > a(y_1)$, a sufficient condition is that the term in the square bracket exceeds 1. This, in turn, follows from decreasing absolute prudence, $\eta'(y) < 0$. Hence, standard risk aversion is a sufficient condition.

To establish that standard risk aversion is not necessary, consider a case that is not standard risk averse. Suppose, in particular, that $v'''(y) < 0, v''''(y) < 0$, that is, the utility function exhibits increasing risk aversion and negative prudence.[10] In this case, it follows

[9]Note that wherever $v'''(y)$ has the same sign for all $y$, the three-state condition in Theorem 1b (i.e. the condition on $y$, $y_1$, and $y_2$) can be replaced by a two-state condition (a condition on $y_1$ and $y_2$).

[10]As an example, consider the utility function

$$\nu(y) = \frac{1-\gamma}{\gamma}\left[A + \frac{y}{1-\gamma}\right]^{\gamma}, \text{where } \gamma \in (1,2), y < A(\gamma - 1)$$

This utility function exhibits *increasing* risk aversion and *negative* prudence. However, despite this, since $\theta(y)$ decreases with income even in this case, the derived risk aversion increases with background risk.

from Theorem 1b that $\partial \hat{a}(x,\sigma)/\partial\sigma > 0, \forall(x,\sigma)$. □

The contribution of Theorem 1b is that it defines a set of utility functions (that is broader than the class of standard risk averse functions) for which the agent becomes more risk averse when faced with an independent, zero-mean background risk. It also shows that weak proper risk aversion (Gollier and Pratt, (1993)) is not a sufficient condition in our case. This is because we are concerned, in Theorem 1b, with the effect of an *increase* in background risk and not just the effect of *introducing* background risk.[11]

An implication of Theorem 1b is that an investor facing the choice between a riskless asset and one risky asset buys less of the risky asset if an increase in background risk increases his/her derived risk aversion. This follows from Pratt (1964, Theorem 7), where it is shown that the investment in the risky asset is smaller, the higher is the coefficient of absolute risk aversion. Similarly, the investment in the risky asset does not depend on background risk, when the condition in Theorem 1b holds as an equality. This is true for exponential and quadratic utility.

Theorem 1a), in conjunction with Theorem 1b) states the conditions under which the derived absolute risk aversion rises with background risk, but declines as income rises. The intuition is that, under these conditions, an increase in background risk is similar to a decrease in marketable income and hence, derived risk aversion behaves in a similar manner under both changes. This raises the question of substitution between these two effects. This takes us to Theorem 1c).

This part of the theorem considers the marginal rate of substitution (which is defined to be positive) between changes in background risk and income $x$ which leave derived marginal utility unaffected. Theorem 1c) states that, given positive prudence, this marginal rate of substitution decreases [stays constant] [increases] with marketable income if and only if absolute prudence is decreasing [stays constant] [increases] in total income, for all levels of income. Hence, positive declining absolute prudence implies that the marginal rate of substitution is lower, the higher is the income $x$. In other words, at higher levels of marketable income, the agent is willing to accept a smaller increase in income to compensate for a small increase in background risk to maintain the same level of derived marginal utility.

[11]Gollier and Pratt (1993) are concerned only with a comparison of the case where $\sigma > 0$ with the case where $\sigma = 0$. Also, since they allow for background risks with negative expected values, they require decreasing absolute risk aversion.

# 4 BACKGROUND RISK AND THE OPTIMAL SHARING RULE

After establishing the basic results on the behavior of an agent who faces background risk, we can now characterize his/her optimal purchases of claims on *marketable* income. We assume that the capital market is perfect and complete with respect to marketable aggregate income, $X$. Specifically, the agent can buy claims on the marketable aggregate income which produce an end-of-period payoff represented by the sharing rule $x = g(X)$. Thus, each agent in the economy solves the following maximization problem[12]

