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Repeated Auctions with Endogenous Selling*

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Abstract

This paper studies trade in repeated auction markets. We show, for conditionally independent signals, that an owner's decision to sell, expected prices, and continuation values are the same for a large class of auction mechanisms, extending the Revenue Equivalence Theorem to a multi-period setting. Further, we derive a robust No-Trade Theorem. For conditionally affiliated signals, we give conditions under which revenue ranking implies volume and welfare ranking. In particular, we show that English auctions have larger volume and welfare than second-price auctions, which in turn have larger volume and welfare than first-price auctions.

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Keywords: auctions, revenue equivalence, no trade, volume, welfare.

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1 Introduction

Fine art is usually sold in a traditional auction, and the same piece of art is often traded repeatedly. Real estate is sometimes traded in auctions, and a house is often held by several owners during its life. Similarly, planes, land, and many other durable goods are repeatedly traded in auctions. Further, most financial securities are traded using auction-like mechanisms, and securities are often turned over many times before they mature. Important examples include blocks of shares, bonds, OTC derivatives, and, recently, the right to use shares for shorting.

This paper studies the effect of the information structure and trading mechanism on prices, volume, and welfare in a dynamic economy with double-sided asymmetric information.

Auctions have been studied extensively, starting with the seminal paper by Vickrey (1961).[1] This literature is based on the assumptions that the current owner must sell, and that the buyer of an asset keeps the asset throughout its life. In many situations, however, such as those mentioned above, the *sale decision is endogenous*, and the buyer of an asset anticipates later *resale* of the asset. Although resale motives are important in many markets, resale has received little attention in the auction literature. Bikhchandani and Huang (1989) consider an auction model in which bidders in the "primary market" must sell immediately after the auction in a competitive "secondary market." This makes the model essentially a single-period model in which interesting issues regarding information revelation through bidding arise. Haile (1999) studies a generalized version of the model by Bikhchandani and Huang (1989), while Haile (2001) finds resale to be important in U.S. timber sales. Further, Haile (2003) considers resale due to new information immediately after the auction, Ausubel and Cramton (1999) and Zheng (2002) consider (multiple unit) auctions with asymmetric agents and efficient secondary markets, and Nyborg and Strebulaev (2000) study short squeezes in the secondary market.

Common among these papers is that they take the initial sale as exogenous, consider an immediate possible resale, and, therefore, the main focus is on the signaling problem associated with the information revealed by bids. These assumptions are natural for Treasury auctions, timber auctions, and in other markets with active secondary markets.

[1] For an overview of auction theory see Klemperer (2000).

A work by Picasso, a plane, a piece of land, or a block a shares, on the other hand, are sold when the owner chooses to do so, and such an asset is usually not bought with the intention of immediate resale. Rather, such an asset is typically held for a significant time period before it is resold, and, at the time of resale, the information revealed in the previous auction is no longer of first-order relevance. In order to study repeated trade of such assets, we consider a multi-period model in which *(i)* every period, the owner and the possible buyers receive private information about the value of owning the asset this period (the asset's agent-specific use value or "dividend"), *(ii)* based on his information, the current owner decides whether to keep the asset or to sell it, and *(iii)* the private information is short lived in the sense that what is known privately this period is made public or becomes irrelevant next period.

These assumptions imply that on owner's sale decision depends on his expected dividend this period, the expected price he can raise in an auction, the value of owning the asset in the future periods, and the value of not owning the asset depends on the possibility of buying the asset at a relatively low price. In case of a sale, bidders are concerned with the information revealed by the owner's decision to sell, the value of the future dividends, the ease with which the object can be sold in the future, and the possible rents that one can extract as a buyer in the future. A straightforward implication is that, in a Markov equilibrium, bidders bid as in a single-period auction in which the "prize" is the next dividend plus the (continuation-)value of being an owner less the value of being a buyer (similarly to Haile (2001) and Haile (2003)). Bidders take into account the (adverse) information revealed by the fact that the owner has decided to sell. Hence, solving our dynamic auction model is similar to standard single-period auctions with the complication that one must determine the equilibrium set of signals for which the owner decides to sell, a fixed point problem.

We give conditions under which the owner's decision to sell, expected prices, and expected continuation values are the same for a large class of auction mechanisms. The most important condition underlying this result is that private signals are independent conditional on the public signal. This generalizes the Revenue Equivalence Theorem[2] to a multi-period setting, and

[2]Vickrey (1961) showed that certain different auction methods have the same expected revenue. General revenue equivalence results were derived first by Myerson (1981) and Riley and Samuelson (1981). That allocation equivalence implies the equivalence of value functions is a general property of games in which agents have (conditionally) independent,

incorporates irrelevance with respect to *allocations* as well as revenue.

Further, under these assumptions, we can investigate the circumstances under which trade is impossible (or liquidity "dries up") without specifying a trading mechanism. It follows intuitively from the No-Trade Theorems of Kreps (1977) and Milgrom and Stokey (1982) that no trade is an equilibrium if the dividend does not depend on who owns the asset (pure common values). We derive a robust version of the No-Trade Theorem by showing that there cannot be trade if the values are "too common." Said differently, trade can only occur if the gains from trade are greater than some threshold. This threshold equals the owner's expected cost of selling, which is due to the rents extracted by buyers because of their private information.

So far, we have discussed the simple benchmark case of (conditionally) independent signals. In common-value auctions, however, it may be the case that agents have correlated signals. In particular, these signals may satisfy the intuitively appealing condition of affiliation, which, informally, means that if one agent's signal is good then the other agents' signals are likely to be good, as well. When private signals are affiliated (conditional on the public signal), Milgrom and Weber (1982) show that the expected revenue of an English auction is higher than that of a second-price auction, which, in turn, has a higher expected revenue than a first-price auction. We give general conditions under which "revenue ranking" implies "volume ranking" and "welfare ranking." By this, we mean that the owner sells on a larger set of signals when using the auction mechanism that generates higher expected revenue *for a fixed (anticipated) sale set*, and that this leads to a more efficient allocation of the asset. In particular, we show that volume and welfare are higher with English auctions than with second-price auctions, and volume and welfare are higher with second-price auctions than with first-price auctions, under certain conditions.

The intuition for these results is as follows. Consider, for instance, the case of the first-price and the second-price auctions, and an equilibrium for the first-price auction with sale set X. If the owner had to use a second-price auction instead, and the bidders still expected X to be the sale set, then the owner would have a higher expected price conditional on any signal he might have. (See the discussion below.) Hence, the owner would still want

short-lived private information, and such a results are derived in a different dynamic settings by Athey, Bagwell, and Sanchirico (2000) and Haile (2003). We note that the allocation equivalence is part of our result, namely that the sale sets are the same.

to sell for signals in X, and also if he had (slightly) better signals, so he would sell on a larger set, $X^2 \supset X$, of signals. If the bidders anticipated that the owner sold if his signal were in X^2, then they would bid more. Taking this into account, the owner would sell on an even bigger set, $X^3 \supset X^2$, of signals, and so on. In the limit we find an equilibrium sale set, $\cup_i X^i$, for the second-price auction, which is clearly larger than the first-price-auction sale set. This argument also shows that the expected price is higher for the second-price auction if the continuation values in the two mechanisms are the same — as is the case in the last period. In earlier periods, however, the expected price may not be higher for the second price auction, as is discussed further in the paper. The second price auction has, nevertheless, the larger sale set in *all* periods.

Note that the argument above uses the fact that, for a given (anticipated) sale set, the second-price auction has a higher expected price than the first-price auction *conditional on any signal the owner might have.* This is a stronger result than that of Milgrom and Weber (1982), who show that the price is higher *when averaging over the owner's signals.* We show that this "strong revenue ranking" applies when comparing the first- and second-price auctions. The strong revenue ranking may not apply, however, when comparing the English and second-price auctions. The English auction has a smaller winner's curse, which increases expected prices (Milgrom and Weber (1982)), but it also has the effect of partially revealing the owner's signal in the course of the auction. Therefore, conditional on a low owner signal the expected price in an English auction could be lower than that of a second-price auction. For high owner signals, on the other hand, we have the standard ranking of expected prices. The highest owner signal in the sale set is the signal for which the owner is indifferent between selling and not selling. This "marginal" signal is what determines the equilibrium, and since revenue ranking applies conditional on this signal, the English auction has a larger equilibrium sale set than the second-price auction.

In addition to this volume ranking, we also rank mechanisms in terms of welfare. The intuition for welfare being higher with a higher-volume mechanism is as follows: When the owner decides to sell, the expected price is higher than his expected utility of keeping the asset. Further, the buyer expects a higher utility than the price. Therefore, the buyer's utility of owning is higher than the seller's, and the trade is welfare improving. This intuition over-simplifies the problem slightly by ignoring conditioning information, but, under certain conditions, it is correct at least for high owner signals, implying that a higher-volume mechanism indeed is associated with higher welfare.

The paper is organized as follows. Section 2 lays out our model of repeated auctions. Section 3 provides our results regarding multi-period revenue equivalence and no trade. Section 4 considers the case of conditionally affiliated signals, and Section 5 concludes. Proofs are in the appendix.

2 A Model of Repeated Auctions

In this section we present a model of repeated auctions and show how to compute its equilibria.

A finite set, $\mathcal{N} = \{0, 1, \ldots, n\}$, of (ex-ante) identical risk-neutral agents trades one object[3] in each of periods $0, 1, \ldots, T-1 \leq \infty$. If agent i owns the security at the beginning of period t, then he receives a dividend given by the real-valued random variable V_t^i. We denote the probability space on which all random variables are defined by $(\Omega, \mathcal{F}, Pr)$, and let $o_t \in \mathcal{N}$ denote the owner at the beginning of period t. After the dividend is paid, each agent, i, receives a private signal, x_t^i, and a public signal, y_t. Here, x_t^i and y_t are random variables with compact supports, $\mathbb{X}$ and $\mathbb{Y}$, respectively, that are subsets of Euclidean spaces. These signals help agents predict the next period's dividend, V_{t+1}^i. After the owner of the security has seen his information, he decides whether to keep the object or to offer it for sale. If the owner decides to sell, the object is sold using an auction mechanism M.

