Dynamic Equicorrelation

Robert Engle *　　Bryan Kelly †

February 27, 2008

Abstract

A new covariance matrix estimator is proposed under the assumption that at every time period all pairwise correlations are equal. This assumption, which is pragmatically applied in various areas of finance, makes it possible to estimate arbitrarily large covariance matrices with ease. The model, called DECO, is a special case of the CCC and DCC models which involve first adjusting for individual volatilities and then estimating the correlations. A QMLE result shows that DECO can continue to give consistent parameter estimates when the equicorrelation assumption is violated. Generalizations to block equicorrelation structures, models with exogenous variables, and alternative specifications are explored and diagnostic tests are proposed. Estimation is evaluated by Monte Carlo and using US stock return data.

*Michael Armellino Professor in the Management of Financial Services, Department of Finance, Stern School of Business, NYU, 44 West Fourth Street, Suite 9-62, New York, NY 10012-1126. E-mail: rengle@stern.nyu.edu.

†Department of Finance, Stern School of Business, NYU, 44 West Fourth Street, Suite 9-197, New York, NY 10012-1126. E-mail: bkelly@stern.nyu.edu.

1 Introduction

Since the first volatility models were formulated in the early eighties there have been efforts to estimate multivariate models. The specification of these models developed over the past thirty years with a range of papers surveyed by Bollerslev, Engle and Nelson (1994) and more recently by Bauwens et al. (2006). A general conclusion from this analysis is that it is difficult to estimate multivariate GARCH models with more than half a dozen return series because the specifications are so complicated.

Recently, Engle (2002) proposed Dynamic Conditional Correlation (DCC), greatly simplifying multivariate specifications. For large systems, there is not only increasing difficulty in estimating dynamic correlations, but also difficulty presenting and analyzing these correlations. DCC is designed for high dimensional systems but has only been successfully applied to up to 50 assets by Engle and Sheppard (2001). Even though there are only a couple of parameters to estimate, the maximum likelihood estimator must invert a 50×50 matrix thousands of times and this is time consuming. With 50 assets there are 1225 correlation time series; this output is large and difficult to store or plot. In a smaller study with 34 assets, by Capiello, Engle and Sheppard (2006), the output was averaged over various regions and asset classes in order to better interpret the findings.

In this paper we propose a dramatic simplification that eliminates both the presentational difficulties and the computational difficulties of high dimension systems. We consider a system where all pairs of returns have the same correlation on a given day but this correlation varies over time; this general structure is called dynamic equicorrelation (abbreviated DECO). DECO estimates only one correlation time series instead of averaging correlations after estimating them.

There is a substantial history of the use of equicorrelation. In early studies of asset allocation Elton and Gruber (1973) found that the assumption that all pairs of assets had the same correlation reduced estimation noise and provided superior portfolio allocations over a wide range of alternative assumptions. This work is still widely referenced and is in the leading investments textbook. The same assumption surfaces in derivatives trading. A popular position is to buy an option on a basket of assets and then sell options on each of the components, sometimes called a dispersion trade. By delta hedging each option, the value of this position can be seen to depend solely on the correlations. Let the basket have weights given by the vector w, and let the implied covariance matrix of components of the basket be given by the matrix S. Then the variance of the basket can be expressed as

$$\sigma^2 = w'Sw.$$

In general we only know about the variances of implied distributions, not the covariances. Hence it is common to assume that all correlations are equal, giving

$$\sigma^2 = \sum_{j=1}^{n} w_j^2 s_j^2 + \rho \sum_{i \neq j} w_i w_j s_i s_j$$

which can be solved for the implied correlation

$$\rho = \frac{\sigma^2 - \sum_{j=1}^{n} w_j^2 s_j^2}{\sum_{i \neq j} w_i w_j s_i s_j}.$$

As a consequence, the value of this position depends upon the evolution of the implied correlation. When each of the variances is a variance swap made up of a portfolio of options, the full position is called a correlation swap. As the implied correlation rises, the value of the basket variance swap rises relative to the component variance swaps.

Another application of this assumption is in the market for credit derivatives such as collateralized debt obligations, or CDO's. A key feature of the risk in loan portfolios is the degree of correlation between the default probabilities. A simple industry valuation model allows this correlation to be one number if the firms are in the same industry and a different and smaller number if they are in a different industry. Hence within each industry an equicorrelation assumption is being made. This assumption may be implemented with our block DECO generalization.

More generally, to price CDO's, an assumption is often made that these are large homogeneous portfolios (LHP's) of corporate debt. As a consequence, each asset will have the same variance, the same covariance with a market factor and the same idiosyncratic variance. In a one factor world we can express the relation between the return on an asset and the market return as

$$r_j = \beta_j r_m + e_j, \quad \sigma_j^2 = \beta_j^2 \sigma_m^2 + v_j. \tag{1}$$

In an LHP, the j subscripts disappear. The correlation between any pair of assets then becomes

$$\rho = \frac{\beta^2 \sigma_m^2}{\beta^2 \sigma_m^2 + v}.$$

In fact, the LHP assumption implies equicorrelation.

In Section 2 we develop the basic DECO model and a variety of extensions. Appealing to quasi-maximum likelihood theory, we provide a result that demonstrates DECO's robustness to an important model misspecification: when equicorrelation is violated, DECO may still provide consistent estimates. In particular, when the true model is DCC, DECO is a quasi-maximum likelihood estimator.

DECO, like many multivariate GARCH models, requires that the cross section composition remain fixed for the full sample. Among the extensions discussed in Section 2 is a modified DECO model with linear correlation evolution, called LDECO, that solves this problem. A further advantage of LDECO is that it fixes the order of computational complexity for any cross section size, making correlation estimation feasible for an arbitrary number of assets. Section 2 goes on to explore questions regarding residual variance dynamics, forecasting and finally the block DECO model. We then propose diagnostic tests in Section 3 which may be used to assess the appropriateness of DECO specifications. Section 4 presents results from several Monte Carlo experiments that examine the model's performance for various equicorrelated and non-equicorrelated generating processes. In Section 5 we apply the DECO and block DECO models to US stock return data.

