Valuation, Linear Information Dynamic, and Stochastic Discount rates

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Abstract

We generalize Ohlson (1995) to stochastic interest rates. Our analysis provides four insights. First, the earnings capitalization multiple depends on the lagged rate, not the current rate. Second, the abnormal earnings persistence parameter increases in the current rate and decreases in the lagged rate. Third, it is not necessary to specify the stochastic process underlying interest rates to relate stock prices and accounting numbers. Finally, only the lagged rate is needed to capitalize current earnings to determine current stock price, while both the lagged and current rates are needed to forecast next-period earnings based on current earnings.

1. Introduction

Ohlson (1995) relates accounting numbers and stock prices under risk neutrality and non-stochastic discount rates. The model specifies abnormal earnings as a first-order autoregressive process.[1] There are two extreme benchmark valuations. In mark-to-market accounting, book values are equal to prices and abnormal earnings have no persistence; in permanent-earnings accounting, prices are equal to capitalized earnings net of dividends and abnormal earnings have a persistence of one. The earnings capitalization multiple equals R/(R-1) where R denotes the risk-free discount rate. The model also allows for convex combinations of the two extremes such that price is a weighted average of book value and capitalized earnings net of dividends. The weight depends on the persistence of abnormal earnings.

We generalize Ohlson (1995) to stochastic discount rates. The natural questions are: How does the earnings capitalization multiple depend on the interest rates? What linear information dynamic sustains the pricing equation under stochastic interest rates? Does one need to specify the stochastic process underlying interest rates? How do the interest rates affect the current earnings and price relation, and current earnings and next-period expected earnings relation?

Our analysis provides the following answers. First, the earnings capitalization multiple depends on the lagged rate, not the current rate. Second, the abnormal earnings persistence parameter in the linear information dynamic increases in the current rate and decreases in the lagged rate. Third, one need ***not*** specify the stochastic process underlying interest rates to model the relationship between stock prices and accounting numbers. Finally, only the lagged rate is needed to capitalize current earnings to determine current stock price. The lagged rate is needed because the earnings rate for the current period is the rate prevailing at the beginning of the period. In contrast, both the lagged and current rates are needed to forecast next-period earnings based on current earnings. Current earnings are divided by the lagged rate to arrive at the current price, which is then multiplied by the current rate to arrive at the forecast of next-period earnings.

We build our analysis of the four issues above by analyzing models with increasing generality and complexity. Section 2 describes the notation and assumptions. Section 3 analyzes the pure mark-to-market model. Section 4 analyzes the pure permanent-earnings model. Section 5 analyzes the weighted average of the two models. Section 6 analyzes the weighted average model with other information. Section 7 summarizes and concludes the paper.

2. Notation and Assumptions

At date t, the "preceding" period refers to the period from date t-1 to date t, and the "forthcoming" period refers to the period from date t to date t+1.

x_t = earnings for the period t-1 to t, i.e., the preceding period

d_t = dividends, net of capital contributions, date t

P_t = ex-dividend market price of equity, date t

b_t = book value, date t

g_t = $P_t - b_t$ = goodwill, date t

r_t = risk free interest rate for the period t to t+1. (At date t, r_t is the current rate and r_{t-1} is the lagged rate.)

R_t = $1 + r_t$

x_t^a = $x_t - r_{t-1}b_{t-1}$ = abnormal or residual earnings for the preceding period.

Assumptions:

1. Risk neutrality,[2] which yields:

$$P_t = \frac{E_t(P_{t+1} + d_{t+1})}{R_t} \qquad \text{(RN)}$$

Note that R_t is random before t.

2. Clean surplus relation:

$$b_{t+1} = b_t + x_{t+1} - d_{t+1} \qquad \text{(CSR)}$$

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[1] See Frankel and Lee (1998), Dechow, Hutton, and Sloan (1999), and Lo and Lys (2000) for an empirical assessment of the Ohlson (1995) model.

[2] For risk aversion, one can replace the expectation operator E by the E* that reflects risk-adjusted probabilities. See Huang and Litzenberger (1988).