An ***auction mechanism***, $M = (\Theta, \pi, z, \alpha)$, consists of a measurable space,[4] Θ, of allowed "bids"[5] and a triplet, (π, z, α), of mappings defined as follows. To an n-tuple of bids,[6] $\pi : \Theta^n \to \Delta_n$ assigns the probabilities with which each of the bidders acquires the object and $z : \Theta^n \to \mathbb{R}^n$ assigns the

[3]Some of our results extend to multi-object models in which each agent can hold one object only.

[4]Also, a mechanism must specify a σ-algebra on Θ. We require throughout that all mappings and subsets considered to be measurable. We use the Borel σ-algebra on Euclidean spaces.

[5]For sealed-bid auctions, a bid is just a real number. However, for other auction mechanisms a "bid" can be a complicated strategy. In an English Auction, for example, a "bid" is a specification of the price level at which to drop out conditionally on the price levels at which other bidders have dropped out. This is why we allow a general specification of Θ.

[6]Here, $\Delta_n = \{(x_1, \ldots, x_n) \in [0,1]^n : \sum_{i=1}^n x_i = 1\}$ is the n-simplex.

amounts to be paid by the bidders. The mapping $\alpha : \Theta^n \times \{1, \ldots, n\} \to \mathcal{I}$, for some measurable space $\mathcal{I}$, is such that $\alpha(\theta_1, \ldots, \theta_n, i)$ is the information revealed by the mechanism to all of the participants when the bids are $\theta_1, \ldots, \theta_n$, and i is the agent who receives the object. We assume that all agents receive the same information through the mechanism. In particular, the information, α, reveals the identity, o_{t+1}, of the new owner. The price received by the seller, $p : \Theta^n \to \mathbb{R}$, is the sum of the transfers from the bidders. That is, $p = \sum_{i=1}^{n} z_i$. We could also assume that there is a transaction cost associated with the auction. While such a feature may add realism, we abstract from it since it does not affect our results.

We do not consider reserve prices for the following reasons. First, if the owner would set a reserve price and the object is not sold, then he would be tempted to hold another auction in the same time period. This inability to commit to a reserve price reduces its effect — see McAfee and Vincent (1997) and Horstmann and LaCasse (1997). Second, Myerson and Satterthwaite (1983) show that double-sided asymmetric information implies that all efficient trades cannot take place no matter what mechanism is used. Third, we can envisage a model in which the owner sells the object to a risk-neutral competitive intermediary, who has no use for the object and who sells the object immediately to the other traders. This assumption means that the current owner has no price risk, and the intermediary has no reason to keep the asset. In such a model, nothing changes in the case of conditionally independent signals, while the same qualitative results obtain[7] when signals are affiliated.

The repeated-auction game is defined as follows. At each time the owner decides whether to sell or keep the asset and buyers decide how much to bid in case there is a sale. A ***strategy*** for agent i is defined as a process $A = \left(A_t\right)_{t=0}^{T-1}$, where $A_t : \Omega \to \{sell, keep\} \cup \Theta$ is measurable with respect to the information[8] $\mathcal{F}_t^i$ available to agent i at time t. A strategy A for agent i is said to be ***feasible*** provided $A_t \in \{sell, keep\}$ if and only if agent i is the owner at the beginning of period t (i.e. $o_t = i$).

[7]In fact, the analysis is cleaner, in that fewer assumptions are necessary.

[8]Formally, $\mathcal{F}_t^i = \sigma(\beta_{-1}, x_0^i, y_0, \beta_0, \ldots, x_{t-1}^i, y_{t-1}, \beta_{t-1}, x_t^i, y_t)$, where for $u \geq 0$, $\beta_u = \alpha(\theta_1, \ldots, \theta_n, i)$ if a sale took place at u with bids $\theta_1, \ldots, \theta_n$ and bidder i receiving the asset, and otherwise β is a fixed constant (conveying no information). We assume that β_{-1} contains the identity, o_0, of the owner at time 0, and the initial value, y_{-1}, of the public signal.

The ***utility***, Π^i, of agent i is given by

$$\Pi^i(A^0, \ldots, A^n) = E\left(\sum_{u=1}^{T} \delta^u V_u^i 1_{(i=o_u)} - \sum_{u=0}^{T-1} \delta^u \bar{z}_u^i \;\middle|\; \beta_{-1} \right),$$

where, $\bar{z}_u^i$ is the net cash payment (receipt if negative) made by agent i at time u. That is, if there is a sale, $\bar{z}_u^i = z_u^i$ for $i \neq o_u$ and $\bar{z}_u^i = -p_u$ for $i = o_u$. If there is no sale, then $\bar{z}_u^i = 0$ for all i. We are ready to define an equilibrium in the repeated auction game.

Definition 1 *An* **equilibrium** *for* (M, x, y, V, T) *is a set of feasible strategies,* $A = (A^0, A^1, \ldots, A^n)$*, such that, for all* $i \in \mathcal{N}$*,*

$$\Pi^i(A) \quad \geq \quad \Pi^i(A^0, \ldots, A^{i-1}, A', A^{i+1}, \ldots, A^n), \tag{1}$$

for all strategies, A'*, feasible for agent* i*.*

It would be natural to impose an equilibrium refinement in the spirit of Perfect Bayesian Nash Equilibrium. This would rule out, for instance, equilibria in which the owner never sells, and conditionally on a sale, all bidders bid "minus infinity." To keep notation simple, however, we will not impose such a refinement since our results do not depend on it.

Assumption 1 *The process* y *is Markov, and given* y_t, $x_t = (x_t^0, \ldots, x_t^n)$ *is independent of* (x_s, y_s) *for* $s < t$*. There exist* $v_t : \mathbb{X} \times \mathbb{X} \times \mathbb{X}^{n-1} \times \mathbb{Y} \to \mathbb{R}$*, and* $u_t : \mathbb{X} \times \mathbb{X}^n \times \mathbb{Y} \to \mathbb{R}$*, such that for* $i \neq o_t$*,*

$$E(V_{t+1}^i \mid x_0, y_0, \ldots, x_t, y_t) = v_t(x_t^i, x_t^{o_t}, (x_t^j)_{j \notin \{i, o_t\}}, y_t), \tag{2}$$

and[9]

$$E(V_{t+1}^{o_t} \mid x_0, y_0, \ldots, x_t, y_t) = u_t(x_t^{o_t}, (x_t^j)_{j \neq o_t}, y_t). \tag{3}$$

This assumption implies that when agents have received the public signal, y_t, at time t, their assessments of future dividends do not depend on beliefs about past private signals. Hence, we are considering "short-lived" private information. Private information is short lived, for instance, if it is only relevant for the next dividend payment. If the private information is relevant

[9]Here, $(x_t^j)_{j \notin \{i, o_t\}}$ can be interpreted as the ordered sequence of signals. Later, however, we assume that v is symmetric in these arguments. The same holds for u. See Footnote 11.

for all future dividends is also short lived, however, if it is made public in the next period regardless if the agents' actions. While this is a strong assumption, it is not unrealistic, for instance, if the potential auction times are relatively far apart. We allow for agent i's dividend, V^i_{t+1}, to depend on his own information, x^i_t, and the current owner's information, $x^{o_t}_t$, in different ways than it depends on the information $(x^j_t)_{j\notin\{i,o_t\}}$ of the other agents.

Additionally, Assumption 1 allows us to focus on Markov equilibria. We further restrict our attention to symmetric equilibria, that is, equilibria in which all agents use the same strategy.[10] For a symmetric equilibrium to obtain naturally, we assume that: v is symmetric in its last $n-1$ arguments taking values in $\mathbb{X}$; u is symmetric in its last n arguments taking values in $\mathbb{X}$;[11] agents' signals, x_t, are symmetrically distributed[12] conditionally on y_t; the auction mechanism M treats agents symmetrically.[13] Finally, we call an equilibrium, $A = (A^0, A^1, \ldots, A^n)$, a ***symmetric Markov equilibrium*** if for all t there exists some $\mathcal{A}_t : \mathbb{X} \times \mathbb{Y} \times \{0,1\} \to \{sell, keep\} \cup \Theta$, such that for all i, $A^i_t = \mathcal{A}_t(x^i_t, y_t, 1_{\{i=o(t)\}})$. In a symmetric Markov equilibrium, the strategies can be characterized by a set, $X_t = X(t, y_t) \subset \mathbb{X}$, of private signals on which an agent decides to sell if he owns the asset at time t, and a bidding strategy, $b_t : \mathbb{X} \times \mathbb{Y} \to \Theta$, used by any agent who does not own the asset. Here, $X : \{0, \ldots, T-1\} \times \mathbb{Y} \to 2^{\mathbb{X}}$ is a correspondence with a measurable graph.[14] When computing any player's optimal strategy, his own future equilibrium strategy (as well as the strategies of all other players) can be taken as given.[15] Moreover, in a symmetric Markov equilibrium, continuation values (at time t) depend only on current signals (x^i_t and y_t) and on whether the agent owns the asset.[16] Hence, we can define $S_t = S(t, x^i_t, y_t)$ as the value at time t, after the dividend is paid and information is received, conditional on owning the object, and $B_t = B(t, x^i_t, y_t)$ as the value function conditionally

[10]Most of our results generalize to a setting in which agents are not symmetric. For simplicity we restrict attention to the symmetric case.

[11] That is, for any permutation, ρ, of $\{1, \ldots, n-1\}$, $v(a, b, c_1, \ldots, c_{n-1}, y) = v(a, b, c_{\rho(1)}, \ldots, c_{\rho(n-1)}, y)$, and for any permutation, σ, of $\{1, \ldots, n\}$, $u(a, b_1, \ldots, b_n, y) = u(a, b_{\sigma(1)}, \ldots, b_{\sigma(n)}, y)$.

[12]This means that for any permutation, ρ, of $\{0, \ldots, n\}$, $(x^0_t, \ldots, x^n_t)$ and $(x^{\rho(0)}_t, \ldots, x^{\rho(n)}_t)$ are identically distributed (conditionally on y_t).