2 The Dynamic Equicorrelation Model

We begin by defining an equicorrelation matrix and presenting a result for its invertibility and positive definiteness that will be useful throughout the paper.

Definition 2.1 *A matrix R_t is an* equicorrelation matrix *of an $n \times 1$ vector of random variables if it is positive definite and takes the form*

$$R_t = (1 - \rho_t)I_n + \rho_t J_{n \times n} \tag{2}$$

where ρ_t is the equicorrelation, I_n denotes the n-dimensional identity matrix and $J_{n\times n}$ is an $n \times n$ matrix of ones.

Lemma 2.1 *The inverse and determinant of the equicorrelation matrix, R_t, are given by*

$$R_t^{-1} = \frac{1}{1-\rho_t}[I_n - \frac{\rho_t}{1+(n-1)\rho_t} J_{n\times n}] \tag{3}$$

and

$$\det(R_t) = (1-\rho_t)^{n-1}[1+(n-1)\rho_t]. \tag{4}$$

Further, R_t^{-1} exists if and only if $\rho_t \neq 1$ and $\rho_t \neq \frac{-1}{n-1}$, and R_t is positive definite if and only if $\rho_t \in (\frac{-1}{n-1}, 1)$.

Proof: For Equations 3 and 4 see Graybill (1983), Theorems 8.3.4 and 8.4.4. Existence of an inverse relies on non-zero denominators in Equation 3. The positive definiteness condition derives from its equivalence with all eigenvalues being positive; this can be seen in 4, which is the product of the eigenvalues of R_t. *Q.E.D.*

Definition 2.2 *A time series of $n \times 1$ vectors $\{\tilde{r}_t\}$ satisfies a* dynamic equicorrelation (DECO) *model if $Var_{t-1}(\tilde{r}_t) = D_t R_t D_t$, where D_t is a diagonal matrix with conditional standard deviations on the diagonal and R_t is given by Equation 2 for all t. The equicorrelation, ρ_t, is a general, potentially time-varying, function.*

DECO is adopted to individual applications by specifying a ρ_t process and specifying a conditional volatility model (i.e., defining the process for D_t, for instance using a GARCH model). We may abstract from the question of conditional volatility model by working with volatility-standardized returns, which we denote by omitting the tilde, $r_t = D_t^{-1}\tilde{r}_t$, so that $Var_{t-1}(r_t) = R_t$. In the empirical exercise of Section 5 we return to the question of modeling individual volatilities.

2.1 DECO-DCC

The basic ρ_t specification we consider derives from the DCC model of Engle (2002) and may be referred to as DECO-DCC. In DCC, the correlation matrix of standardized returns, R_t^{DCC}, is given by[1]

$$Q_t = \bar{Q}(1 - \alpha - \beta) + \alpha r_{t-1} r_{t-1}' + \beta Q_{t-1} \tag{5}$$

[1] The formulation in which the intercept of the Q process is $\bar{Q}(1 - \alpha - \beta)$ is referred to as "correlation targeting". Often $\bar{Q}$ is the unconditional return correlation estimated in a preliminary step rather than

$$R_t^{DCC} = \text{diag}\{Q_t\}^{-1} Q_t \text{ diag}\{Q_t\}^{-1} \tag{6}$$

where $\text{diag}\{Q_t\}$ replaces the off-diagonal elements of Q_t with zeros.

The DCC model is adapted to the equicorrelation framework by setting ρ_t equal to the pairwise average of off-diagonal elements of R_t^{DCC},

$$\rho_t = \frac{1}{n(n-1)} \sum_{i \neq j} \frac{q_{i,j,t}}{\sqrt{q_{i,i,t} q_{j,j,t}}} \tag{7}$$

where $q_{i,j,t}$ is the i,jth element of Q_t. As we show in subsequent sections, DECO possesses a simpler likelihood function than DCC so that the model is estimable for large cross sections while remaining a consistent (QMLE) estimator of the original DCC model.

The Q matrix (and hence the equicorrelation) will be mean reverting under the condition $\alpha + \beta < 1$. For a well behaved multivariate model, the correlation matrix should always be positive definite. The next result shows that this is the case for DECO.

Lemma 2.2 *The correlation matrices generated by every realization of a DECO process are positive definite and invertible.*

Proof: Following from Lemma 2.1 it is sufficient to show that $\rho_t \in (\frac{-1}{n-1}, 1)\ \forall t$. Q is a weighted average of positive definite matrices and therefore positive definite. This implies R_t^{DCC} is also positive definite with ones along the main diagonal. For any positive definite matrix Ω and non-zero vector x, $x'\Omega x > 0$, from which we obtain that the pairwise average of off-diagonal elements of R_t^{DCC} satisfies $\rho_t = \frac{1}{n(n-1)} \sum_{i \neq j} (R_t^{DCC})_{i,j} > \frac{-1}{n-1}$. Further, using the fact that for any positive definite matrix Ω, $\omega_{i,j}^2 < \omega_{i,i}\omega_{j,j}$, we may obtain the upper bound, $\rho_t < 1$. *Q.E.D.*

via maximum likelihood since it contains $n(n-1)/2$ parameters. An alternative specification is to assume $\bar{Q}$ is equicorrelated, which contributes only one additional parameter that can be easily estimated in the maximum likelihood step.

2.2 Estimation

To use maximum likelihood estimation we begin by assuming joint normality of returns. Let $\tilde{r}_{t|t-1} \sim N(0, H_t)$, with $H_t = D_t R_t D_t$.