$$\max_{x=g(X)} E[\nu(x+\sigma\varepsilon)] \tag{11}$$

subject to

$$w = E[\Phi(X)x]$$

where $w$ is the agent's initial endowment and $\Phi(X)$ is the market pricing function for state-contingent claims on $X$. Note that the market pricing function gives the market price of a state contingent claim divided by the probability of the state occurring. $\Phi(X)$ is determined in equilibrium and is taken as given by the agent. The first order condition for a maximum is

$$E[\nu'(x+\sigma\varepsilon)|x] = \lambda\Phi(X) \tag{12}$$

where $E(\cdot|x)$ is the conditional expectation given $x$, the payoff, and $\lambda$ is a positive state independent Lagrange multiplier reflecting the tightness of the budget constraint. Notice that the derived marginal utility of the agent is proportional to the price $\Phi(X)$. If all agents are risk averse, it follows immediately that $\Phi'(X) < 0$. To see this, if we differentiate equation (12) with respect to X, we have

$$\frac{\partial x}{\partial X} = \frac{\partial g(X)}{\partial X} = \frac{\lambda\Phi'(X)}{E[\nu''(x+\sigma\varepsilon)|x]} \tag{13}$$

Thus, $\partial g(X)/\partial X$ has the same sign for each agent and is positive, since all the marketable aggregate income has to be allocated, in equilibrium. Since $\partial g(X)/\partial X > 0, \lambda > 0$ and

[12] We drop the subscript $i$ for notational simplicity.

$\nu'' < 0$, it follows immediately that $\Phi'(X) < 0$. This establishes our first result regarding the optimal sharing rule for the individual agent in this economy. Agents buy state-contingent claims that are increasing in the level of marketable aggregate income X. The sharing rule has a positive slope as it does in the absence of background risk [see Rubinstein (1974)].

We now investigate the effect of changing background risk on the agent's sharing rule assuming that the market pricing function $\Phi(X)$ is given.

Theorem 2 (Background Risk and The Optimal Sharing Rule).

*Consider an agent with positive risk aversion. Then, if $\Phi(X)$ is exogenous*

*a)* $\partial^2 g(X)/\partial\sigma\partial X < [=][>]\ 0,\ \forall(X,\sigma) \iff \partial\hat{a}(x,\sigma)/\partial\sigma > [=][<]\ 0,\ \forall(x,\sigma)$

*b) If* $\partial\hat{a}(x,\sigma)/\partial\sigma > 0$, $\exists\ X^*$, *such that* $\partial g(X)/\partial\sigma > [=][<]\ 0$ *for* $X < [=][>]\ X^*$.

*If* $\partial\hat{a}(x,\sigma)/\partial\sigma < 0$, *then the inequality signs for* $\partial g(X)/\partial\sigma$ *are reversed.*

Proof

a) Equation (12) can be written using the precautionary premium, as

$$\nu'[x - \psi(x,\sigma)] = \lambda\Phi(X) \tag{14}$$

where $\nu'(.)$ is the derived marginal utility of the agent. Differentiating the logarithm of equation (14) with respect to $X$, and using equation (3), we find, since $\lambda$ is not dependent on $X$,

$$-\hat{a}(x,\sigma)\partial g(X)/\partial X = \partial ln\Phi(X)/\partial X$$

Now, differentiating again with respect to $\sigma$, since $\Phi(X)$ is exogenous, we have

$$\frac{\partial\hat{a}(x,\sigma)}{\partial\sigma}\frac{\partial g(X)}{\partial X} + \frac{\hat{a}(x,\sigma)\partial^2 g(X)}{\partial X\partial\sigma} = 0$$

Theorem 2a follows immediately from the assumption that $a(y)$, and hence also $\hat{a}(x,\sigma)$, is positive and $\partial g(X)/\partial X > 0$.

b) Suppose that $\partial\hat{a}(x,\sigma)/\partial\sigma > 0$. Then, from part a) of the theorem, an increase in $\sigma$ results in a relatively large increase in the demand for claims in low states of $X$. Since the purchase and sale of state-contingent claims must be self-financing, there must exist a critical level $X^*$ such that $\partial g(X)/\partial\sigma$ is positive for $X < X^*$, and negative for $X > X^*$. A similar argument can be used to establish the second part of Theorem 2b. □