[13]That is, for any permutation, ρ, of $\{1, \ldots, n\}$, we require that $\pi_i(x_1, \ldots, x_n) = \pi_{\rho(i)}(x_{\rho(1)}, \ldots, x_{\rho(n)})$ for all i. Similar conditions are required for z and α.

[14]Measurability is with respect to the product σ-algebra.

[15]This is called the one-stage-deviation principle (see Fudenberg and Tirole (1998)).

[16]This follows from Assumption 1.

on not owning the object.[17] Supposing, without loss of generality, that agent 0 owns the asset and agent 1 considers how much to bid, the respective Bellman equations are:

$$\begin{aligned} S_t(x_t^0, y_t) &= \sup_{a \in \{sell, keep\}} E\left(1_{(a=sell)}\left[p_t + \delta B_{t+1}\right]\right. \quad (4) \\ &\quad \left. +1_{(a=keep)}\delta\left[V_{t+1}^0 + S_{t+1}\right] \;\middle|\; x_t^0, y_t\right) \quad (a.s.) \\ B_t(x_t^1, y_t) &= \sup_{\hat{b}\in\Theta} E\left(1_{(x_t^0 \in X_t)}\left[\delta\hat{\pi}(\hat{b})(V_{t+1}^1 + S_{t+1}) - \hat{z}(\hat{b})\right.\right. \quad (5) \\ &\quad \left.\left. +\delta\left(1 - \hat{\pi}(\hat{b})\right) B_{t+1}\right] + 1_{(x_t^0 \notin X_t)}\delta B_{t+1} \;\middle|\; x_t^1, y_t\right) \quad (a.s.), \end{aligned}$$

where the supremum is to be taken state by state. We have used the simplifying notation: $p_t = p(b_t(x_t^1, y_t), \ldots, b_t(x_t^n, y_t))$, $\hat{\pi}(\hat{b}) = \pi_1(\hat{b}, b_t(x_t^2, y_t), \ldots, b_t(x_t^n, y_t))$, and $\hat{z}(\hat{b}) = z_1(\hat{b}, b_t(x_t^2, y_t), \ldots, b_t(x_t^n, y_t))$. Equation (4) states that the owner chooses optimally whether or not to sell. When he sells he receives the price, p_t, paid in the auction, and the value of being a buyer at the next period. The price depends on the buyers' strategies, $b_t(\,\cdot\,, y_t)$, which are taken as given by the seller. When the owner does not sell, he receives the dividend and the value of being the owner next period. Similarly, Equation (5) states that the buyer must choose his bidding strategy optimally.

It can be shown by standard dynamic-programming arguments that a symmetric Markov equilibrium, (b, X), is characterized by the following properties:

1. S_t is the maximum in (4), and selling for any (x_t^0, y_t) with $x_t^0 \in X_t(y_t)$ maximizes the right-hand side of (4), taking b_t, S_{t+1}, and B_{t+1} as given.

2. B_t is the maximum in (5), and b_t is a measurable selection from the maximand of the right-hand side of (5), taking X_t, S_{t+1}, and B_{t+1} as given, as well as the fact that the other bidders use b_t.

3. $S_T = B_T = 0$ if $T < \infty$ and $\lim_{t\to\infty} \delta^t S_t = \lim_{t\to\infty} \delta^t B_t = 0$ otherwise.

We note that (5) shows that the equilibrium bidding strategy, b_t, is a solution to the optimization problem

$$\sup_{\hat{b}\in\Theta} E\left(\hat{\pi}(\hat{b})\delta(V_{t+1}^1 + S_{t+1} - B_{t+1}) - \hat{z}(\hat{b}) \;\middle|\; x_t^1, y_t, x_t^0 \in X_t\right), \quad (6)$$

which yields the following proposition.

[17]Hence, S and B are mappings from $\{0, 1, \ldots, T-1\} \times \mathbb{X} \times \mathbb{Y}$ into $\mathbb{R}$.

Proposition 1 *The equilibrium bidding strategies form an equilibrium in a standard*[18] *(single-period) auction in which the asset is worth* $\delta(V^i_{t+1} + S_{t+1} - B_{t+1})$ *to agent* i*, bidders' signals are drawn from the distribution conditional on* y_t *and* $x^0_t \in X_t$*, and it is common knowledge that* $x^0_t \in X_t$*.*

Proposition 1 is straightforward but useful, for it means that solving this dynamic auction equilibrium is not much harder than solving standard single-period auctions — the only added complications are the derivations of the set, $X_t(y_t)$, on which the owner chooses to sell, and the value functions. The equilibrium sale set solves a fixed-point problem: The sale set must be the owner's best response given that the bidders bid as if the owner is using this sale set. The value functions are easily computed recursively.

3 Conditionally Independent Private Signals

The case in which private signals are independent conditionally on y_t is an important benchmark. We show that, under this condition, we can generalize the Revenue Equivalence Theorem (RET)[19] to our multi-period model with endogenous trade, and derive a robust No-Trade Theorem. Although independence is strong assumption, it may not be unreasonable in some cases, especially since private signals are allowed to be marginally correlated. What must be independent is the agents' information over and above what is public information. (This is also true in the standard RET.) In Section 4 we consider the case of conditionally correlated (affiliated) signals.

Assumption 2 *Conditional on* y_t*, the random variables* $x^0_t, x^1_t, \ldots, x^n_t$ *are iid, and* x^i_t *has a strictly positive density on* $\mathbb{X} = \{x \in \mathbb{R}^m : \underline{\chi} \le x \le \overline{\chi}\}$*, for some* $m \ge 1$*, where* $\underline{\chi} \in \mathbb{R}^m$ *is strictly smaller, coordinate-wise, than* $\overline{\chi} \in \mathbb{R}^m$*.*

Whenever there is a sale, buyers are bidding as in a single-period auction in which the prize (to agent $i > 0$) is

$$\begin{aligned} &w(x^i_t, (x^j_t)_{j\notin\{0,i\}}, t, y_t, X_t, D_t) = \\ &\quad \delta E(v_t(x^i_t, x^0_t, (x^j_t)_{j\notin\{0,i\}}, y_t) \mid x^0_t \in X_t, (x^j_t)_{j>0}, y_t) + D_t, \end{aligned} \tag{7}$$

[18] See, for instance, Klemperer (2000).
[19] See Myerson (1981).

where $D_t = \delta E(S_{t+1} - B_{t+1} \mid y_t)$, $w : \mathbb{X} \times \Xi \to \mathbb{R}$ is a mapping defined (a.s.) by (7), and Ξ is the product of $\{0, \ldots, T-1\}$, $\mathbb{Y}$, the set of measurable subsets of $\mathbb{X}$, and $\mathbb{R}$. We note that under Assumption 2, the distribution of the bidders' signals is not affected by conditioning on $(x_t^0 \in X_t)$. We let $\xi_t = (t, y_t, X_t, D_t)$, and let ξ denote a generic element of Ξ. We make a technical assumption on the (expected) dividend value.

Assumption 3 *The function w is continuously differentiable in its first argument in $\mathbb{X}$.*

We say that a bidding strategy, $b^\xi : \mathbb{X} \to \Theta$, is a symmetric equilibrium in the single-period auction $(M, x_t, w(\cdot, \xi))$ given[20] $y_t = y$ if, for (almost) all x_t^1,

$$b^\xi(x_t^1) \in \arg\max_{\hat{b} \in \Theta} E\left(\hat{\pi}(\hat{b}) w(x_t^1, (x_t^j)_{j>1}, \xi) - \hat{z}(\hat{b}) \;\middle|\; x_t^1, y_t = y\right), \qquad (8)$$

where $\hat{\pi}(\hat{b}) = \pi_1(\hat{b}, b^\xi(x_t^2), \ldots, b^\xi(x_t^n))$, and $\hat{z}(\hat{b}) = z_1(\hat{b}, b^\xi(x_t^2), \ldots, b^\xi(x_t^n))$. We will not focus on the question of solving a standard single-period auction but, rather, we take such a solution as given, and show how standard results generalize to a multi-period setting. Hence, we consider the following condition.

Condition 1 *For the mechanism M, for each $\xi = (t, y, X, D) \in \Xi$, there exists a symmetric equilibrium, b^ξ, in the single-period auction $(M, x_t, w(\cdot, \xi))$, given $y_t = y$.*

Condition 1 is satisfied by many standard auction mechanisms under standard distributional assumptions — see, for instance, Milgrom and Weber (1982), Klemperer (2000), and references therein.

For any mechanism M (that satisfies Condition 1), we fix an equilibrium bidding strategy b^ξ, and denote by $\pi^M : \mathbb{X}^n \times \Xi \to \Delta_n$ the auction allocation (corresponding to M),[21] by $z^M : \mathbb{X}^n \times \Xi \to \mathbb{R}^n$ the payments, and by $U^M : \mathbb{X} \times$

[20] Here, we use the concept of regular conditional probabilities. A regular conditional probability always exists on Euclidian spaces (see Breiman (1968), page 79). This allows us to write "$E(\cdot \mid y_t = y)$" with some abuse of notation even when the set $\{y_t = y\}$ has zero probability.

[21] We are implicitly using the Revelation Principle. (See, for instance, Myerson (1981).)