Subsequent derivations are based on the following goodwill equation (GE), which holds if and only if one assumes risk neutrality and CSR:

$$g_t = \frac{E_t(g_{t+1} + x^a_{t+1})}{R_t} \qquad \text{(GE)}$$

3. The Mark-to-Market Model

We start with the simple but important benchmark -- the pure mark-to-market model. As described in the introduction, we now examine the following four aspects of the mark-to-market model:

1. The behavior of abnormal earnings: Since there is no goodwill, the goodwill equation (GE) yields $E_t x^a_{t+1} = 0$.
2. The pricing equation: $P_t = b_t$.
3. The role of the stochastic process underlying interest rates: Interest rates play no role here because the book value subsumes information about interest rates. An analogy to an investment fund is helpful. The prices of securities held by the fund will generally depend on interest rates, but since mark-to-market accounting sets the book value of each security to its market price, the book value will variations in market value due to interest rates without having to model stochastic interest rates.
4. The role of current and lagged rates: In mark-to-market accounting, goodwill and expected abnormal earnings are zero. It also follows that $E_t x_{t+1} = r_t b_t = r_t P_t$. Thus, the expected forthcoming earnings depend only on the current rate.[3] The lagged rate plays no direct role in the analysis because the book value captures all the information.

4. The Permanent-Earnings Model

We now analyze the permanent-earnings model along these four dimensions. In contrast to the mark-to-market model, the permanent-earnings model is more subtle and complex because relating earnings to prices requires a specification of the earnings

[3] See Nissim and Penman (2000) for an empirical relationship between interest rates and accounting rates of return.

capitalization multiple.[4] Ohlson (1995) specifies the permanent-earnings model with non-stochastic interest rates as:

$$P_t = \frac{R}{r} x_t - d_t$$

The earnings capitalization multiple equals R/r where R is the risk-free rate and r = R-1.

4.1 The Pricing Equation under Stochastic Discount Rates

In the permanent-earnings model, price equals capitalized earnings minus dividends. The main question is: When interest rates are stochastic, should the earnings capitalization multiple be defined as R_{t-1}/r_{t-1} or R_t/r_t? It is important that the choice also apply to the case of certainty, i.e., a savings account. We show that only the former satisfies this criterion.

If at date t we observe x_t as the earnings for the period t-1 to t, we can infer that the savings account balance at t-1 was x_t/r_{t-1}. By t, the balance grows to $x_t + \frac{x_t}{r_{t-1}} = \frac{R_{t-1}}{r_{t-1}} x_t$. The balance after the withdrawal d_t is the price P_t. The earnings rate for the period t-1 to t is the rate prevailing at t-1, not t, so the capitalization factor used to interpret earnings for the preceding period depends on the lagged rate not the current rate. Therefore, one obtains the following pricing equation under certainty:

$$P_t = \frac{R_{t-1}}{r_{t-1}} x_t - d_t$$

From the perspective of our analysis, the major difference between certainty and uncertainty is that abnormal earnings are zero under certainty, but not under uncertainty. It remains to be seen whether the above earnings capitalization multiple extends to stochastic discount rates in the spirit of the Ohlson (1995) model.

4.2 The Behavior of Abnormal Earnings and Earnings

Ohlson (1995) shows that in a permanent-earnings model under constant interest rates the abnormal earnings persistence parameter is constant.

[4] See Ryan (1988).

$x_{t+1}^a = x_t^a + \varepsilon_{t+1}$,

where $E_t(\varepsilon_{t+1}) = 0$.

We now specify the linear information dynamic that sustains the pricing equation under stochastic interest rates to see how the abnormal earnings persistence parameter depends on interest rates. We hypothesize the following linear information dynamic:

$x_{t+1}^a = \omega_t x_t^a + \varepsilon_{t+1}$,

where ω_t can depend only on the history of interest rates. The two main issues are: Does ω_t depend on the entire history of interest rates or is a smaller subset sufficient? Does ω_t oscillate around 1, which is its value when interest rates do not change across time?