The consistent two-step estimation procedure for conditional correlations suggested by Engle (2002) is applicable to the DECO model. As shown in that paper, the multivariate Gaussian log likelihood function L can be decomposed as

$$\begin{aligned} L &= -\frac{1}{2}\sum_t (n\log(2\pi) + \log|H_t| + \tilde{r}_t' H_t \tilde{r}_t) \\ &= -\frac{1}{2}\sum_t \big(n\log(2\pi) + \log|D_t|^2 + \tilde{r}_t' D_t^{-2}\tilde{r}_t - r_t' r_t + \log|R_t| + r_t' R_t^{-1} r_t\big). \end{aligned} \tag{8}$$

Denote as Θ the vector of parameters for the univariate volatility processes (i.e., the parameters contributing to the evolution of elements in the diagonal D_t matrix). Denote as Φ the parameters affecting the correlation process. The log likelihood separates additively into two terms,

$$L = L_{Vol}(\Theta) + L_{Corr}(\Theta, \Phi). \tag{9}$$

The first term, L_{Vol}, is a function of the asset-specific volatility parameters. The second term, L_{Corr}, is a function of the equicorrelation parameters that are shared by all assets, in addition to the individual volatility parameters.

When each of the returns series obeys a univariate GARCH model, L_{Vol} is in fact the sum of the individual GARCH likelihoods and is maximized by separately maximizing each term. Assuming this is the case, the first step in the two-step procedure is to estimate individual GARCH models for each $\{\tilde{r}_{i,t}\}$ series. The resulting standardized variables are input into the above likelihood where the fitted D_t matrix is treated as a constant and R_t is parameterized according to DECO. In the second step the likelihood is maximized to obtain estimates of the parameters for the ρ_t process,

$$\max_{\Phi} L(\Phi, r|\hat{\Theta})$$

where

$$\begin{aligned} L &= -\frac{1}{2}\sum_t (n\log(2\pi) + \log|R_t| + r_t{\prime}R_t^{-1}r_t) \qquad (10) \\ &= -\frac{1}{2}\sum_t \Big(n\log(2\pi) + \log\big([1-\rho_t]^{n-1}[1+(n-1)\rho_t]\big) \\ &\quad + \frac{1}{1-\rho_t}\big[\sum_i (r_{i,t}^2) - \frac{\rho_t}{1+(n-1)\rho_t}(\sum_i r_{i,t})^2\big]\Big) \end{aligned}$$

with n being the size of the cross section and ρ_t obeying Equation 7.

The payoff from making the equicorrelation assumption can now be appreciated. In DCC, the conditional correlation matrices must be recorded and inverted for all t and their determinants calculated; further, these T inversions and determinant calculations are repeated for each of the many iterations required in a numeric optimization program. This is costly for small cross sections and potentially infeasible for very large ones. With DECO, only the scalar equicorrelation parameter for each t is recorded, and the compact analytical forms for the determinant and inverse of a covariance matrix under the assumption of equicorrelation, as presented in Lemma 2.1, make the computational demands for solving the likelihood optimization problem manageable for large cross sections. When ρ_t follows Equation 7, the extent of computation is reduced to n-dimensional vector outer products with no matrix inversion or determinant computation required.

2.3 DECO, DCC and the Average Cross Sectional Correlation

Often the equicorrelation assumption fails so that there is cross sectional variation in pairwise correlations, as in DCC. In this case the DECO model remains a powerful tool. Indeed, DECO is a quasi-maximum likelihood estimator (QMLE) of DCC models, providing consistent DCC parameter estimates despite misspecification.

To formally show that DECO is a consistent estimator when DCC is the true generating process we must demonstrate that the expectation of the score of the (misspecified) DECO likelihood is equal to zero under the true model, DCC.

Let f_t^{DECO} denote the time t conditional density under DECO. The log likelihood is

$$\sum_t \log(f_t^{DECO}) = \sum_t \big(-\frac{1}{2}\log(2\pi) - \frac{1}{2}\log|R_t| - \frac{1}{2} r_t' R_t^{-1} r_t \big). \tag{11}$$

The score at each time t is a 2-vector

$$\frac{\partial \log(f_t^{DECO})}{\partial(\alpha,\beta)} = \frac{\partial \log(f_t^{DECO})}{\partial \rho_t} \frac{\partial \rho_t}{\partial(\alpha,\beta)}. \tag{12}$$

Note that the second term of the product on the LHS of Equation 12 depends only on information up to $t-1$. Thus, since each of the partials of ρ_t with respect to the model parameters is $t-1$ measurable, the conditional expectation of the time t score is simply

$$E_{t-1}\Big[\frac{\partial \log(f_t^{DECO})}{\partial \rho_t}\Big] \frac{\partial \rho_t}{\partial(\alpha,\beta)}. \tag{13}$$

The expression inside the expectation is the derivative of the time t contribution to likelihood Equation 11, which is given by

$$\begin{aligned}\frac{\partial \log(f_t^{DECO})}{\partial \rho_t} = (1-\rho_t)^{-2}(1+[n-1]\rho_t)^{-2}\Big[&(n-1)(1-\rho_t)^2(1+[n-1]\rho_t) \\ &-(n-1)(1-\rho_t)(1+[n-1]\rho_t)^2 + (1+[n-1]\rho_t)^2 \sum_i r_{i,t}^2 \\ &-(1+[n-1]\rho_t^2)(\sum_i r_{i,t})^2\Big].\end{aligned} \tag{14}$$