$\Xi \to \mathbb{R}$ the expected surplus. That is, assuming (without loss of generality) that agent 0 is the owner,

$$\begin{aligned} \pi^M(x^1, \ldots, x^n, \xi) &= \pi(b^\xi(x^1), \ldots, b^\xi(x^n)) \\ z^M(x^1, \ldots, x^n, \xi) &= z(b^\xi(x^1), \ldots, b^\xi(x^n)), \end{aligned}$$

and

$$\begin{aligned} U^M(x_t^i, \xi_t) \;=\; & E\big(w(x_t^i, (x_t^j)_{j \notin \{i,0\}}, \xi_t)\, \pi_i^M(x_t^1, \ldots, x_t^n, \xi_t) \\ & - z_i^M(x_t^1, \ldots, x_t^n, \xi_t) \;\big|\; x_t^i, y_t \big). \end{aligned}$$

We note that U^M is the expected "rent" from the auction for any bidder (given that there is a sale), whereas B is each bidder's value function before the bidder knows whether the owner will decide to sell. From (5),

$$B_t = Pr(x_t^0 \in X_t \;\big|\; y_t) U(x_t^i, \xi_t) + \delta E(B_{t+1} \;\big|\; y_t). \tag{9}$$

Our Multi-Period Revenue Equivalence Theorem states that the auction allocations, π^M, and the surplus of the lowest type, $U^M(\underline{\chi}, \cdot)$, determine the decision to sell, the value functions, and the expected price. As a consequence, many auctions used in practice are "multi-period revenue equivalent" under the assumption of conditional independence of signals. This is due to the fact that the following two properties are shared by most auction mechanisms: *(i)* The bidder with the "best news" wins the auction. *(ii)* A bidder with the worst possible news neither gets the object nor must pay anything. (For general multi-dimensional signals, however, there may not be a clear concept of what "the best news" is.)

Theorem 2 (Multi-Period Revenue Equivalence) *Suppose that Assumptions 1–3 hold. Let M be a mechanism that satisfies Condition 1, and suppose that $\{(X_t, b_t)\}$ is a symmetric Markov equilibrium for (M, x, y, V, T). Then, for any other mechanism, M', that satisfies Condition 1, with $\pi^M = \pi^{M'}$ and $U^M(\underline{\chi}, \cdot) = U^{M'}(\underline{\chi}, \cdot)$, there exists b'_t such that $\{(X_t, b'_t)\}$ is a symmetric Markov equilibrium for (M', x, y, V, T). Moreover, these equilibria have the same value functions (S, B, and U), and the same conditional expected prices (given y_t).*

We note that the theorem has the following implications about the equilibria of mechanisms with the same auction allocation, π, and lowest-type surplus. First, (under the assumptions of the theorem) if M has a unique symmetric

Markov equilibrium, then M' also has a unique symmetric Markov equilibrium. Second, if there are multiple equilibria then the set of symmetric Markov equilibria for M (as characterized by X, S, B, and U) is identical to the set of symmetric Markov equilibria for M'.

It is difficult to weaken our assumptions in Theorem 2. First, if information is long-lived (that is, if Assumption 1 is not satisfied) then agents have an incentive to use their bids to signal information. Since different auction mechanisms create different incentives to signal, revenue equivalence may fail to apply. See Bikhchandani and Huang (1989) for this effect. Second, Assumption 1, that signals are y_t-conditionally independent across agents, cannot be eliminated. For instance, Milgrom and Weber (1982) show (in a single-period auction) that the expected price varies for different auctions when agents have correlated signals. (Assumption 3 is an innocuous technical assumption.)[22]

The next result shows that trade occurs only if agents have significantly different values for the object. If these values are "too common," then there can be no trade. To make this statement precise, we parameterize (in the most general way) the commonality of agents' values.

Assumption 4 *For each t, there exist functions $g_t : [0,1] \times \mathbb{X} \times \mathbb{X} \times \mathbb{X}^{n-1} \times \mathbb{Y} \to \mathbb{R}$, $h_t : [0,1] \times \mathbb{X} \times \mathbb{X}^n \times \mathbb{Y} \to \mathbb{R}$, and $f_t : \mathbb{X} \times \mathbb{X}^n \times \mathbb{Y} \to \mathbb{R}$, increasing in their arguments in $\mathbb{X}$, and symmetric in their last $n-1$, n, and n arguments in $\mathbb{X}$, respectively, such that for all $(x^0, \ldots, x^n, y) \in \mathbb{X}^{n+1} \times \mathbb{Y}$,*

$$\begin{aligned} &v_t(x^1, x^0, (x^2, \ldots, x^n), y) = \\ &\qquad g_t(\lambda, x^1, x^0, (x^2, \ldots, x^n), y) + f_t(x^0, (x^1, x^2, \ldots, x^n), y) \\ &u_t(x^0, (x^1, \ldots, x^n), y) = \\ &\qquad h_t(\lambda, x^0, (x^1, \ldots, x^n), y) + f_t(x^0, (x^1, x^2, \ldots, x^n), y). \end{aligned}$$

Further, g and h converge uniformly to 0 as λ approaches 0.

Here, f is the common value, g is a non-owner's private value, and h is the owner's private value. The parameter λ measures the extent to which the asset has a private-value component. If $\lambda = 0$, then the asset has the same

[22] We also note that, as long as the allocation ensuing from two mechanisms is the same, Theorem 2 also holds if reserve prices are allowed. In order to get the same allocations, though, stronger assumptions on the values need to be made because, in general, different mechanisms reveal different information and consequently give rise to a sale on different subsets of the state space.

value to all agents (common values). We also use some regularity conditions given in Assumption $4'$ in the appendix.[23]

Theorem 3 (No Trade) *Under Assumptions 1–4 and $4'$, there exists*[24] *a real number $\underline{\lambda} > 0$ such that if $\lambda < \underline{\lambda}$, then, for any symmetric Markov equilibrium, and for all t, there is no trade, that is, $Pr\left(x_t^0 \in X_t(y_t)\right) = 0$.*

This is related to the No-Trade Theorems of Kreps (1977) and Milgrom and Stokey (1982). Our result is robust, however, in that it shows that there cannot be trade when agents have valuations that are sufficiently close to each other. This result is driven by the fact that we are considering an economy with (a finite number of) strategic agents who have private information. This private information implies that buyers can extract positive rents — we show these rents to be bounded below away from zero independently of the mechanism — which lowers the owner's willingness to sell. Hence, the gains from trade must be greater than the threshold implied by any frictions in the economy.

The robust No-Trade Theorem presented here relies on the analysis used to prove the Multi-Period Revenue Equivalence Theorem. One might wonder whether it applies more generally. Indeed, if the value is common ($\lambda = 0$), no trade is an equilibrium regardless of distributions. The robustness result, however, does not apply generally because there exist mechanisms that extract all surplus from bidders when signals are correlated (see Crémer and McLean (1988)). It has been noted, though, that the implementation of such mechanisms is demanding, since they rely on perfect knowledge of the distributions of signals.

4 Conditionally Affiliated Private Signals

In this section we compare allocations, welfare, and revenues associated with different auction mechanisms when signals are affiliated. We build on the work of Milgrom and Weber (1982), who show that the expected revenue in a first-price auction is dominated by that generated by a second-price auction, which in turn is dominated by the expected revenue in an English

[23]We require that a bidder's probability of winning increases in her signal, that bidders do not lose money, on average, by participating, and some boundedness.

[24]Here, $\underline{\lambda}$ depends on g and f of Assumption 4, and on the distribution of signals, but not on t.

auction. It is important to ask how auction mechanisms differ in terms of their welfare and volume of trade, too. Answering these question requires a dynamic model with endogenous trade, such as the one presented here. We show that a mechanism that has higher revenue for a fixed sale set generates more trade in our endogenous-sale equilibrium (volume ranking). We further show that, under certain conditions, the equilibrium trades are efficient, so that higher volume implies higher welfare (welfare ranking).

We proceed by first providing a set of general conditions that capture the intuition behind the results, and then show that these conditions are met for the first-price, second-price, and English auctions under natural distributional assumptions.

4.1 General Results

We consider only one-dimensional private signals here because the characterization of optimal sale sets is otherwise more complicated. We introduce some natural notation for a mechanism, M, that satisfies Condition 1 and an associated fixed equilibrium. First, the price paid to the seller is $p^M(x_t^1, \ldots, x_t^n, \xi_t)$ with $\xi_t = (t, y_t, X_t, D_t)$, where X_t is the sale set, bidders have signals $x_t^1, \ldots, x_t^n$, and $D_t = \delta E(S_{t+1} - B_{t+1} \mid y_t)$. Second,

$$P^M(x_t^0, \xi_t) = E\left[p_t^M(x_t^1, \ldots, x_t^n, \xi_t) \mid x_t^0, y_t\right]$$

is (a.s.) the conditional expected price, given the owner's signal, x_t^0. Third,

$$\bar{u}(x_t^0, \xi_t) = E\left[u_t(x_t^0, \ldots, x_t^n) \mid x_t^0, y_t\right] + D_t$$

is the expected value associated with keeping the asset for an owner with signal x_t^0, and, fourth,

$$\bar{v}(x_t^0, \xi_t) = E\left[v_t(x_t^{(1)}, x_t^0, (x_t^{(j)})_{j\geq 2}) \mid x_t^0, y_t\right] + D_t$$

is the owner's belief about the asset value to the bidder with highest signal. ($x_t^{(j)}$ denotes the k^{th} highest among $x_t^1, \ldots, x_t^n$.) This quantity is the expected value to the auction winner in the equilibria that we consider.

An owner with private signal x_t^0 wants to sell if and only if the expected price $P^M(x^0, \xi)$ is greater than the value of keeping the asset, $\bar{u}(x^0, \xi)$. Hence, the following condition implies that the owner chooses to sell on a connected subset of $\mathbb{X}$ containing its smallest element.

Condition 2 *For all* $\xi \in \Xi$*, the set* $\{x^0 \in \mathbb{X} : \bar{u}(x^0, \xi) \le P^M(x^0, \xi)\}$ *is either of the form* $[\underline{\chi}, a]$ *for some* $a \in \mathbb{X}$*, or is empty.*

Another natural condition is that enlarging the sale-set by adding higher signals increases the expected price:

Condition 3 *For any* $t < T$*,* $y \in \mathbb{Y}$*,* $D \in \mathbb{R}$*,* $x^0 \in \mathbb{X}$*, and* (a, b) *such that* $a < b \le \overline{\chi}$*, it holds that* $P^M(x^0, t, y, [\underline{\chi}, a], D) \le P^M(x^0, t, y, [\underline{\chi}, b], D)$*.*

We also impose the following innocuous condition stating that the (expected) price increases by D if the prize is increased by a commonly known constant D.