Proposition 1: Given risk neutrality and clean surplus, $P_t = \frac{R_{t-1}}{r_{t-1}} x_t - d_t$ implies

$\omega_t = \frac{r_t}{r_{t-1}}$.

Proof: See Appendix I.

The abnormal earnings persistence parameter depends only on the lagged and current rate, not the entire history of interest rates. It decreases in the lagged rate and increases in the current rate. If the distribution of interest rates satisfies reasonable regularity conditions, then the median abnormal earnings persistence parameter is 1, which is its value when the interest rates are constant.

The intuition underlying the functional form of the earnings persistence parameter can be briefly stated as follows. The current abnormal earnings are first divided by the lagged rate as a capitalization factor and are then multiplied by the current rate to compute forecasted forthcoming abnormal earnings. Further details are in section 4.4.

The Random Walk of Earnings

Ohlson (1995) implies the following stochastic process for earnings:

$E_t x_{t+1} = x_t + r \ \Delta \ b_t$

The first term represents the standard random walk model of earnings and is valid only if there is no new investment and there are no changes in interest rates. The second term represents the adjustment to expected earnings due to changes in investment levels

(Δb_t). It is easy to see that r will be replaced by r_t when interest rates are stochastic because expected earnings depends on the current rate applied to new investments. The following corollary reveals changing r to r_t is not enough; stochastic interest rates introduce an additional term in the standard random walk model.

Corollary 1: $E_t x_{t+1} = x_t + r_t \Delta b_t + \%\Delta r_t x_t$

Proof: See Appendix I.

The third term, which has not been recognized in prior research, shows the direction of a change in interest rate, nor just the level of interest rates, affects earnings forecasts; an uptick in interest rates lead to higher earnings forecasts, and vice versa.

4.3 The Lack of Need To Specify the Stochastic Process Underlying Interest Rates

The permanent-earnings model does not require a specification of the stochastic process underlying interest rates because earnings subsume information about interest rates. In the case of a savings account discussed in section 4.1, the lagged rate is sufficient to infer the savings account balance from observed earnings and the current rate is sufficient to compute the growth in the balance over the forthcoming period. Expectation of future interest rates is not needed.

4.4 The Role of Current and Lagged Rates

A key insight of the paper is that only the lagged rate is needed to capitalize current abnormal earnings and only the current rate is needed to capitalize expected forthcoming abnormal earnings.

Corollary 2: $g_t = \frac{E_t x_{t+1}^a}{r_t}$ and $g_t = \frac{x_t^a}{r_{t-1}}$.

Proof: See Appendix I.

The corollary brings out the crucial intuition that the earnings rate for a period is the interest rate prevailing at the beginning of that period.

From the corollary, we get $E_t\, x^a_{t+1} = r_t\, g_t = r_t \dfrac{x^a_t}{r_{t-1}}$, i.e., the abnormal earnings persistence parameter $\omega_t = \dfrac{r_t}{r_{t-1}}$. Given current abnormal earnings, the higher the lagged rate, the lower the current goodwill; the higher the current rate, the higher the abnormal earnings that this goodwill is expected to generate.

5. A Weighted-Average of the Two Models

We now extend the weighted average of the permanent-earnings model and the mark-to-market model presented in Ohlson (1995) to stochastic interest rates. To facilitate comparison, we continue to study the four aspects listed in the introduction.

5.1 The Pricing Equation

Ohlson (1995) expresses price as a weighted average of the two models as follows:

$$P_t = k\ (\frac{R}{r} x_t - d_t) + (1 - k\ ) b_t$$

We specify the pricing equation as a weighted average of the permanent-earnings model and the mark-to-market model under stochastic interest rates as follows:

$$P_t = k\ (\frac{R_{t-1}}{r_{t-1}} x_t - d_t) + (1 - k\ ) b_t$$

where $k \in [0,1]$.

Our objective is to derive the linear information dynamic and the modification to the random walk of earnings that are implied by such a representation.