All the terms of this expression are known at time $t-1$ with the exception of $\sum_i r_{i,t}^2$ and $\big(\sum_i r_{i,t}\big)^2$. Under DECO, these have conditional expectations of n and $n(n-1)\rho_t + n$, respectively. Plugging these into 14 we see that the score under DECO (Equation 13) reduces to zero,

$$
\begin{aligned}
E_{t-1}^{DECO}\left[\frac{\partial \log(f_t^{DECO})}{\partial \rho_t}\right] = (1-\rho_t)^{-2}(1+[n-1]\rho_t)^{-2}\Big[&(n-1)(1-\rho_t)^2(1+[n-1]\rho_t) \\
&-(n-1)(1-\rho_t)(1+[n-1]\rho_t)^2 + n(1+[n-1]\rho_t)^2 \\
&-n(1+[n-1]\rho_t)(1+[n-1]\rho_t^2)\Big] = 0
\end{aligned} \tag{15}
$$

which ensures that the DECO score is also zero in expectation,

$$
E_{t-1}^{DECO}\left[\frac{\partial \log(f_t^{DECO})}{\partial \rho_t}\right]\frac{\partial \rho_t}{\partial(\alpha,\beta)} = 0
$$

where E^{DECO} makes explicit that the expectation is taken under DECO. For QML to be invoked, is must also be true that the expectation of the DECO score is zero when the expectation is taken under DCC. That is, QML requires

$$
E_{t-1}^{DCC}\left[\frac{\partial \log(f_t^{DECO})}{\partial \rho_t}\right]\frac{\partial \rho_t}{\partial(\alpha,\beta)} = 0. \tag{16}
$$

Equation 16 is satisfied if the conditional expectations of $\sum_i r_{i,t}^2$ and $\left(\sum_i r_{i,t}\right)^2$ under DCC are the same as the expectations under DECO. Indeed

$$
E^{DCC}[\big(\sum_i r_{i,t}\big)^2] = \sum_{i\neq j}\rho_{i,j,t} + n = n(n-1)\rho_t + n \tag{17}
$$

and

$$
E^{DCC}[\sum_i r_{i,t}^2] = n \tag{18}
$$

the same as their DECO expectations.

We have shown that the expectation of the score under both models is zero, which, following from White (1994)[2], proves the following result.

Proposition 2.1 *The maximum likelihood estimate of DECO is consistent for DCC parameters α and β when DCC is the true model.*

[2]This is based on the QMLE consistency results in Chapter 3, particularly Theorem 3.5.

2.4 Exogenous Variables and Asymmetric Terms

Prior research suggests that stock return correlations are subject to a gamut of influences beyond past correlations. Ang and Chen (2002) find that downside moves are associated with higher correlation levels than upside moves, suggesting inclusion of an asymmetric market response term in the equicorrelation evolution. The result of Ribeiro and Veronesi (2002) predicts an inverse relation between correlations and the state of the macroeconomy. Veldkamp (2006) suggests that correlation levels will be affected by cost and quality of information acquisition in a market. In a similar vein, Morck, Yeung, and Yu (2000) find that the degree of financial development of an economy affects the amount of comovement among its stocks. Further suggestions for determinants of correlation abound in the literature. These effects may be easily added to the DECO evolution in Equation 5,

$$Q_t = \bar{Q}(1 - \alpha - \beta - \gamma'\bar{z}) + \alpha r_{t-1}r'_{t-1} + \beta Q_{t-1} + \gamma' z_{t-1}\bar{Q} \tag{19}$$

where $z_i > 0$ with mean $\bar{z}_i$, and γ_i is its partial effect on Q. Since the terms incorporating z_i's are positive definite, Q continues to be a weighted average of positive definite matrices and is hence positive definite. To ensure that Q is mean reverting the model requires that $\omega + \alpha + \beta + \gamma'\bar{z} < 1$.[3]

[3] For the LDECO form (see Section 2.5), exogenous terms are added to the equicorrelation dynamics with similar ease, $\rho_t = \omega + \alpha u_{t-1} + \beta\rho_{t-1} + \gamma' z_{t-1}$. In that context, when $z_i \in (\frac{-1}{n-1}, 1)$ and $\gamma_i > 0$, R_t will be positive definite if $\omega + \alpha + \beta + \sum_i \gamma_i < 1$. For more general z variables, a transformation to a variable that lies in $(\frac{-1}{n-1}, 1)$ is useful. The logit function $f(z_i) = (1 + \frac{1}{n-1})\frac{exp(z_i)}{1+exp(z_i)} - \frac{1}{n-1}$ is an example of a mapping for this purpose. To obtain $\gamma_i > 0$, the appropriate sign change of z_i may be applied as part of the mapping.

2.5 Alternative Specification: Linear DECO

It is common when working with large cross sections for the consituent assets to change over time. Assets are frequently added and deleted from benchmark indices like the S&P 500 or the CDX. Investors drop and add assets in their portfolios in the course of rebalancing or in response to firm delistings, acquisitions and new issues. Multivariate GARCH models have difficulty accommodating changes of cross section elements; in DCC and DECO-DCC, assets must remain uninterruptedly in the cross section for the duration of the sample due to the structure of the process in Equation 5.

A linear variation of the DECO ρ_t process (abbreviated LDECO) overcomes this difficulty. The key in this approach is extracting a measurement of the equicorrelation in each time period using a statistic that is insensitive to the indexing of assets in the return vector. Consider the expression

$$u_t = \frac{[\big(\sum_i r_{i,t}\big)^2 - \sum_i (r_{i,t}^2)]/n(n-1)}{\sum_i (r_{i,t}^2)/n} = \frac{\sum_{i \neq j} r_{i,t} r_{j,t}}{(n-1)\sum_i (r_{i,t}^2)}. \tag{20}$$

This statistic estimates equicorrelation at time t without the need to maintain assets' indices in different time periods, nor does it require that the number of assets in the cross section remains fixed. The LDECO updating equation based on u_t is

$$\rho_{t+1} = \omega + \alpha u_t + \beta \rho_t. \tag{21}$$

The update expression takes an intuitive form. The numerator is an estimate of the covariance of returns (which is feasibly estimated from a single cross section thanks to the equicorrelation assumption), while the denominator is an estimate of the variance for all assets. Since returns have been volatility-standardized, they should have unit variance and therefore the numerator should be a correlation estimate. The numerator is not restricted to the range that ensures positive definiteness of R_t, however, and it lacks robustness to deviations from unity for the conditional variance estimate. u_t standardizes this covariance estimate by an estimate of the common variance. This lends some robustness to deviations from unit conditional variance[4] and, as shown in Lemma 2.3, ensures that u_t lies within the positive definite range.