Condition 4 *For any* $t < T$*,* $y \in \mathbb{Y}$*,* $D \in \mathbb{R}$*,* $x^0 \in \mathbb{X}$*, and* a*, it holds that* $P^M(x^0, t, y, [\underline{\chi}, a], D) = P^M(x^0, t, y, [\underline{\chi}, a], 0) + D$*.*

Finally, in order to make welfare comparisons, we state one more condition on the equilibrium and make an additional distributional assumption.

Condition 5 *The winner of the auction is the bidder with the highest signal (a.s.) and* $P^M(a, t, y, [\underline{\chi}, a], D) \le \bar{v}(a)$ *for all* $a \in \mathbb{X}$*.*

The first part of Condition 5 simply says that the bidder with the best news wins. The second part says that the expected price is lower than the winning bidder's utility if the owner has the best possible selling signal. The price is lower in part because the bidders extract rents and in part because the bidder's face an adverse-selection problem and expect the owner's signal to be lower than a.

Assumption 5 *The function* $\bar{u}(x^0, \xi) - \bar{v}(x^0, \xi)$ *increases in* x^0*.*

This assumption states that the owner's utility is more dependent on his signal than is the winning bidder's utility. This is a natural assumption, and is satisfied, for instance, if the signals are distributed according to a (truncated) normal distribution with all correlations positive, implying affiliation, and if

$$\begin{aligned} u(z_0, \ldots, z_n, y) &= \alpha(y) z_0 + h(z_0, \ldots, z_n, y) \\ v(z_0, \ldots, z_n, y) &= \beta(y) z_0 + h(z_0, \ldots, z_n, y), \end{aligned} \tag{10}$$

where h is symmetric and increasing in the first $n+1$ arguments and $\alpha(y) \ge \beta(y) > 0$.

We are ready to make precise our claim that revenue ranking implies volume and welfare ranking:

Theorem 4 (Volume and Welfare Ranking) *Suppose Assumption 1 holds. Consider two mechanisms, M and N, satisfying Conditions 1–4, such that*

$$P^M(a,t,y,[\underline{\chi},a],D) \geq P^N(a,t,y,[\underline{\chi},a],D) \tag{11}$$

for all $a \in \mathbb{X}$, $t < T$, $y \in \mathbb{Y}$, and $D \in \mathbb{R}$. If $\{(X_t^M, b_t^M)\}$ is a symmetric Markov equilibrium for M, then there exists a symmetric Markov equilibrium, $\{(X_t^N, b_t^N)\}$, for N with $X_t^N \subseteq X_t^M$ for all t. Conversely, if $\{(X_t^N, b_t^N)\}$ is a symmetric Markov equilibrium for N, then there exists a symmetric Markov equilibrium, $\{(X_t^M, b_t^M)\}$, for M with $X_t^M \supseteq X_t^N$ for all t. The equilibrium $\{(X_t^M, b_t^M)\}$ has a higher single-period revenue than $\{(X_t^N, b_t^N)\}$, in the sense that

$$E\left(P^M(x^0,t,y,X^M,D) \;\middle|\; x^0 \in X^M\right) \geq E\left(P^N(x^0,t,y,X^N,D) \;\middle|\; x^0 \in X^N\right).$$

If, in addition, Assumption 5 holds and M satisfies Condition 5, the welfare in the M-equilibrium is higher than that in the N-equilibrium.

From now on, we shall find it convenient to say, whenever the conclusion of the theorem regarding the existence and comparison of the sale sets of equilibria holds, that mechanism M has ***higher volume*** then mechanism N. If the conclusion regarding welfare holds, we shall say that mechanism M has ***higher welfare*** than mechanism N.

In the statement of the theorem, inequality (11) states that for any fixed anticipated sale set, mechanism M yields a higher expected revenue than mechanism N, conditionally on the owner's signal being the best possible signal for which he sells. The intuition for the volume ranking is as follows. Suppose that $[\underline{\chi}, a^1]$ is an equilibrium sale set for mechanism N in the last time period. Consider the owner's best response if the trading mechanism is changed to M, but bidders keep anticipating the same sale set. Then, because of the revenue ranking, the owner will sell for a larger set of signals, say $[\underline{\chi}, a^2]$. Now, suppose that the bidders anticipate the sale set $[\underline{\chi}, a^2]$. Then, by Condition 3, they will bid more since now they think that the owner might have a better signal. This, in turn, will lead the owner to sell with even better signals, and by iterating this argument, we end up (in the limit) with an equilibrium that has larger sale set and expected price than for mechanism N.

The same argument shows that mechanism M has higher volume in earlier periods, $t < T-1$, because the sale decision does not depend on the value functions. Further, it shows that prices are higher in the M-equilibrium as long as the value functions are such that $D_t^M = S_t^M - B_t^M$ is at least at high as D_t^N. This is trivially the case in the last period since $D_T^M = D_T^N = 0$, but in earlier periods, D_t^M can be higher or lower than D_t^N depending on the model specifics. Hence, mechanism M may have lower expected prices, for $t < T-1$, than does mechanism N. This can happen because, in earlier periods, the buyers bid not only for the dividend next period, but also for the value of owning the asset over and above the opportunity cost of not owning it (Proposition 1). Using a mechanism that generates higher average revenues makes owning the asset more valuable (that is, yield a higher S), but has an ambiguous effect on the value, B, of not owning: A higher-revenue mechanism allows buyers to extract fewer rents if there is a sale, but makes a sale more likely. Hence, whether $S - B$ is higher for the higher-revenue mechanism depends on the relative benefits from the efficiency gain to owners and non-owners. For instance, if (y_t) is iid over time, then $D_t^M - D_t^N = (D_{T-1}^M - D_{T-1}^N)(1 - \delta^{(T-t)})/(1-\delta)$, and there are two possible price patterns: *(i)* M has higher expected prices than N for all t, or *(ii)* M has lower expected prices than N for small t and higher expected prices for large t, when $D_{T-1}^M - D_{T-1}^N < 0$ and δ is high enough.[25] The former case seems the more natural.

The intuition for the welfare result is also very simple. Condition 5 implies that trade is welfare improving given that the owner has the highest signal for which he would sell. This is because the owner's utility is smaller than the price, which in turn is smaller than the buyer's utility. Assumption 5, then, implies that trade is welfare improving for all equilibrium sales, whence a smaller sale set is welfare reducing.

We briefly note how Theorem 4 extends to a setting in which the owner sells to an uninformed intermediary, who then resells to the informed bidders. In that case, Condition 2 is satisfied trivially, since the price does not depend on the owner's information, and the theorem goes through as stated. Furthermore, the sale set is always smaller than the one that obtains when the owner sells directly, which implies a lower welfare.

[25]In an infinite-horizon model, with iid (y_t), the expected prices are the same at each point in time.

4.2 First-Price, Second-Price, and English Auctions

Milgrom and Weber (1982) show that the average revenues yielded by first-price and second-price auctions can be ranked when bidders' signals are affiliated.[26] To examine how this result generalizes to our setting, we make the following assumption.

Assumption 6 *Conditional on y_t, the random variables $x_t^0, x_t^1, \ldots, x_t^n$ are symmetrically and absolutely continuously distributed and affiliated. The support of their y_t-conditional joint distribution is a subset of $\mathbb{X}^n$, with $\mathbb{X} = \left[\underline{\chi}, \overline{\chi}\right]$. Further, v_t and u_t are increasing in their first $n+1$ arguments.*

In order to use the results of Milgrom and Weber (1982) we would want bidders' signals to be affiliated *conditional on the event that the owner sells,* which is implied by our next result.

Lemma 5 *Suppose $Z_0, Z_1, \ldots, Z_n$ are affiliated random variables in $\mathbb{R}$, and A is a measurable subset of $\mathbb{R}$ with $Pr(Z_0 \in A) > 0$. Then, given $Z_0 \in A$, $Z_1, \ldots, Z_n$ are affiliated.*

First, we verify that the first-price, second-price, and English auctions all have the property that bidders bid more if the owner is assumed to sell for a larger set of signals (as required in Condition 3), that prices are additive in constants (Condition 4), and that the highest-signal bidder wins the auction and that the price is lower than the buyer's value for large x_0 (Condition 5).

Lemma 6 *If Assumptions 1 and 6 hold, each of the first-price, second-price, and English auctions has an equilibrium that satisfies Conditions 3, 4, and 5.*

In order to apply Theorem 4 above, we must show that (11) holds, that is, that revenue ranking holds *conditional on an owner's highest signal that would make him sell*, i.e., conditional on $x_t^0 = a$. (We note that this is different from the revenue ranking *on average* across an owner's signals as in Milgrom and Weber (1982).) When comparing the first-price and second-price auctions, we get an even stronger result, namely that revenue ranking holds conditionally on *any* seller signal.

Lemma 7 *For all $x^0 \in \mathbb{X}$ and $\xi \in \Xi$,*

$$P^2(x^0, \xi) \geq P^1(x^0, \xi), \tag{12}$$

[26] See Milgrom and Weber (1982) for a definition of affiliation.

where P^1 and P^2 are the expected prices in the symmetric equilibria of the first-price and second-price auctions, respectively.

Hence, the following corollary follows immediately from our previous results.

Corollary 8 *Suppose Assumptions 1 and 6 hold, and that the first-price and second-price auctions satisfy Condition 2. Then the second-price auction has higher volume than the first-price auction. If, in addition, Assumption 5 is satisfied, then the second-price auction has higher welfare than the first-price auction.*

Now, we turn to the comparison of the second-price and English auctions. Milgrom and Weber (1982) show that the winner's curse is smaller for the English auction, and therefore that the revenue is higher on average across the seller's signals. We find, however, that this revenue ranking may *not* apply conditional on all seller signals. Fortunately, it does apply for the owner's highest selling signal, as required in Theorem 4, (11). Here is the intuition. During the English auction, the information of those bidders with low signals is revealed. This not only diminishes the winner's curse, but also reveals information about the owner's signal. If the owner has a low signal, then this information revelation is bad for him. If he has the best possible selling signal, however, the information revelation is good for the seller, and reinforces the (conditional) revenue ranking.