5.2 The Behavior of Abnormal Earnings

Ohlson (1995) shows that the above pricing equation under non-stochastic rates implies the following linear information dynamic:

$$x^a_{t+1} = \omega x^a_t + \varepsilon_{t+1},$$

where $E_t(\varepsilon_{t+1}) = 0$ and

$$\omega\ = \frac{1+r}{k+r} k\ .$$

We hypothesize the following linear information dynamic:

$x^a_{t+1} = \omega_t \, x^a_t + \varepsilon_{t+1}$.

As before, ω_t can depend only on the history of interest rates. One can ask whether ω_t continue to increase in the current rate and decrease in the lagged rate, as in the permanent earnings model.

Proposition 2: Given risk neutrality and clean surplus, $P_t = k \left(\frac{R_{t-1}}{r_{t-1}} x_t - d_t\right) + (1-k)\, b_t$ implies $\omega_t = \frac{1+r_t}{k+r_t} r_t \frac{k}{r_{t-1}}$.

Proof: See Appendix I.

Similar to the permanent-earnings model, the abnormal earnings persistence parameter decreases in the lagged rate and increases in the current rate (For k > 0, $\frac{\partial \omega_t}{\partial r_t} > 0$ and $\frac{\partial \omega_t}{\partial r_{t-1}} < 0$.) As the weight assigned to earnings in the pricing equation increases, the abnormal earnings persistence parameter increases ($\frac{\partial \omega_t}{\partial k} > 0$). In the mark-to-market model (k=0), $\omega_t = 0$, while in the permanent earnings model (k=1), $\omega_t = r_t/r_{t-1}$.

Although the sensitivity of the abnormal earnings persistence parameter to interest rates may be expected, its functional form is not obvious. Rearranging the terms in ω_t highlights the impact of changing interest rates on ω_t.

$$\omega_t = \frac{r_t}{r_{t-1}} \frac{(1+r_t)k}{k+r_t}$$

The first term reflects the "correction" due to the changing interest rates while the second term equals ω under constant interest rates. A further understanding of this relationship requires a specification of how current goodwill relates to current earnings and expected forthcoming earnings. These relationships are examined in Section 5.4.

So far, we have assumed that k, the weight assigned to permanent-earnings model, is constant. One can question the extent to which our results depend on this assumption.

The robustness of our results is examined in Appendix II, which allows k to vary across time. It shows that ω_t continues to increase in the current rate and decrease in the lagged rate when k varies over time but is known at the beginning of a period.

The Random Walk of Earnings

Ohlson (1995) implies the following expression for expected forthcoming earnings in the weighted-average model:

$$E_t x_{t+1} = \omega(x_t + r\Delta b_t) + (1-\omega) r b_t$$

Two features of the expression above are noteworthy. First, expected forthcoming earnings are a weighted average of the expected forthcoming earnings under the two models. Second, the weight assigned to permanent earnings equals the abnormal earnings persistence parameter (ω). The corollary below shows that under stochastic interest rates the expected earnings continue to be a weighted average of earnings under the permanent-earnings model and the mark-to-market model. It turns out, however, that the weight is no longer equal to the abnormal earnings persistence.

Corollary 3: $E_t x_{t+1} = \theta_t(x_t + r_t \Delta b_t + \%\Delta\ r_t\ x_t) + (1-\theta_t) r_t b_t$ where $\theta_t = \frac{1+r_t}{k+r_t} k$.

Proof: See Appendix I.

In contrast to the non-stochastic case, now the weight, θ_t, assigned to permanent earnings in the random walk equation differs from the abnormal earnings persistence parameter, ω_t. In fact, $\theta_t = \frac{r_{t-1}}{r_t} \omega_t$, and θ_t depends only on the current rate while ω_t depends on both the current and the lagged rate.

There is, however, a key similarity between the non-stochastic and stochastic case. In both cases, the weight assigned to permanent earnings in the expected earnings equation increases with k. (Both ω_t and θ_t increase in k.)

5.3. The Lack of Need to Specify the Stochastic Process of Interest Rates

The weighted-average model does not require that we specify the stochastic process underlying interest rates because the earnings and book value subsume this information. This is not because k is time independent in our model. Appendix II shows that we do not need a specification of the stochastic process even if k varies through time but is known at the beginning of a period.