Lemma 2.3 *The LDECO update u_t lies in the interval $(\frac{-1}{n-1}, 1)$ almost surely.*

Proof: By Holder's Inequality, $\left(\sum_i(a_i b_i)\right)^2 \leq \sum_i a_i^2 \sum_i b_i^2$. Applying this to the term $\sum_{i\neq j} r_{i,t}r_{j,t}$, we obtain $(\sum_{i\neq j} r_{i,t}r_{j,t})^2 \leq (n-1)^2(\sum_i r_{i,t}^2)^2$. This ensures that $u_t \in [-1, 1]$. Next, $\left(\sum_i r_{i,t}\right)^2 = \sum_{i\neq j} r_{i,t}r_{j,t} + \sum_i r_{i,t}^2 \geq 0$, which implies $\frac{\sum_{i\neq j} r_{i,t}r_{j,t}}{\sum_i(r_{i,t}^2)} \geq -1$, further implying $\frac{\sum_{i\neq j} r_{i,t}r_{j,t}}{(n-1)\sum_i(r_{i,t}^2)} \geq \frac{-1}{n-1}$. Since the cross sectional elements have continuous densities, this completes the proof. *Q.E.D.*

Other update forms may be used in place of expression 20. One such form is the time t cross sectional variance of the random variables. If all variables have unit conditional variance, their cross sectional dispersion at any time will be informative about their correlation that period. This is easy to see in the case of a factor model such as Equation 1. The realization of the factor determines the average return, but the dispersion around the mean is determined solely by correlations, that is, the relative magnitudes of the factor loadings and the error variances, both of which are assumed equal for all i.

The update expression based on the cross sectional variance is

$$u_t^{var} = 1 - \frac{1}{n-1}\sum_i (r_{i,t} - \bar{r}_t)^2.$$

The attractions of u_t^{var} are that it is an unbiased estimate of ρ_t and that it has a smaller variance than u_t in Equation 20. The drawbacks of u_t^{var} are three-fold. First, while bounded

[4] The problem of predictable deviations from unit variance is addressed in more detail in Section 2.6.

above by one, this form may fall below the lower bound of $\frac{-1}{n-1}$ and thus requires truncation to ensure positive definiteness of R_t. Truncation introduces bias into the update, undermining one of its benefits. Second, it is sensitive to non-unit conditional variances. As variances of returns deviate from one, the accuracy of u_t^{var} deteriorates as it cannot distinguish non-unit variance effects from equicorrelation effects. This second point can be a serious shortcoming in light of dynamics that remain in variances of large cross sections even after "de-GARCHing", a consideration discussed in more detail in the following section. Lastly, the relative efficiency of u_t^{var} may be sensitive to non-normality. In the presence of excess kurtosis, it is no longer necessarily true that u_t^{var} has a smaller variance than u_t. Simulations from an equicorrelated Student t distribution for 100 assets and four degrees of freedom show that at low levels of correlation (at or below approximately 0.4) u_t is in fact the more efficient form.[5] This is potentially problematic for financial returns, which tend to exhibit fat tails, especially in light of the fact that the unconditional equicorrelation of the datasets in Section 5 are between 0.2 and 0.3.

In consideration of these points we focus throughout the paper on the more robust

[5]Simulations are unreported and available from the authors upon request.

LDECO update in Equation 20.[6]

While the DECO process in 7 (which we may call DECO-DCC) yields a large simplification in computational complexity, the linear alternative in Equation 21 simplifies computation even further. DECO-DCC requires n-vector outer product calculations in the matrix Equations 5 and 6.[7] While DECO-DCC model speeds up dynamic correlation estimation making it feasible for very large cross sections, the computing time continues to increase with n. In contrast, the LDECO evolution and resulting likelihood rely only on two easily computed scalar series: the sum of squared standardized returns, $\sum_i r_{i,t}^2$, and their squared sum, $\left(\sum_i r_{i,t}\right)^2$. u_t can be computed from these two series, thus these are the only required data inputs for the entire likelihood (Equation 10).

There are three caveats to the simplification afforded by LDECO. First, the model is no longer a QML estimator for the DCC model. Second, a Monte Carlo experiment in Section

[6]Yet another update that may be employed is derived from the maximum likelihood estimate of ρ_t based on time t returns only. The log likelihood in any period t is

$$\log\left(f_r(r_t, \rho_t)\right) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\log|R_t| - \frac{1}{2}r_t' R_t^{-1} r_t.$$

Substituting expressions for the inverse and determinant of R_t based on Equations 3 and 4 and maximizing with respect to ρ_t yields the first order condition for a maximum

$$\begin{aligned}(n-1)(1-\rho_t)^2(1+[n-1]\rho_t) - (n-1)(1-\rho_t)(1+[n-1]\rho_t)^2 \\ + \sum_i r_{i,t}^2(1+[n-1]\rho_t)^2 - \left(\sum_i r_{i,t}\right)^2(1+[n-1]\rho_t^2) = 0.\end{aligned}$$

The solution to this cubic function in ρ_t can serve as an update to the ρ process. Simulations (not reported) show that this estimator behaves similarly to u_t^{var} in terms of unbiasedness, variance, and lack of robustness to excess kurtosis and deviations from unit conditional variance.

[7]Since the Q matrix is being pre- and post-multiplied by diagonal matrices, the matrix multiplication may be written as a Hadamard product of an n-vector outer product and an n-matrix.