Lemma 9 *For all $a \in \mathbb{X}$, $t < T$, $y \in \mathbb{Y}$, and $D \in \mathbb{R}$,*

$$P^E(a, t, y, [\underline{\chi}, a], D) \geq P^2(a, t, y, [\underline{\chi}, a], D)\,, \tag{13}$$

where P^2 and P^E are the expected prices in the symmetric equilibria of the second-price and English auctions, respectively.

These results yield the following corollary.

Corollary 10 *Suppose Assumptions 1 and 6 hold, and that the second-price and English auctions satisfy Condition 2. Then the English auction has higher volume than the second-price auction. If, in addition, Assumption 5 is satisfied, then the English auction has higher welfare than the second-price auction.*

Finally, let us see that Condition 2 is not excessively restrictive and is, in fact, satisfied by a large class of models. It is clear that this condition is satisfied if the owner's expected value for the dividend is more sensitive, in the sense of having a larger derivative, to the owner's signal than is the expected price. This intermediate condition is itself natural. One way to ensure it, independently of the mechanism M, is to require that the owner's utility u depends more strongly on his signal x_t^0 than does the conditional distribution of $(x_t^1, \cdots, x_t^n)$ given (x_t^0, y_t). The following lemma[27] makes this precise. We note that, for any distribution of (x_t^i, y_t), the utility u can be chosen such that this condition is satisfied. Further, a very similar condition would ensure that Assumption 5 holds. (We do not state this condition because Assumption 5 is already stated independent of the mechanism.) For instance, the parametric models given in (10) satisfy Assumption 5 and Condition 2 when α is large enough.

Lemma 11 *Suppose that, for all t (a.s.),*

$$E\left(\frac{\partial u_t}{\partial x^0}(x_t^0, \ldots, x_t^n, y_t) \;\middle|\; x_t^0, y_t\right) > k_t \; E\left(\left(\frac{\partial \log \zeta_t}{\partial x^0}(x_t^1, \ldots, x_t^n \mid x_t^0, y_t)\right)^2 \;\middle|\; x_t^0, y_t\right)^{1/2}, \tag{14}$$

where[28] ζ_t *is the conditional density of* $x_t^1, \ldots, x_t^n$ *given* (x_t^0, y_t)*, and*

$$k_t = E\left(\left(v(x_t^{(1)}, a_t, x^{(1)} \ldots, x_t^{(1)}, y_t)\right)^2 \;\middle|\; x_t^0, y_t\right)^{1/2},$$

with a_t *the largest sale signal. Then Condition 2 is satisfied for the first-price, second-price, and English auctions.*

5 Conclusion

This paper investigates repeated trade in a dynamic economy in which both buyers and sellers have private information. The model shows how the information structure and market mechanism affect prices, volume of trade, and welfare.

[27]The lemma could be stated so as to apply generally to any mechanism satisfying a boundedness condition on the price.

[28]Of course, part of the assumption is that u_t and $\log \zeta_t$ are differentiable (almost everywhere), and that the conditional expectations in (14) exist as finite random variables.

The model's features — endogenous sale decisions, repeated trade, and short-lived information — make it a realistic and useful tool for studying markets for (illiquid) durable goods, blocks of securities, and other assets. For example, the model yields the policy implication that if an organization (for instance, a government) needs to sell a long-lived asset then it must not only choose the mechanism with the lowest winner's-curse problem, but should also ensure the existence of an environment in which subsequent trade is easy. Such a trading environment is especially helpful if it makes allocations more efficient and if it is "seller-friendly." Buyers bid more today if they know *(i)* that they can easily sell the asset in the future, should they not need it anymore, and *(ii)* that possible future sales by others do *not* generate the opportunity to buy the asset at a large discount (in "fire sales"). An additional benefit from such trading environments is increased welfare due to the efficiency of the allocations. Consistently, it is common for investment banks that underwrite initial public offerings to also play a role as a market-maker in the secondary market, and when Drexel Burnham Lambert began issuing junk bonds, it also put in place a secondary market for these bonds (Cornell (1992)).

A Appendix

Proof of Theorem 2 (Multiperiod Revenue Equivalence): Proceed by backward induction. Assume that there exists b'_s, $s > t$, such that $\{(X_s, b'_s)\}_{s=t+1}^{T-1}$ is a repeated-auction equilibrium for M', and such that the value functions for $s > t$ are given by S_s and B_s. This claim holds for $t = T-1$, since $S_T = B_T = 0$ for any mechanism. It will be apparent that the induction step only requires that S_{t+1} and B_{t+1} be the value functions for the mechanism M', as well.

In order to verify that $X_t(y_t)$ is an equilibrium decision to sell, at t, when the mechanism M' is employed, we make use of the fact, which we prove below, that the two mechanisms yield the same expected price if the sale set, denoted by X, is the same in the two cases. Let that expected price be $P(X)$. Since the owner finds it optimal to sell for $x^0 \in X_t(y_t)$ under mechanism M, he also does under mechanism M', because in both cases the owner is faced with the same problem: The value to the owner of not selling is $\delta E\left(V_{t+1}^0 \;\middle|\; x_t^0, y_t\right) + \delta S_{t+1}$ in both cases, and the expected value of selling is $P(X) + \delta B_{t+1}$ in both cases. Thus $X_t(y)$ is an equilibrium sale set, and S_t is the owner's value function, when M' is employed.

The argument that, for a given public signal y_t, a given sale set X, and given value functions at $t+1$, the two mechanisms yield the same expected price is a standard envelope theorem one. Let $D = E(S_{t+1} - B_{t+1} \mid y_t = y)$, $\xi = (t, y, X, D)$, and let $P^N(\xi)$ denote the expected price for mechanism N, where $N \in \{M, M'\}$. Note that

$$
\begin{aligned}
P^N(\xi) &= E\left(\sum_{i=1}^n z_i^N(x_t^{-0}, \xi) \;\middle|\; y_t = y, x_t^0 \in X\right) \\
&= E\left(\sum_{i=1}^n \delta V_{t+1}^i \pi_i^N(x_t^{-0}, \xi) \;\middle|\; y_t = y, x_t^0 \in X\right) \\
&\quad - E\left(\sum_{i=1}^n U^N(x_t^i, \xi) \;\middle|\; y_t = y\right).
\end{aligned}
$$

Here, the first term is the social surplus and the second is the sum of the bidders' surpluses. Note that only the second part depends on z. A sufficient condition for $P^M = P^{M'}$ is, therefore, $U^M(x, \xi) = U^{M'}(x, \xi)$. This condition is also sufficient for the bidders' value function, B_t, to be the same for M and M', which follows from (9).

We define $W^N : \mathbb{X} \times \mathbb{X} \times \Xi \to \mathbb{R}$ so that $W^N(\hat{x}, x, \xi)$ is the expected surplus at time t of a bidder of with private signal x who plays the equilibrium strategy of an agent with signal $\hat{x}$. That is,

$$\begin{aligned} W^N(\hat{x}, x, \xi) &= E\left(w(x, (x_t^j)_{j>1}, \xi)\pi_1^N(\hat{x}, (x_t^j)_{j>1}, \xi)\right. \\ &\quad \left. -z_1^N(\hat{x}, (x_t^j)_{j>1}, \xi) \;\middle|\; y_t = y\right) . \end{aligned}$$

Fix $x \in \mathbb{X}$ and take $\gamma : [0,1] \to \mathbb{X}$ continuously differentiable such that $\gamma(0) = \underline{\chi}$ and $\gamma(1) = x$. A particular case of agent i's problem is to maximize ${}^\gamma W^N(\hat{r}, r, \xi) := W^N(\gamma(\hat{r}), \gamma(r), \xi)$ over $\hat{r}$ in [0,1], for each $r \in [0,1]$. If w is continuously differentiable, on the compact set $\mathbb{X}$, in its first argument, then ${}^\gamma W^N$ is continuously differentiable in its second argument, and the envelope theorem can be applied to yield

$$\begin{aligned} U^N(x, \xi) &= U^N(\underline{\chi}, \xi) + \int_0^1 E\left[D_1 w(\gamma(r), (x^j)_{j>1}, \xi)\right. \\ &\quad \left. \times \pi_1^N(\gamma(r), (x^j)_{j>1}, \xi) \;\middle|\; y_t = y\right] \cdot D\gamma(r)\, dr. \end{aligned} \tag{A.1}$$

Here $D_1 w$ denotes the differential of w with respect to the first argument, while $D\gamma$ the differential of γ. By assumption, neither $U^N(\underline{\chi})$, nor π^N depends on N, and certainly nor does γ, proving that $U^M(x, \xi) = U^{M'}(x, \xi)$.

□

The following assumption is used in the statement of Theorem 3.

Assumption 4′ *(i) The probability of acquiring the object, conditional on all signals, increases in one's own signal; that is,* $Q^M(x, \xi) := E\left(\pi_1^M(x, x_t^2, \ldots, x_t^n, \xi)\right)$ *increases in* $x \in \mathbb{X}$ *for all* $\xi \in \Xi$.
(ii) The density $\zeta(x_t^i; y_t)$ *of the distribution of* x_t^i *has a finite* L^2 *norm, bounded uniformly in* y_t.
(iii) The norm of the differential $D_1 f(a, b_1, \ldots, b_n, y)$ *is uniformly bounded away from zero, (which follows if* f *is* C^1*.)*
(iv) A participation constraint is satisfied, in that an agent with any signal weakly prefers participating in the auction to not participating. Precisely, $U^M(x, \xi) \geq 0$ *for all* $(x, \xi) \in \mathbb{X} \times \Xi$.