Theorem 2 allows us to analyze the effect of a marginal increase in a zero-mean, independent background risk, given that this increase has a negligible impact on the equilibrium prices of state-contingent claims.[13] In Theorem 1b, we established a necessary and sufficient condition for $\partial\hat{a}(x,\sigma)/\partial\sigma > 0$. Suppose now that this condition is fulfilled for a particular agent. Theorem 2a) says that an increase in $\sigma$ will reduce the slope of this agent's optimal sharing rule. As can be seen from Theorem 2b), the agent reacts to an increase in $\sigma$ by purchasing more claims in states of low marketable income, financing the purchase by selling some claims in states of high marketable income. Theorem 2 can also be interpreted by comparing, within an equilibrium, the demand of agents, who differ only in the size of their respective background risks. Theorem 2 suggests that agents with high background risk will buy more state-contingent claims on low states and fewer claims on high states. Comparing agents with no background risk to those with positive background risk, the former will tend to sell part of their claims on low states to the latter. Agents with high background risk will buy "insurance" (i.e. claims on low states) from those with low background risk.

It is important to emphasize that our optimal sharing rule result does not require standard risk aversion, although standard risk aversion clearly is a sufficient condition. However, standard risk aversion permits us to decompose the total effect of an increase in background risk on the sharing rule into an income effect and a substitution effect, both of which have the same sign. Totally differentiating the left hand side of equation (14) with respect to $\sigma$ yields

$$\frac{d\nu'(\cdot)}{\cdot\, d\sigma} = \frac{\partial\nu'(\cdot)}{\partial\sigma} + \frac{\partial g(X)}{\partial\sigma}\frac{\partial\nu'(\cdot)}{\partial g(X)}.$$

It also follows from equation (14) that $d\nu'(\cdot)/d\sigma = \nu'(\cdot)\partial\ln\lambda/\partial\sigma$. Hence, the effect of the background risk on the sharing rule is given by

$$\frac{\partial g(X)}{\partial\sigma} = \frac{[\partial\ln\lambda/\partial\sigma]\nu'(\cdot)}{\partial\nu'(\cdot)/\partial g(X)} - \frac{\partial\nu'(\cdot)/\partial\sigma}{\partial\nu'(\cdot)/\partial g(X)} \tag{15}$$

The effect of a change in the background risk on the sharing rule shown in equation (15) is the sum of two factors. The first term is the income effect and the second term is the substitution effect of the change in $\sigma$. We now investigate the behavior of these two effects as a function of the size of the background risk.

[13]Note that the conditions of Theorem 2 (derived from Theorem 1b) ) are weaker than those required for standard risk aversion.

Corollary 2

*Consider an agent with positive risk aversion and positive prudence. Suppose that the scale of the agent's background risk, $\sigma$, increases. Then*

*a)*

$$\frac{\partial}{\partial X}\left[\frac{[\partial \ln \lambda/\partial\sigma]\nu'(\cdot)}{\partial\nu'(\cdot)/\partial g(X)}\right] < 0 \iff \frac{\partial a(y)}{\partial y} < 0, \forall y$$

*b)*

$$\frac{\partial}{\partial X}\left[-\frac{\partial\nu'(\cdot)/\partial\sigma}{\partial\nu'(\cdot)/\partial g(X)}\right] < 0 \iff \frac{\partial\eta(y)}{\partial y} < 0, \forall y$$

*c) Suppose that* $\frac{\partial a(y)}{\partial y} < 0$ *and* $\frac{\partial\eta(y)}{\partial y} < 0$*, i.e. the agent is standard risk-averse. Then* $\partial^2 g(X)/\partial\sigma\partial X < 0$*. Also,* $\exists X^*$ *such that* $\partial g(X)/\partial\sigma > [=][<]0$ *for* $X < [=][>]X^*$*.*

Proof

a) Corollary 2a) states that the income effect of an increase in background risk declines in $X$ if and only if $a(y)$ is declining. To prove this note that the income effect can be written as

$$-\frac{1}{\lambda}\frac{\partial\lambda}{\partial\sigma}\frac{1}{\hat{a}(x,\sigma)}$$