4 demonstrates that LDECO may be a less precise correlation estimator than DECO-DCC. Lastly, the model's simple regularity conditions on parameter values no longer hold exactly. If u_t were unbiased, stationarity and positive definiteness would be implied by the condition $\omega/(1-\alpha-\beta) \in (-\frac{1}{n-1}, 1)$. However, u_t is a ratio of correlated random variables and hence downward biased. One implication of this bias is that the model can be stationary for $\alpha+\beta$ slightly in excess of 1. In practice, numeric optimization procedures can be altered to ensure that the fitted equicorrelation process obeys the bounds $(-\frac{1}{n-1}, 1)$ without imposing explicit constraints on the sum of the parameters.

2.6 Alternative Specification: Residual Dynamic Equivariance

Estimating a conditional variance model such as GARCH for individual asset return series and then standardizing the returns by the fitted conditional volatilities yields a collection of time series of random variables, each with approximately constant conditional variance. If univariate variance standardizations fully capture the variance dynamics of the cross section, then the resulting conditional covariance matrices would in fact be correlation matrices. This implies that the sum of squared standardized returns should be serially uncorrelated. While univariate GARCH standardization generally removes the statistical detectability of serial correlation in individual squared returns, an empirical regularity in financial data is that the sum of squares for large groups of assets tends to demonstrate serial correlation even after asset-by-asset standardization. The severity of this autocorrelation appears to increase with the number of assets. Evidently, the conditional covariance matrices for large cross sections of standardized returns are not exactly correlation matrices as over time they demonstrate small predictable deviations from unity on the main diagonal. Including an equivariance paramater in the covariance matrix of raw returns allows us to take this fact into account and improve correlation estimation. We can define an equicovariance matrix, S_t, which is simply the DECO matrix multiplied by a conditional equivariance component, $S_t = \sigma_t^2 R_t$.

While we are particularly interested in the equicorrelation component, the equicovariance structure adds an important flexibility to correlation models. Without this consideration, dynamics in the sum of squares may be misattributed to the correlation process, thus affecting the quality of correlation estimates. We propose an extension of the model that includes a dynamic equivariance component in addition to the base univariate volatility dynamics.

The time-varying equivariance parameter is assumed to obey

$$\sigma_{t+1}^2 = \gamma + \eta v_t + \phi \sigma_t^2. \tag{22}$$

with update term $v_t = \sum_i r_{i,t}^2/n$. This can be combined with any equicorrelation process ρ_t. Under this specification, the covariance matrix of returns takes the form $H_t = D_t S_t D_t = D_t(\sigma_t^2 R_t)D_t$, so that the likelihood becomes

$$L = -\frac{1}{2}\sum_t \left(n\log(2\pi) + \log|D_t|^2 + \tilde{r}_t' D_t^{-2}\tilde{r}_t - r_t' r_t + \log|R_t| + n\log\sigma_t^2 + \frac{1}{\sigma_t^2} r_t' R_t^{-1} r_t\right).$$

The likelihood decomposition of Equation 9 remains valid with the parameters of the equivariance process included in Φ and consistently estimated in the second step maximization.

The appropriateness of dynamic equivariance is an empirical question. Information criteria may be used to choose between DECO specifications with and without it; similarly, specification tests such as likelihood ratio and Lagrange multiplier tests may be the basis for the choice.

When equivariance is included in LDECO, the model's updates possess a favorable optimality property. We can show that u_t from Equation 21 and v_t are the values for ρ_t and

σ_t^2 that maximize the log likelihood of the time t cross section, as shown in the following Lemma.

Lemma 2.4 *The solutions to the optimization problem*

$$\max_{\rho_t, \sigma_t^2} \log\left(f_r(r_t, \rho_t, \sigma_t^2)\right) = \max_{\rho_t, \sigma_t^2} \quad -\frac{n}{2}\log(2\pi) - \frac{1}{2}\log|S_t| - \frac{1}{2}r_t' S_t^{-1} r_t$$

are given by

$$\sigma_t^{2*} = \frac{1}{n}\sum_i r_{i,t}^2 = v_t, \qquad \rho_t^* = \frac{(\sum_i r_{i,t})^2 - \sum_i (r_{i,t}^2)}{(n-1)\sum_i (r_{i,t}^2)} = u_t.$$

Proof: We first substitute expressions for the inverse and determinant of S_t based on Equations 3 and 4 and suppress t subscripts. The first order conditions for an optimum are

$$n - \frac{\sum_i r_i^2}{\sigma^2(1-\rho)} + \frac{\rho(\sum_i r_i)^2}{\sigma^2(1-\rho)(1+[n-1]\rho)} = 0 \tag{23}$$

and

$$\frac{1-n}{(1-\rho)} + \frac{n-1}{1+[n-1]\rho} + \frac{\sum_i r_i^2}{\sigma^2(1-\rho)^2} - \frac{(1+\frac{\rho}{1-\rho} - \frac{\rho(n-1)}{1+[n-1]\rho})(\sum_i r_i)^2}{(1-\rho)(1+[n-1]\rho)} = 0. \tag{24}$$

Equation 23 may be rewritten as

$$\sigma^2 = \frac{(1+[n-1]\rho)\sum_i r_i^2 - \rho(\sum_i r_i)^2}{n(1-\rho)(1+[n-1]\rho)}. \tag{25}$$

Subsituting this into 24 and solving for σ^2 we obtain ρ^*, which can then be used in 25 to obtain σ^{2*}. The second order conditions for an optimum are satisfied by the concavity of the likelihood in both arguments. *Q.E.D.*

Lemma 2.4 say that the updates to LDECO with equivariance are the period-by-period ML estimates. It is in this sense that u_t and v_t optimally extract correlation information from each return realization.