Proof of Theorem 3 (No Trade):

Consider any mechanism, a given λ, and an equilibrium for which $Pr\left(X_t(y)\right) > 0$ for some $y \in \mathbb{Y}$. Let $x_t = (x_t^1, \ldots, x_t^n)$. If the owner sells the object, then her expected surplus, conditionally on her signals, is:

$$\begin{aligned}
& P^N(\xi) + \delta B_{t+1}(y) \\
= \;& \delta E\left(\sum_{i=1}^{n} V_{t+1}^i \pi_i^M(x_t,\xi) \;\Big|\; x_t^0 \in X_t(y), y_t = y\right) \\
& + \delta S_{t+1}(y) - E\left(\sum_{i=1}^{n} U^M(x_t^i,\xi) \;\Big|\; y_t = y\right) \\
= \;& \delta E\left(f(x_t^0, x_t, y) \;\big|\; x_t^0 \in X_t(y), y_t = y\right) \\
& + \delta E\left(\sum_{i=1}^{n} \pi_i^M(x_t,\xi) g(\lambda, x_t^i, x_t^0, x_t^{-(0,i)}, y) \;\Big|\; x_t^0 \in X_t(y), y_t = y\right) \\
& + \delta S_{t+1}(y) - E\left(\sum_{i=1}^{n} U^M(x_t^i,\xi) \;\Big|\; y_t = y\right).
\end{aligned}$$

A bidder's surplus, $U^M(x_t^i,\xi)$, is non-negative, since otherwise he will not participate. In fact, we show below that $E\left(\sum_{i=1}^n U^M(x_t^i,\xi) \;\big|\; y_t = y\right)$ is bounded below away from 0 independently of λ and the mechanism, uniformly in ξ. Hence, there exists $\omega > 0$ (not depending on λ, ξ, or the mechanism) such that:

$$\begin{aligned}
& P^{y,D}(x_t^0) + \delta B_{t+1}(y) < \\
& \quad \delta E\left(f(x_t^0, x_t, y) \;\big|\; x_t^0 \in X_t(y), y_t = y\right) + \delta S_{t+1}(y) - \omega \\
& \quad + \delta E\left(\sum_{i=1}^{n} \pi_i^M(x_t,\xi) g(\lambda, x_t^i, x_t^0, x_t^{-(0,i)}, y) \;\Big|\; x_t^0 \in X_t(y), y_t = y\right).
\end{aligned}$$

Consider now that the owner keeps the object; then, she gets the surplus

$$E(f(x_t^0, x_t, y) + h(\lambda, x_t^0, x_t, y) \;\big|\; x_t^0, y_t = y) + \delta S_{t+1}(y).$$

Taking expectations over the set on which the sale occurs and writing out explicitly the condition that the sale occurs, we obtain:

$$\begin{aligned}
& \delta\; E\Bigg(\sum_{i=1}^{n} \pi_i^M(x_t,\xi) g(\lambda, x_t^i, x_t^0, x_t^{-(0,i)}, y) \\
& -h(\lambda, x_t^0, x_t, y) \;\Bigg|\; x_t^0 \in X_t(y), y_t = y\Bigg) \;>\; \omega.
\end{aligned}$$

Since g, h, and π are bounded, and g and h approach 0 as λ approaches 0, a symmetric equilibrium with sale can exist only if λ is large enough (which follows by dominated convergence). This proves the theorem.

It remains to be shown that the $E\left(\sum_{i=1}^{n} U^M(x_t^i, \xi) \;\middle|\; y_t = y\right)$ can be bounded below away from 0 independently of λ and the mechanism, uniformly in $\xi \in \Xi$. It suffices to prove that $U^M(x_t^i, \xi)$ is bounded below, away from 0, on a set of values of x_t^i of non-zero probability, uniformly in ξ. To that end, we use the Envelope Theorem as in (A.1), and show that the integrand is bounded away from 0. First, take a point $\bar{x}$ such that $\underline{\chi} \ll \bar{x} \ll \overline{\chi}$ and consider the set $C = \{x \in \mathbb{X} : x \geq \bar{x}\}$. Fix a point x^1 in the interior of C. Choose γ to parameterize the straight line joining $\underline{\chi}$ and x^1, which makes $D\gamma$ into a constant vector. Note that each entry of $D\gamma$ is bounded away from 0 uniformly in $x^1 \in C$. Then for $\gamma(r) \in C$,

$$
\begin{aligned}
& E_t\left(D_{x^1} v_{t+1}(x_t^1, x_t^0, (x_t^j)_{j>1}) D\gamma(r) \pi_1^M(x_t, \xi) \;\middle|\; x_t^0 \in X, x_t^1 = \gamma(r), y_t = y\right) \\
= \;& E_t\Bigg((D_{x^1} f(x_t^0, x_t, y) + D_{x^1} g(\lambda, x_t^1, x_t^0, x_t^{-(0,1)}, y)) \\
& \quad \times D\gamma(r) \pi_1^M(x_t, \xi) \;\Bigg|\; x_t^0 \in X, x_t^1 = \gamma(r), y_t = y \Bigg) \\
\geq \;& \phi_1 E_t\left(\pi_1^M(x_t, \xi) \;\middle|\; x_t^1 = \gamma(r), y_t = y\right) \\
= \;& \phi_1 \, Q^M(\gamma(r), \xi)\,,
\end{aligned}
$$

where $\phi_1 > 0$ is a uniform lower bound on $D_{x^1} f(x_t^0, x_t, y) D\gamma(r)$ for $\gamma(r) \in C$. (The term containing $D_{x^1} g$ is always positive.)

We now show that, on a set of positive probability, Q^M is bounded below uniformly away from 0. We first note, using the Cauchy-Schwartz inequality and symmetry, that

$$
\int_{\mathbb{X}} \left(Q^M\right)^2 (x, \xi)\, dx \int_{\mathbb{X}} \zeta^2(x, y)\, dx \geq \left(E\left(Q^M(\,\cdot\,, \xi)\right)\right)^2 = \frac{1}{n^2}.
$$

Then we use the fact that Q^M is increasing and bounded above (by 1) and the uniform bound on $\int_{\mathbb{X}} \zeta^2(x, y)\, dx$ to deduce the existence of a point $\bar{\bar{x}} \ll \overline{\chi}$ such that $Q^M(\bar{\bar{x}}) > \phi_2 > 0$ for any mechanism. Consequently, invoking the envelope-theorem result, on the set of points in $\mathbb{X}$ that are larger than both $(\bar{x} + \overline{\chi})/2$ and $(\bar{\bar{x}} + \overline{\chi})/2$, $U(x, \xi)$ is bounded below. This finishes the proof.

□

In the proofs to follow we omit from the notation the dependence on any other variable, including time, than the private signals.

Proof of Theorem 4:

We prove the result for any given values of t, y_t, and $D_t = \delta E(S_{t+1} - B_{t+1} \mid y_t)$, and suppress notational dependence on these variables. Although D_t depends on the mechanism, the sale decision does not depend on it because both the owner's utility, $\bar{u}$, and the expected price, P, are additive in D_t (Condition 4).

Let the sale set associated with mechanism N be $X^N = [\underline{\chi}, a^N]$, with $a^N \in \mathbb{X}$. We define $\mathcal{S} : \mathbb{X} \to \mathbb{X}$ by $\mathcal{S}(x) = z$, where z satisfies $P^M(z, [\underline{\chi}, x]) = \bar{u}(z)$ if $P^M(\overline{\chi}, [\underline{\chi}, x]) \geq \bar{u}(\overline{\chi})$, and $z = \overline{\chi}$ otherwise. That is, if buyers anticipate the sale set $[\underline{\chi}, x]$ then the owner will sell on the set $[\underline{\chi}, \mathcal{S}(x)]$. Hence, a sale set, $[\underline{\chi}, x]$, is and equilibrium if x is a fixed point for $\mathcal{S}$, that is, if $\mathcal{S}(x) = x$. We note that S is well-defined because of Condition 2. We are looking for a fixed point, a^M, which is larger than a^N. It follows from Condition 3 that $\mathcal{S}$ is (weakly) increasing. Further, it follows from (11) that $\mathcal{S}(a^N) \geq a^N$, and it is obvious that $\mathcal{S}(\overline{\chi}) \leq \overline{\chi}$. Now, Lemma 12 (below) implies that $\mathcal{S}$ has a fixed point in $[a^N, \overline{\chi}]$. One proves analogously that, given an equilibrium for M with sale set $[\underline{\chi}, a^M]$, an equilibrium exists for N with a smaller sale set $[\underline{\chi}, a^N]$.

The welfare claim follows immediately. The welfare is higher for mechanism M than for mechanism N if and only if

$$E[(\bar{v}(x^0) - \bar{u}(x^0))1_{x^0 \in (a^N, a^M]}] \geq 0.$$

Condition 5 ensures that the value of the integrand is positive at $x^0 = a^M$, while Assumption 5 states that it is decreasing. Consequently, the expectation is positive.

Lemma 12 *Suppose that $f : [a, b] \to \mathbb{R}$, for $a, b \in \mathbb{R}$, is (weakly) increasing, and that $f(a) \geq a$ and $f(b) \leq b$. Then, f has a fixed point in $[a, b]$.*

Proof: This lemma is a special case of the lattice fixed-point theorem of Knaster-Tarski. See, for instance, Dugundji and Granas (1982). We provide a simple proof for the readers' convenience.

If $f(b) = b$ then we are done, so assume $f(b) < b$. Then, $z = \inf\{x \in [a, b] : f(x) < x\}$ is well-defined. We claim that z is a fixed point. Consider a sequence, (y_i), in $\{x \in [a,b] \,:\, f(x) < x\}$ converging to z and, an increasing sequence, (z_i), also converging to z. Then, we have the inequalities

$$z_i \leq f(z_i) \leq f(z) \leq f(y_i) < y_i,$$

and the proof is completed by letting i approach infinity.