5.4 The Role of Current and Lagged Rates

The permanent-earnings model showed that one needs only the lagged rate to capitalize current abnormal earnings and only the current rate to capitalize expected forthcoming abnormal earnings. The corollary below shows that this intuition extends to the weighted-average model.

Corollary 4: $g_t = \frac{k + r_t}{1 + r_t} \frac{E_t x^a_{t+1}}{r_t}$ **and** $g_t = k \frac{x^a_t}{r_{t-1}}$**.**

Proof: See Appendix I.

There is a key difference between the weighted-average model and its two extremes (the permanent-earnings model and the mark-to-market model). At both extremes, $E_t x^a_{t+1} = r_t g_t$ and $E_t x_{t+1} = r_t P_t$. This, however, is no longer true in the weighted average of the two models. The following restatement of the relationship between expected forthcoming abnormal earnings and current goodwill reveals why this is so:

$E_t x^a_{t+1} = \frac{1+r_t}{k + r_t} r_t g_t$.

r_t is the earnings rate from the current goodwill over the forthcoming period. Since $k \le 1$, $\frac{1+r_t}{k + r_t} = 1 + \frac{1-k}{k + r_t} \ge 1$. When k = 1 (the permanent-earnings model), $E_t x^a_{t+1} = r_t g_t$. When k <1 (the weighted-average model), $E_t x^a_{t+1} \ge r_t g_t$ in addition to the earnings from the current goodwill, a part of the current goodwill itself ($\frac{1-k}{k + r_t}$) is expected to be booked as earnings, i.e., the current goodwill is expected to decay over time as it is gradually transformed into book value through earnings. (In the mark-to-market model, goodwill is identically zero and so are expected abnormal earnings.)

6. The Role of Other Value Relevant Information

So far, we have generalized the Ohlson (1995) model without "other" information. We have established how stock prices and forecasts of forthcoming earnings depend on accounting numbers alone when interest rates are stochastic. The main insight from the preceding analysis is that both the lagged and current rates are needed to forecast forthcoming earnings based on current earnings. Current earnings are first divided by the lagged rate to capitalize them and are then multiplied by the current rate to arrive at the forecast of forthcoming earnings.

We now extend our analysis to include the Ohlson (1995) model with "other" information. It is interesting to determine whether the current and future rates continue to play the same role in the presence of such other value relevant information.

Ohlson (1995) allows for non-accounting value-relevant information and expresses price as follows

$$P_t = k\,(\frac{R}{r}x_t - d_t) + (1-k)\,b_t + \beta\,\upsilon_t$$

The linear information dynamic is specified as follows

$$\begin{aligned} x^a_{t+1} &= \omega\, x^a_t + \upsilon_t + \varepsilon_{1,t+1} \\ \upsilon_{t+1} &= \quad\quad + \gamma\,\upsilon_t + \varepsilon_{2,t+1} \end{aligned}$$

where $E_t(\varepsilon_{1,t+1}) = 0$, $E_t(\varepsilon_{2,t+1}) = 0$. Ohlson (1995) then derives $\omega = \frac{kR}{k+r}$ and $\gamma = R - \frac{k+r}{r\beta}$.

We allow stochastic interest rates and specify price as follows:

$$P_t = k\,(\frac{R_{t-1}}{r_{t-1}}x_t - d_t) + (1-k)\,b_t + \beta\,\upsilon_t$$

The linear information dynamic is as follows:

$$\begin{aligned} x^a_{t+1} &= \omega_t\, x^a_t + \upsilon_t + \varepsilon_{1,t+1} \\ \upsilon_{t+1} &= \quad\quad \gamma_t\,\upsilon_t + \varepsilon_{2,t+1} \end{aligned}$$

where $E_t(\varepsilon_{1,t+1}) = 0$ and $E_t(\varepsilon_{2,t+1}) = 0$. We hypothesize that ω_t and γ_t depend only on the history of interest rates.