2.7 Forecasting

Often the correlation between two returns must be forecast several periods ahead. In many GARCH models, the variance evolution is linear and the expected value of the evolution update is equal to the variance itself. When this is the case, analytical expressions for multi-period ahead forecasts can be obtained by recursively solving forward the evolution equation. The nonlinearity of DCC precludes the convenient recursive solution method used for GARCH. Engle and Sheppard (2001) suggest analytical forecast approximation techniques for DCC and demonstrate their accuracy in simulations. Likewise, DECO does not give exact analytical correlation forecasts, though the approximation methods of Engle and Sheppard (2001) may be applied in this context as well. The first analytical forecast method begins with the approximation $E_t[r_{t+1}r'_{t+1}] \approx Q_t$, which leads to the K-step ahead forecast for the Q matrix

$$E_t[Q_{t+K}] = \sum_{k=0}^{K-2} \bar{Q}(1-\alpha-\beta)(\alpha+\beta)^k + (\alpha+\beta)^{K-1} Q_{t+1}.$$

This is used to obtain the K-step ahead ρ forecast

$$E_t[\rho_{t+K}] = \frac{1}{n(n-1)} \sum_{i \neq j} \frac{E_t[q_{i,j,t+K}]}{\sqrt{E_t[q_{i,i,t+K}]E_t[q_{j,j,t+K}]}}.$$

The second analytical forecast derives from the approximation $E_t[R_{t+1}] \approx E_t[Q_{t+1}]$, which is then used to calculate

$$E_t[R_{t+K}] = \sum_{k=0}^{K-2} \bar{Q}(1-\alpha-\beta)(\alpha+\beta)^k + (\alpha+\beta)^{K-1} R_{t+1}$$

which gives K-step ahead ρ forecast

$$E_t[\rho_{t+K}] = \sum_{k=0}^{K-2} \bar{q}(1-\alpha-\beta)(\alpha+\beta)^k + (\alpha+\beta)^{K-1} \rho_{t+1}$$

where $\bar{q}$ is the average off-diagonal element of the correlation target matrix, $\bar{Q}$.

An alternative to analytical approximation is simulation-based forecasting. In this approach, the estimated model through date t is used as the data generating process to repeatedly simulate the model an additional $K-1$ periods forward, using r_t and the fitted value of ρ_t to initiate the process. For each iteration z, data is generated according to

$$Q^z_{t+k} = \bar{Q}(1-\hat{\alpha}-\hat{\beta}) + \hat{\alpha} r^z_{t+k-1} r^{z\prime}_{t+k-1} + \hat{\beta} Q^z_{t+k-1},\ k = 2, ..., K,$$

where parameters with hats are estimated using data through time t. The simulated K-period ahead equicorrelation is

$$\rho^z_{t+K} = \frac{1}{n(n-1)} \sum_{i \neq j} \frac{q^z_{i,j,t+K}}{\sqrt{q^z_{i,i,t+K} q^z_{j,j,t+K}}}$$

and the simulation-based K-period forecast is the average of the equicorrelation over all simulations,

$$E^{Sim}_t[\rho_{t+K}] = \frac{1}{Z} \sum_{z=1}^{Z} \rho^z_{t+K}.$$

2.8 The Block Dynamic Equicorrelation Model

For some applications it is useful to consider more flexible correlation structures yet retain some of the tractability and robustness of the equicorrelation model. For instance, one may model the correlation of common stock returns with particular interest in intra- and inter-industry correlation dynamics by imposing the restriction of equicorrelation within and between industries. Each industry has its own single dynamic equicorrelation parameter and each industry pair has a dynamic cross-equicorrelation parameter. With block equicorrelations, richer cross-sectional variation is accomodated while still greatly reducing the dimensionality of the estimation problem. In this section we develop an extension to block DECO models and examine their properties. We now define the class of block dynamic equicorrelation (BDECO) models.

Definition 2.3 *R_t is a* k-block equicorrelation matrix *if it is invertible and positive definite and takes the form*

$$R_t = \begin{pmatrix} (1-\rho_{1,1,t})I_{n_1} & 0 & \cdots \\ 0 & \ddots & 0 \\ \vdots & 0 & (1-\rho_{k,k,t})I_{n_k} \end{pmatrix} + \begin{pmatrix} \rho_{1,1,t}J_{n_1\times n_1} & \rho_{1,2,t}J_{n_1\times n_2} & \cdots \\ \rho_{2,1,t}J_{n_2\times n_1} & \ddots & \\ \vdots & & \rho_{k,k,t}J_{n_k\times n_k} \end{pmatrix} \tag{26}$$

where $\rho_{l,m,t} = \rho_{m,l,t}\ \ \forall l,m$.

As in the DECO model, the BDECO model for a cross section of random variables specifies that, conditional on the past, each variable is identically Gaussian distributed with mean zero, variance one, and correlations taking the structure in Equation 26. The return vector r_t is partitioned into k sub-vectors; each sub-vector r_l contains $n_l \times 1$ elements. Cross-correlations between elements of sub-vectors $r_{l,t}$ and $r_{m,t}$ are also assumed to be equicorrelated. Thus, a separate DECO process is specified for each of the k diagonal blocks and each of the $k(k-1)/2$ unique off-diagonal blocks. For DECO-DCC, blocks on the main diagonal have equicorrelations that follow

$$\rho_{l,l,t} = \frac{1}{n_l(n_l-1)} \sum_{i\in l, j\in l, i\neq j} \frac{q_{i,j,t}}{\sqrt{q_{i,i,t}q_{j,j,t}}}, \tag{27}$$

and for blocks off the main diagonal

$$\rho_{l,m,t} = \frac{1}{n_l n_m} \sum_{i\in l, j\in m} \frac{q_{i,j,t}}{\sqrt{q_{i,i,t}q_{j,j,t}}}. \tag{28}$$

The configuration of coefficient matrices in the Q process is a specification choice. The simplest form assumes that parameters are the same for all subgroups, which implies scalar coefficients and thus the Q process is identical to that in Equation 5. This allows for block specific equicorrelations while imposing the restriction that the equicorrelation response to past correlations and return cross products is the same for all blocks.