$\square$

Proof of Lemma 5:
Let $Z = (Z_1, \ldots, Z_n)$. Consider any nondecreasing function $g : \mathbb{R}^n \to \mathbb{R}$, and any sublattice S of $\mathbb{R}^n$. Then, since a product of sublattices is a sublattice, and since $Z_0, Z_1, \ldots, Z_n$ are affiliated, Theorem 23, $(i) \Rightarrow (ii)$, of Milgrom and Weber (1982) gives

$$E\left[g(Z)h(Z) \;\middle|\; Z \in S, Z_0 \in A\right] \geq E\left[g(Z) \;\middle|\; Z \in S, Z_0 \in A\right] E\left[h(Z) \;\middle|\; Z \in S, Z_0 \in A\right],$$

which is equivalent to

$$E^{(Z_0 \in A)}\left[g(Z)h(Z) \;\middle|\; Z \in S\right] \geq E^{(Z_0 \in A)}\left[g(Z) \;\middle|\; Z \in S\right] E^{(Z_0 \in A)}\left[h(Z) \;\middle|\; Z \in S\right],$$

where $E^{(Z_0 \in A)}$ denotes expectation with respect to the conditional distribution given $(Z_0 \in A)$. The latter inequality shows, using Theorem 23, $(ii) \Rightarrow (i)$, of Milgrom and Weber (1982), that the conditional distribution of $Z_1, \ldots, Z_n$ given $(Z_0 \in A)$ is affilated.

$\square$

Notation:

In the following proofs, we make use of some results in Milgrom and Weber (1982), to which we refer as "MW." For this reason we use notation that is very close to that of MW. The analysis relies on Lemma 5, which implies that the random variables $x^1, \ldots, x^n$ are affiliated conditionally on a sale, that is, $x^0 \in X$. We isolate bidder 1 and let $Y_1, \ldots, Y_{n-1}$ be the bids of the other $n-1$ bidders, arranged in *descending* order.

Proof of Lemma 6:

The fact that Condition 4 is satisfied is trivial to verify. Contion 5 is also clearly satisfied by the equilibria displayed by Milgrom and Weber (1982). Hence, we turn to Condition 3.

First, we show that Condition 3 applies for the second-price auction. This follows from the fact that the equilibrium bids, which are given by

$$E\left(v(x,x^0,x,(Y_j)_{j>2}) \;\middle|\; x^1=x, Y_1=x, x^0\leq a\right),$$

increase in a, which is an immediate consequence of Theorem 5 in Milgrom and Weber (1982).

Second, showing that Condition 3 applies for the English auction is analogous to the argument given for the second-price auction.

Lastly, we show that Condition 3 holds for the first-price auction. We work under the (unrestrictive) assumption that the optimal bids in the first-price auction, denoted by $b(x,X)$, where x is the bidder's signal and X the sale-set, are differentiable in the first argument. Let b_1 designate the derivative of b with respect to the first argument. MW show that the optimal bid in the first-price auction must obey the differential equation

$$b_1(x,X) = (\hat{v}(x,x,X) - b(x,X))\frac{f_{Y_1}(z|z,X)}{F_{Y_1}(z|z,X)}, \tag{A.2}$$

where $f_{Y_1}(\cdot|z,X)$, respectively $F_{Y_1}(\cdot|z,X)$ is the probability density function, respectively cumulative distribution function, of Y_1 conditionally on $x^1=z$ and on the owner's signal being in the sale set X, and

$$\hat{v}(x,y,X) = E\left(v(x^1,x^0,(x^j)_{j\notin\{0,1\}}) \;\middle|\; x^1=x, Y_1=y, x^0\in X\right).$$

Now, let a and a' be such that $\underline{\chi}\leq a' < a \leq \overline{\chi}$. Then, the equilibrium condition $b(\underline{\chi},X) = \hat{v}(\underline{\chi},\underline{\chi},X)$ and Theorem 5 in MW imply that $b(\underline{\chi},[\underline{\chi},a'])\leq b(\underline{\chi},[\underline{\chi},a])$. Suppose, in order to apply Lemma 14 below that $b(x,[\underline{\chi},a])\leq b(x,[\underline{\chi},a'])$ for some x. Then, by (A.2) and Lemma 13 (below), we conclude that $b_1(x,[\underline{\chi},a'])\leq b_1(x,[\underline{\chi},a])$. Hence, using Lemma 14 we see that Condition 3 is also satisfied by the first-price auction.

□

Lemma 13 *$F_{Y_1}(\,\cdot\,|\,z,[\underline{\chi},a])/f_{Y_1}(\cdot|z,[\underline{\chi},a])$ is decreasing in a.*

Proof: Use the notation $f_{Y_1,x^0}(\,\cdot\,,\,\cdot\,|\,z)$ for the joint probability density function of the signals Y_1 and x^0 conditional on the signal of bidder 1, $x^1 = z$. Take $a' \leq a$ and $x' \leq x$, and integrate the affiliation inequality to obtain

$$\int_{\underline{\chi}}^{a'} f_{Y_1,x^0}(x,u|z)\,du \int_{a'}^{a} f_{Y_1,x^0}(x',u|z)\,du$$
$$\leq \int_{\underline{\chi}}^{a'} f_{Y_1,x^0}(x',u|z)\,du \int_{a'}^{a} f_{Y_1,x^0}(x,u|z)\,du.$$

By adding $\int_{\underline{\chi}}^{a'} f_{Y_1,x^0}(x,u|z)\,du \int_{\underline{\chi}}^{a} f_{Y_1,x^0}(x',u|z)\,du$ to both sides, we get

$$\frac{\int_{\underline{\chi}}^{a} f_{Y_1,x^0}(x',u|z)\,du}{\int_{\underline{\chi}}^{a} f_{Y_1,x^0}(x,u|z)\,du} \leq \frac{\int_{\underline{\chi}}^{a'} f_{Y_1,x^0}(x',u|z)\,du}{\int_{\underline{\chi}}^{a'} f_{Y_1,x^0}(x,u|z)\,du}.$$

Now integrate both sides over $x' \in [\underline{\chi}, x]$ to finish the proof.

□

Lemma 14 *Let g and h be differentiable functions for which* (i) $g(\underline{\chi}) \geq h(\underline{\chi})$ *and* (ii) $g(x) < h(x)$ *implies* $g'(x) \geq h'(x)$. *Then $g(x) \geq h(x)$ for all $x \geq \underline{\chi}$.*

Proof: This is Lemma 2 in Milgrom and Weber (1982).

□

Proof of Lemma 7:
We denote by $W^M(x,z,X)$ the conditional expected payment made by bidder 1 in auction mechanism M if (i) the other bidders follow their equiibrium strategies, (ii) bidder 1's estimate is z, (iii) he bids as if it were x, (iv) he wins, and (v) all bidders believe that the owner's signal lies in X. To prove inequality (12), we use that $W^2(z,z,X) \geq W^1(z,z,X)$. (See MW's Theorem 15 and its proof.) Hence, (12) follows because

$$P^M(x^0) \;=\; E\left(W^M(x^1,x^1,X) \;\middle|\; x^1 > Y_1, x^0\right),$$

for $M \in \{1,2\}$.

□

Proof of Lemma 9:
It follows from MW, that when the anticipated sale set is X, and $X_1 > Y_1$, then the price in the second price auctions is $v(Y_1, Y_1, X)$, where

$$v(x, y, X) = E(V^1 \mid X_1 = x, Y_1 = y, X_0 \in X),$$

and the price in an English auction is $w(Y_1, Y_1, (Y_2, \ldots, Y_n), X)$, where

$$w(x, y, z, X) = E(V^1 \mid X_1 = x, Y_1 = y, (Y_2, \ldots, Y_n) = z, X_0 \in X).$$

Let $X = [\underline{\chi}, a]$ and $x > y$, and consider the inequality

$$\begin{aligned} & v(y, y, X) \\ & \quad = \; E(w(Y_1, Y_1, (Y_2, \ldots, Y_n), X) \mid X_1 = y, Y_1 = y, X_0 \in X) \\ & \quad \leq \; E(w(Y_1, Y_1, (Y_2, \ldots, Y_n), X) \mid X_1 = x, Y_1 = y, X_0 = a). \end{aligned}$$

This implies that

$$\begin{aligned} P^2(a, X) &= E(v(Y_1, Y_1, X) \mid X_1 > Y_1, X_0 = a) \\ &\leq E(w(Y_1, Y_1, (Y_2, \ldots, Y_n), X) \mid X_1 > Y_1, X_0 = a) \\ &= P^E(a, X). \end{aligned}$$

□

Proof of Lemma 11:
It suffices to show that $\bar{u}(x^0, \xi) - P^M(x^0, \xi)$ is a strictly increasing function of x^0, since in that case there can surely be no more than one solution to the equation $\bar{u}(x^0, \xi) - P^M(x^0, \xi) = 0$. Evaluating the expressions below at $\hat{x}^0 = x^0$, we have

$$\begin{aligned} \frac{\partial \bar{u}(\hat{x}^0, \xi)}{\partial \hat{x}^0} &= E\left(\frac{\partial u}{\partial \hat{x}^0}(\hat{x}^0, \ldots, x^n) \;\middle|\; x^0\right) + \frac{\partial}{\partial x^0} E\left(u(\hat{x}^0, \ldots, x^n) \;\middle|\; x^0\right) \\ &\geq E\left(\frac{\partial u}{\partial \hat{x}^0}(\hat{x}^0, \ldots, x^n) \;\middle|\; x^0\right) \\ &> E\left(\left(v(x^{(1)}, a, x^{(1)} \ldots, x^{(1)})\right)^2 \;\middle|\; x^0\right)^{1/2} \\ &\quad \times E\left(\left(\frac{\partial \log \zeta}{\partial \hat{x}^0}(x^1, \ldots, x^n \mid \hat{x}^0)\right)^2 \;\middle|\; x^0\right)^{1/2} \\ &\geq E\left[\left(p(x^1, \ldots, x^n, \xi_t) - D\right) \frac{\partial \log \zeta}{\partial \hat{x}^0}(x^1, \ldots, x^n \mid \hat{x}^0) \;\middle|\; x^0\right] \\ &= \frac{\partial P^M(x^0, \xi)}{\partial x^0}, \end{aligned}$$

where the first inequality follows because u is increasing and the signals affiliated, the second follows from the assumption of the lemma, and the third follows from Cauchy-Schwartz and the fact that

$$p(x^1, \ldots, x^n, (t, y, [\underline{\chi}, a], D)) < v(x^{(1)}, a, x^{(1)}, \ldots, x^{(1)}, y) + D$$

for the three mechanisms considered.

□

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