One can ask whether the introduction of "other" information change the functional form of ω_t, and whether γ_t depends on the lagged rate. Note that ω_t depends on the lagged rate because the lagged rate is needed to interpret current earnings. γ_t is, however, *not* expected to depend on the lagged rate.

Proposition 3: Given risk neutrality and clean surplus,

$P_t = k\ (\frac{R_{t-1}}{r_{t-1}} x_t - d_t) + (1-k\ ) b_t + \beta \upsilon_t$ implies $\omega_t = \frac{r_t + 1}{r_t + k} r_t \frac{k}{r_{t-1}}$ and $\gamma_t = R_t - \frac{r_t + k}{r_t \beta}$.

Proof: See Appendix I.

The proposition shows that the functional form of abnormal earnings persistence (ω_t) is unaffected by the introduction of "other" information. The persistence of other information (γ_t) depends only on the current rate, not the lagged rate.

Thus, this paper generalizes Ohlson (1995) to stochastic interest rates and highlights the role of current and lagged rates in valuation and forecasting. The next section describes the empirical implications of our results.

7. Summary and Implications

The analysis in this paper yields a number of striking observations. First, the generalization of Ohlson [1995] hinges on a thorough understanding of how the benchmark settings – mark-to-market and permanent-earnings accounting – can allow for stochastic interest rates. Neither of these two cases leaves any choice as to how one models value as it relates to book value and earnings, respectively, when interest rates change. In particular, with respect permanent earnings it is clear that the capitalization depends solely on the lagged interest rate. Second, given the two benchmarks it is reasonably straightforward to expand the modeling to weighted-average settings, and to include so-called "other information". Third, in all of these cases the lagged interest rates serves the critical role of scaling current earnings so one can infer how current value relates to current earnings. Fourth, current interest rates enter the analysis by influencing the forecast of next-period's expected earnings. Whether one considers current book value or current capitalized earnings, the current interest rate thus determines the earnings rate in a traditional sense.

From an empirical perspective, it may seem unsatisfactory that current rates do not show up explicitly in the valuation function. It is, after all, well known that unexpected changes in interest rates correlate with market returns. But this observation is actually entirely consistent with this paper's analysis. Interest rate changes are relevant because they modify perceptions about long run earnings relative to the current interest rate. The most general version of the Ohlson [95] model here subnames this case. Simply consider the possibility of having other information (ν_t) depend on the current interest rate; that is, the innovation (ε_{2t+1}) may correlate negatively with unexpected changes in interest rates. This aspect of the model completes the analysis in that the model developed is fully consistent with the idea that current rates should influence current market values.

Appendix I: Proofs

Proof of Proposition 1

We can restate the expression for P_t as:

$$P_t = b_t + \frac{x_t^a}{r_{t-1}}$$

That is:

$$g_t = \frac{x_t^a}{r_{t-1}}$$

From the goodwill equation (GE) we get,

$\frac{R_t x_t^a}{r_{t-1}} = E_t(\frac{x_{t+1}^a}{r_t} + x_{t+1}^a)$, which simplifies to

$E_t\, x_{t+1}^a = \frac{r_t}{r_{t-1}} x_t^a$. Thus, $\omega_t = \frac{r_t}{r_{t-1}}$ QED.

Proof of Corollary 1

From Proposition 1 we get, $E_t\, x_{t+1}^a = \frac{r_t}{r_{t-1}} x_t^a$. Substituting the expression for abnormal earnings, we get

$E_t\, x_{t+1} - r_t\, b_t = \frac{r_t}{r_{t-1}}(x_t - r_{t-1} b_{t-1})$, which simplifies to

$E_t x_{t+1} = x_t + r_t(b_t - b_{t-1}) + (r_t - r_{t-1})\frac{x_t}{r_{t-1}}$,or

$$E_t x_{t+1} = x_t + r_t\, \Delta\; b_t + \%\Delta\; r_t\, x_t$$

Proof of Corollary 2

From Proposition 1 we get, $E_t\, x_{t+1}^a = \frac{r_t}{r_{t-1}} x_t^a$. From the proof of proposition 1, we get $g_t = \frac{x_t^a}{r_{t-1}}$. Substituting, we get $E_t\, x_{t+1}^a = r_t\, g_t$. QED