If $\partial\lambda/\partial\sigma > 0$, the income effect is decreasing in $x$, and hence, in $X$, if and only if the derived risk aversion is decreasing in income i.e. $\partial\hat{a}(x,\sigma)/\partial x < 0$. Using Theorem 1a), this is the case if and only if $\partial a(y)/\partial y < 0$. It remains to be shown that $\partial\lambda/\partial\sigma > 0$. To establish this, note that the budget constraint requires $\partial g(X)/\partial\sigma$ to be positive in some states and negative in others, with a state being defined by the level of aggregate marketable income. From equation (15), $\partial g(X)/\partial\sigma < 0$, in some states at least, would require that $\partial \ln\lambda/\partial\sigma > 0$, and hence, $\partial\lambda/\partial\sigma > 0$.

b) Corollary 2b) states that the substitution effect of an increase in background risk declines in $X$, if and only if absolute prudence declines in $y$. Since $\partial g(X)/\partial X$ is positive, Corollary 2b) follows directly from Theorem 1c).

c) If both absolute risk aversion and absolute prudence decline in income, $y$, it follows immediately from Corollaries 2a) and 2b) that the total (income plus substitution) effect of an increase in background risk declines in $X$. This establishes the first part of Corollary 2c). Then, the second part of Corollary 2c) follows as it does in Theorem 2b). □

Corollary 2 allows us to analyze separately the income and the substitution effects of a marginal increase in background risk, given that this increase has a negligible impact on equilibrium prices. Corollary 2a) analyses the income effect of an increase in background risk. This decreases with marketable aggregate income if and only if the agent's absolute risk aversion declines. Corollary 2b) then analyses the substitution effect. This decreases with $X$ if and only if the agent's absolute prudence declines. Then, Corollary 2c) states that standard risk aversion is a sufficient condition for the change in the demand for contingent claims to be declining in $X$.

Kimball (1993, p.594) has shown that globally decreasing absolute prudence implies globally decreasing absolute risk aversion and, hence, standard risk aversion. It follows, then, that positive decreasing absolute prudence is the essential assumption for Corollary 2 to hold.

# 5 PRICING EFFECTS OF CHANGES IN BACKGROUND RISK

In the preceding section, sharing rules of agents have been investigated taking the prices of contingent claims as given. Theorem 2 established that an agent demands more claims in the states with low marketable aggregate income and less claims in the states with high marketable aggregate income, if derived risk aversion increases. Suppose now that there is an aggregate shock such that background risk increases the derived risk aversion for most agents in the economy. This increase in background risk for most agents in the economy can be measured by an index $\sigma$, such that an increase in the index $\sigma$ signifies that the background risk for all agents increases or stays the same.[14] Then, these agents wish to buy more claims which pay off in the "low" states at the *pre-shock* prices. But, this excess demand cannot be satisfied since the agents' additional demand for marketable claims must sum to zero for each and every state. Hence, the prices of all claims must change to reflect the change in demand. Theorem 2 suggests that claims in "low" states will become relatively more expensive compared to claims in "high" states. A somewhat weak qualification is necessary, however, for this implication to go through. This qualification is a

[14] Strictly speaking, we could even allow for the background risk for a *few* agents to go down.

restriction on the harmonic mean of the agents' risk aversion. We now state the equilibrium implication of an increase in background risk.

<u>Theorem 3</u>: (The Pricing Effect of a General Increase in Background Risk)

*Consider a small rise in aggregate background risk indexed by $\sigma$. Then*

$$\frac{\partial(\Phi(X_s)/\Phi(X_t))}{\partial\sigma} > [=][<]0, \quad \forall[(s,t): X_s < X_t] \Longleftrightarrow \frac{\partial\sum_i 1/\hat{a}_i(x_i,\sigma_i)}{\partial\sigma} < [=][>]\ 0, \quad \forall X.$$

<u>Proof</u>:

The first order condition for an optimal sharing rule of agent $i$ is

$$\nu_i'(g_i(X) - \psi_i(X,\sigma)) = \lambda_i\Phi(X).$$

Taking logarithms of this equation and differentiating with respect to X, we have

$$\frac{\partial g_i(X)}{\partial X} = -1/\hat{a}_i(x_i,\sigma_i)\frac{\partial \ln\Phi(X)}{\partial X}, \quad \forall X$$

Then, since the change in the portfolio demands have to equal to the change in the aggregate marketable income, aggregation over all agents yields the equilibrium condition

$$1 = -\frac{\partial \ln\Phi(X)}{\partial X}\sum_i 1/\hat{a}_i(x_i,\sigma_i).$$

Hence, $-\partial\ln\Phi(X)/\partial X$ equals the harmonic mean of the agent's derived risk aversion. An increase in background risk affects both terms equally. Clearly, an increase in background risk across many agents decreases $\partial\ln\Phi(X)/\partial X$ if and only if it increases the harmonic mean. Thus $\partial^2\ln\Phi(X)/\partial X\partial\sigma < 0, \forall X$, means that the growth rate of $\Phi(X_t)$ is less than that of $\Phi(X_s)$ for *every* pair (s,t) with $X_s < X_t$. Since, $\partial^2\ln\Phi(X)/\partial X\partial\sigma < [=][>]\ 0, \forall X$ is true, if and only if $\partial\sum_i 1/\hat{a}_i(x_i,\sigma_i)/\partial\sigma < [=][>]\ 0, \forall X$. Hence, Theorem 3 follows. □

The intuition behind Theorem 3 is straightforward. If an increase in background risk generates an excess demand in state s which is higher than that in state t, the price relative $\Phi(X_s)/\Phi(X_t)$ must change in order to make these excess demands disappear. Clearly, we would expect the price relative to increase. But price changes can have various feedback effects, for example, on the agents' initial endowments. Moreover, the price change of one state interacts with the price changes in other states. Therefore, these feedback effects need to be constrained in order to get an unambiguous answer [Deaton and Muellbauer (1981)].

The necessary and sufficient condition established in Theorem 3 is that the harmonic mean of the agents' derived risk aversion is an increasing [constant] [decreasing] function of background risk in every state. This condition is quite plausible since, without trade and without endowment changes, an increase in an agent's background risk changes his/her derived risk aversion in a specified direction according to Theorem 1b). If this change is similar for all agents, then the condition of Theorem 3 is satisfied. If all agents are "reasonably" similar in terms of endowments, risk aversion and background risk, then trade and endowment effects cannot overturn this. But, if agents are very different in terms of endowments or background risk, it is possible that, after trade, the harmonic mean of the derived risk aversions increases in some states, and decreases in others. This possibility is ruled out by the condition that the harmonic mean changes in the same direction in every state.

An interesting implication of Theorem 3 is that the reward for risk-bearing increases with background risk if prices for claims in the "low" states increase at a higher rate than prices for claims in the "high" states. If the prices of all claims adjusted by the probability of occurrence of the states were the same, then the pricing would be risk-neutral, so that the reward for risk-bearing would be zero. With risk aversion, $\Phi(X)$ is a decreasing function. If these price differentials are reinforced by background risk, then the reward for risk-bearing is increased.

# 6 CONCLUSIONS

In this paper, we have considered the effect of an increase in background risk on an agent's risk attitudes and on his/her optimal purchase of state-contingent claims on aggregate income. This framework has several applications to problems in financial economics. The common feature of all these applications is that there are some background risks that are non-marketable, but the agent is able to manage his/her overall risk by buying and selling marketable claims.

Consider first the case of the owner of a firm whose shares are not traded, but whose profit is dependent on several economy-wide variables such as interest rates, foreign exchange rates and commodity prices which can be hedged against by using marketable claims. In addition to these economy-wide risks, suppose the profit of the firm is also affected by firm-specific factors which cannot be hedged. The question is how the entrepreneur will hedge against the economy-wide risks, given the exposure to the background risks. If the entrepreneur's risk aversion is increased by background risk, his/her optimal response will be to reduce exposure to the economy-wide, hedgeable factors.