It is interesting to examine the relationship between expected forthcoming earnings and current stock price. Substituting the expression for abnormal earnings in $E_t\, x^a_{t+1} = \frac{r_t}{r_{t-1}} x^a_t$ we get, $E_t\, x_{t+1} - r_t\, b_t = \frac{r_t}{r_{t-1}}(x_t - r_{t-1} b_{t-1})$. Using CSR, we can restate this as

$$E_t\, x_{t+1} = r_t\left(\frac{x_t}{r_{t-1}} + x_t - d_t\right) = r_t\left(\frac{R_{t-1}}{r_{t-1}} x_t - d_t\right) = r_t P_t .$$

An analogy to the savings account brings out the relationship between prices and expected earnings. The earnings x_t for the period (t-1, t) imply that the savings account balance at t-1 was x_t/r_{t-1}. The balance at t equals the balance at t-1 plus the earnings over the period (t-1, t) minus the withdrawals over that period (x_t-d_t). The earnings rate for the period (t, t+1) is r_t.

Proof of Proposition 2

The pricing equation $P_t = k\left(\frac{R_{t-1}}{r_{t-1}} x_t - d_t\right) + (1-k)\, b_t$ can be restated as follows:

$$P_t = k\left(\frac{R_{t-1}}{r_{t-1}} x_t - d_t - b_t\right) + b_t .$$

From the clean surplus relation, we get $b_t + d_t = x_t + b_{t-1}$. Substituting for $b_t + d_t$ in the expression above, we get

$$P_t = k\left(\frac{R_{t-1}}{r_{t-1}} x_t - x_t - b_{t-1}\right) + b_t = k\left(\frac{x_t}{r_{t-1}} - b_{t-1}\right) + b_t$$

Substituting for the expression of abnormal earnings, we get $P_t = k\,\frac{x^a_t}{r_{t-1}} + b_t$, which implies $g_t = k\,\frac{x^a_t}{r_{t-1}}$.

Using the goodwill equation (GE) we get,

$$R_t k\,\frac{x^a_t}{r_{t-1}} = E_t\left(k\,\frac{x^a_{t+1}}{r_t} + x^a_{t+1}\right)$$

$E_t x^a_{t+1} = \frac{1+r_t}{r_t+k}\, r_t k\,\frac{x^a_t}{r_{t-1}}$, which implies $\omega_t = \frac{1+r_t}{r_t+k}\, r_t \frac{k}{r_{t-1}}$ QED

Proof of Corollary 3

From Proposition 2 we get, $E_t\, x_{t+1}^a = \frac{1+r_t}{r_t+k}\, r_t \frac{k}{r_{t-1}}\, x_t^a$. Substituting for abnormal earnings we get, $E_t x_{t+1} = \frac{1+r_t}{r_t+k}\, r_t\, k\; \frac{(x_t - r_{t-1} b_{t-1})}{r_{t-1}} + r_t b_t$

Define $\theta_t = \frac{1+r_t}{k+r_t} k$

Thus, $E_t x_{t+1} = \theta_t \left(\frac{r_t}{r_{t-1}} (x_t - r_{t-1} b_{t-1}) + r_t b_t \right) + (1-\theta_t) r_t b_t$, which can be restated as follows:

$E_t x_{t+1} = \theta_t (x_t + r_t \Delta b_t + \%\Delta\; r_t\; x_t) + (1-\theta_t) r_t b_t$ QED.