As a second example, consider a firm which owns a portfolio of real options whose value depends upon the level of an economy-wide factor and the volatility of that factor. Suppose that the firm can hedge against the economy-wide factor, but is not able to hedge the volatility, which is, therefore, a background risk. The response of the firm to an increase in volatility will be, again, to reduce its exposure to the economy-wide factor.

As a third example, consider the behavior of the manager of a firm, whose compensation is based on the profit of the firm relative to the profits of competing firms. The manager can affect the risk of the firm's own profits by buying claims which are negatively related to these profits. However, here additional risk to his income is created by the profitability of the competing firms. Assuming that this risk is non-hedgeable, it becomes a background risk for the manager. The "normal" response predicted by our analysis here is that the manager will favor hedging policies that reduce the risk of the firm's profits.

We have analyzed the generic problem relating to decision-making by an agent facing background risk by first considering the properties of the derived utility function, i.e. the utility function modified by the presence of background risk. We characterized the derived utility function of the agent in several ways. First, there is a similarity between the risk aversion exhibited by the original utility function and the risk aversion of the derived utility function. Second, we derived a condition under which the effect of background risk on the derived risk aversion of the agent could be predicted. Third, the agent's marginal rate of substitution between changes in background risk and total income which preserve the level of derived marginal utility is determined by the change in the coefficient of absolute prudence in relation to income.

We then used the above general results to characterize the "portfolio" decisions of the agent who faces an increase in background risk and reacts by buying and selling claims on marketable aggregate income. If an increase in background risk raises the agent's derived risk aversion, this would induce the agent to buy a type of insurance to optimize risk bearing: the insurance involves the purchase of claims that pay off in "low" states of marketable aggregate income financed by the sale of claims that pay off in the "high" states. The amount of insurance purchased grows with the rise of the background risk.

When background risk increases for many agents, then prices of marketable claims must adjust. Under a mild condition, we have shown that prices of claims in the "low" states of marketable aggregate income increase relative to the prices of claims in the "high" states. This price change raises the reward for risk-bearing, and motivates agents whose background risk has not increased to take more risk in the market.

## Appendix

### Theorem 1b: Proof of Sufficiency

We use a method similar to that used by Pratt and Zeckhauser (1987) and by Gollier and Pratt (1993).

a) We first show

$$\begin{aligned} v'''(y_1) - v'''(y_2) &> a(y)\left[v''(y_2) - v''(y_1)\right], \forall y_1 \leq y \leq y_2 \\ \Longrightarrow f(x,\sigma) &> 0, \ \ \forall (x,\sigma) \end{aligned} \tag{16}$$

We need to show that $f(x,\sigma) > 0$, for all non-degenerate probability distributions. Hence, we need to prove that the minimum value of $f(x,\sigma)$ over *all* possible probability distributions $\{p_i\}$, with $E(\varepsilon) = 0$, must be positive. In a manner similar to Gollier and Pratt (1993), this can be formulated as a mathematical programming problem, where $f(x,\sigma)$ is minimized, subject to the constraints that all $p_i$ are non-negative and sum to one, and $E(\varepsilon) = 0$. Equivalently, this can be reformulated as a parametric linear program where the non-linearity is eliminated by writing $\bar{a}$ as a parameter

$$\min_{\{p_i\}} f(x,\sigma) = \sum_i p_i \left[\epsilon_i \left\{-v'''(y_i) - v''(y_i)\bar{a}\right\}\right] \tag{17}$$

s.t.

$$\sum_i p_i \epsilon_i = 0 \tag{18}$$

$$\sum_i p_i = 1 \tag{19}$$

the definitional constraint for the parameter $\bar{a}$

$$\bar{a} \sum_i p_i v'(y_i) = -\sum_i p_i v''(y_i) \tag{20}$$

and the non-negativity constraints

$$p_i \geq 0, \quad \forall i \tag{21}$$

A sufficient condition for $\partial\hat{a}/\partial\sigma > 0$ is that $f(x,\sigma)$ as defined by (17) is positive for *any* probability distribution $\{p_i\}$ subject to $E(\varepsilon) = 0$ and the definition of $\bar{a}$ given in (20).