Proof of Corollary 4

From Proposition 2 we get, $E_t\, x_{t+1}^a = \frac{1+r_t}{r_t+k}\, r_t \frac{k}{r_{t-1}}\, x_t^a$. From the proof of proposition 2, we get $g_t = k\; \frac{x_t^a}{r_{t-1}}$. Substituting we get, $E_t x_{t+1}^a = \frac{1+r_t}{r_t+k}\, r_t\, g_t$. QED

Substituting for abnormal earnings and goodwill in the equation above, we get:

$$E_t x_{t+1} - r_t b_t = \frac{r_t}{r_t+k} R_t (P_t - b_t)$$

Upon simplification, we get:

$$E_t x_{t+1} = \frac{r_t}{r_t+k} (R_t P_t - (1-k) b_t)$$

Proof of Proposition 3

$$g_t = P_t - b_t = k\; (\frac{R_{t-1}}{r_{t-1}} x_t - d_t - b_t) + \beta \upsilon_t$$

Substituting for b_t from the clean surplus relation, $b_t + d_t = x_t + b_{t-1}$, and using the definition of abnormal earnings we get:

$$g_t = k\; \frac{x_t^a}{r_{t-1}} + \beta \upsilon_t$$

Using the goodwill equation (GE) we get,

$$R_t k \frac{x_t^a}{r_{t-1}} + R_t \beta \upsilon_t = E_t(k \frac{x_{t+1}^a}{r_t} + \beta \upsilon_{t+1} + x_{t+1}^a)$$

Since $\upsilon_{t+1} = \gamma_t \upsilon_t + \varepsilon_{2,t+1}$, we get

$$E_t x_{t+1}^a = \frac{r_t + 1}{r_t + k} r_t k \frac{x_t^a}{r_{t-1}} + \frac{r_t}{r_t + k} \beta \left(R_t - \gamma_t\right) \upsilon_t$$

This implies,

$$\frac{r_t}{r_t + k} \beta \left(R_t - \gamma_t\right) = 1$$

$$\gamma_t = R_t - \frac{r_t + k}{r_t \beta} \text{ QED}$$

Appendix II: The Weighted Average Model with Variable but Known Weights

We now examine a setting where the weights can vary over time, but are known at the beginning of a period. Thus, price is expressed as follows:

$$P_t = k_t (\frac{R_{t-1}}{r_{t-1}} x_t - d_t) + (1 - k_t) b_t$$

From the above equation, it is clear that $g_t = k_t (\frac{R_{t-1}}{r_{t-1}} x_t - d_t - b_t)$

Using CSR, $b_t + d_t = x_t + b_{t-1}$, we get $g_t = k_t \frac{x_t^a}{r_{t-1}}$

Using the goodwill equation (GE) we get,

$$R_t k_t \frac{x_t^a}{r_{t-1}} = E_t(k_{t+1} \frac{x_{t+1}^a}{r_t} + x_{t+1}^a)$$

Since kt+1 is known at time t, we get: [5]

$$E_t x_{t+1}^a = \frac{r_t + 1}{r_t + k_{t+1}} r_t k_t \frac{x_t^a}{r_{t-1}}$$

Thus, the abnormal earnings persistence parameter is represented by

$$\omega_t = \frac{r_t + 1}{r_t + k_{t+1}} r_t \frac{k_t}{r_{t-1}}$$

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[5] If k_{t+1} is not known at time t, we would need to know the covariance of k_{t+1} and $\frac{x_{t+1}^a}{r_t}$.

As discussed earlier, ω_t continues to depend on the current and lagged rates. Specifically, it increases in the current rate and decreases in the lagged rate. We do not perceive a constant ω_t to be plausible scenario. This can be seen by restating the expression above in terms of k_{t+1}.

$$k_{t+1} = \frac{r_t}{\omega_t} R_t \frac{k_t}{r_{t-1}} - r_t$$

A constant ω_t implies that the expression for weights used in the pricing equation is recursive (k_{t+1} depends on k_t), which implies that generally the weights depend on the entire history of interest rates. Only in the special case of $k_t = mr_{t-1}$, (where m is a constant) we get the following expression where the persistence parameter depends only on the current rate.

$$\omega_t = \frac{m}{1+m} R_t$$

Although, this results in a simple specification of the linear information dynamic, there is no straightforward economic interpretation of this scenario. The analysis above shows that Ohlson (1995) can be generalized to allow for variable weights in the pricing equation.

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