Since we are looking for a sufficient condition for $f(x,\sigma) > 0$, we can relax the non-negativity constraint for $p_0$ in the above linear program. In case even this (infeasible) resulting minimum is positive, then we know that the solution of the above linear program is always positive. We drop the non-negativity constraint on $p_0$, the probability of the zero $\varepsilon$ state in the following manner. We define $p_0^+$ and $p_0^-$ such that

$$p_0 = p_0^+ - p_0^- \tag{22}$$

where both $p_0^+$ and $p_0^-$ are non-negative. These new variables replace $p_0$ in the program.

The modified linear program has three variables in the basis since there are three constraints in the program. In the optimal solution, one basis variable is either $p_0^+$ or $p_0^-$. Hence, the optimal solution of the modified linear program is $(p_0, p_1, p_2)$ and and the objective function is

$$f^*(x,\sigma) = p_1\epsilon_1\left[-v'''(y_1) - v''(y_1)\bar{a}\right] + p_2\epsilon_2\left[-v'''(y_2) - v''(y_2)\bar{a}\right] \tag{23}$$

Since $p_1\epsilon_1 + p_2\epsilon_2 = 0$, it follows that (23) can be rewritten as

$$f^*(x,\sigma) = p_1\epsilon_1\left[(-v'''(y_1) - v''(y_1)\bar{a}) - (-v'''(y_2) - v''(y_2)\bar{a})\right] \tag{24}$$

Hence

$$v'''(y_1) - v'''(y_2) - \left[v''(y_2) - v''(y_1)\right]\bar{a} > 0 \tag{25}$$

is a sufficient condition for $f^* > 0$, given $\bar{a}$.

As shown in equation (9), $\bar{a}$ is a convex combination of $a(y_0)$, $a(y_1)$ and $a(y_2)$, hence $\bar{a} \in \{a(y) | y \in [y_1, y_2]\}$. Hence, a sufficient condition for (25) is that

$$v'''(y_1) - v'''(y_2) - a(y)\left[v''(y_2) - v''(y_1)\right] > 0 \tag{26}$$

for all $\{y_1 \le y \le y_2\}$ as given by the condition of the theorem.

b) By an analogous argument, it can be shown that

$$\begin{aligned} v'''(y_1) - v'''(y_2) &< a(y)\left[v''(y_2) - v''(y_1)\right], \forall y_1 \le y \le y_2 \\ \Longrightarrow f(x,\sigma) &< 0 \quad \forall (x,\sigma) \end{aligned} \tag{27}$$

c) We now show directly that

$$\begin{aligned} v'''(y_1) - v'''(y_2) &= a(y)\left[v''(y_2) - v''(y_1)\right], \forall y_1 \le y \le y_2 \\ \Longrightarrow f(x,\sigma) &= 0 \quad \forall (x,\sigma) \end{aligned} \tag{28}$$

A sufficient condition for $f(x,\sigma)=0, \forall(x,\sigma)$ is that $\min_{\{p_i\}} f(x,\sigma)=\max_{\{p_i\}} f(x,\sigma)=0$, subject to (18)-(20) and the nonnegativity condition for every $p_i$ except $p_0$. The minimum and maximum involve three basis variables, one of which is either $p_0^+$ or $p_0^-$. Therefore, $f^*(x,\sigma)$ is always determined by (24). Hence, the minimal and maximal value of $f^*(x,\sigma)$ are zero if the bracketed term in (24) is zero. This is the case if

$$v'''(y_1)-v'''(y_2)=a(y)\left[v''(y_2)-v''(y_1)\right], \quad \forall y_1 \le y \le y_2. \tag{29}$$

As $y_1 \rightarrow y_2$, equation (29) becomes

$$-v''''(y)=a(y)v'''(y) \quad \forall y \tag{30}$$

The differential equation (30) is satisfied by a family of utility functions of which the quadratic and exponential functions are the most familiar ones. □

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