A richer parameterization allows each block to have its own set of parameters. The elements of the Q matrix in block l, m follow

$$q_{i,j,t} = \bar{q}_{i,j}(1 - \alpha_{l,m} - \beta_{l,m}) + \alpha_{l,m} r_{t-1} r'_{t-1} + \beta_{l,m} q_{i,j,t-1} \tag{29}$$

with $(\alpha_{l,m}, \beta_{l,m}) = (\alpha_{m,l}, \beta_{m,l})$. This can be written in matrix form as

$$Q_t = \bar{Q}(I - A - B) + A \odot r_{t-1} r'_{t-1} + B \odot Q_{t-1}$$

where A and B are matrices containing the α's and β's, respectively, and $\odot$ denotes Hadamard product. For the two block case these matrices are

$$A = \begin{bmatrix} \alpha_{1,1} J_{n_1 \times n_1} & \alpha_{1,2} J_{n_1 \times n_2} \\ \alpha_{1,2} J_{n_2 \times n_1} & \alpha_{2,2} J_{n_2 \times n_2} \end{bmatrix}, \; B = \begin{bmatrix} \beta_{1,1} J_{n_1 \times n_1} & \beta_{1,2} J_{n_1 \times n_2} \\ \beta_{1,2} J_{n_2 \times n_1} & \beta_{2,2} J_{n_2 \times n_2} \end{bmatrix}.$$

For the remainder of the section we focus on the two block case to simplify exposition.

Block equicorrelations will be required to satisfy conditions ensuring the invertibility and positive definiteness of the correlation matrix for each t. The following lemma establishes these conditions and demonstrates that the same analytic tractability provided by the equicorrelation assumption extends to the block structure. We suppress t subscripts in the following lemma since all terms are contemporaneous.

Lemma 2.5 *If R is a two block equicorrelation matrix, that is, if*

$$R = \begin{bmatrix} (1-\rho_{1,1}) I_{n_1} & 0 \\ 0 & (1-\rho_{2,2}) I_{n_2} \end{bmatrix} + \begin{bmatrix} \rho_{1,1} J_{n_1 \times n_1} & \rho_{1,2} J_{n_1 \times n_2} \\ \rho_{1,2} J_{n_2 \times n_1} & \rho_{2,2} J_{n_2 \times n_2} \end{bmatrix}$$

then,

i. the inverse is given by

$$R^{-1} = \begin{bmatrix} b_1 I_{n_1} & 0 \\ 0 & b_2 I_{n_2} \end{bmatrix} + \begin{bmatrix} c_1 J_{n_1 \times n_1} & c_3 J_{n_1 \times n_2} \\ c_3 J_{n_2 \times n_1} & c_2 J_{n_2 \times n_2} \end{bmatrix} \tag{30}$$

where

$$b_i = \frac{1}{1-\rho_{i,i}},\ i = 1,2$$

and

$$c_1 = \frac{\rho_{1,1}\big(\rho_{2,2}(n_2-1)+1\big) - \rho_{1,2}^2 n_2}{(\rho_{1,1}-1)\big([\rho_{1,1}(n_1-1)+1][\rho_{2,2}(n_2-1)+1] - n_1 n_2 \rho_{12}^2\big)}$$

$$c_2 = \frac{\rho_{2,2}\big(\rho_{1,1}(n_1-1)+1\big) - \rho_{1,2}^2 n_1}{(\rho_{2,2}-1)\big([\rho_{1,1}(n_1-1)+1][\rho_{2,2}(n_2-1)+1] - n_1 n_2 \rho_{1,2}^2\big)}$$

$$c_3 = \frac{\rho_{1,2}}{n_1 n_2 \rho_{12}^2 - \big(\rho_{1,1}(n_1-1)+1\big)\big(\rho_{2,2}(n_2-1)+1\big)}$$

ii. the determinant is given by

$$\det(R) = (1-\rho_{1,1})^{n_1-1}(1-\rho_{2,2})^{n_2-1}\big[(1+[n_1-1]\rho_{1,1})(1+[n_2-1]\rho_{2,2}) - \rho_{1,2}^2 n_1 n_2\big]$$

iii. R is invertible if and only if

$$\rho_{i,i} \neq \frac{-1}{n_i-1} \text{ and } \rho_{i,i} \neq 1,\ \ i = 1,2$$

and

$$\rho_{1,2} \neq \sqrt{\frac{\big(\rho_{1,1}(n_1-1)+1\big)\big(\rho_{2,2}(n_2-1)+1\big)}{n_1 n_2}}$$

iv. R is positive definite if and only if

$$\rho_i \in \big(\frac{-1}{n_i-1}, 1\big),\ i = 1,2$$

and

$$\rho_{12} \in \left(-\sqrt{\frac{\big(\rho_{1,1}(n_1-1)+1\big)\big(\rho_{2,2}(n_2-1)+1\big)}{n_1 n_2}}, \sqrt{\frac{\big(\rho_{1,1}(n_1-1)+1\big)\big(\rho_{2,2}(n_2-1)+1\big)}{n_1 n_2}}\right).$$

Proof: Part (i) can be shown by multiplying R and R^{-1}. The restriction that the product equals the identity matrix gives a system of five equations in five unknowns. The result follows as the solution to this system. Part (ii) follows from Graybill (1983) Theorem 8.2.1. Part (iii) follows from the form given in part (i), and part (iv) is equivalent to the statement that all eigenvalues of R are positive, which is equivalent to R being positive definite. *Q.E.D.*

With this result in hand, the likelihood function of a two block dynamic equicorrelation model can, as in the simple equicorrelation case, be written to avoid costly inverse and determinant calculations.

$$
\begin{aligned}
L &= -\frac{1}{2}\sum_t \left(n\log(2\pi) + \log|R_t| + r_t' R_t^{-1} r_t\right) \qquad (31)\\
&= -\frac{1}{2}\sum_t \Bigg[n\log(2\pi) + \log\Bigg((1-\rho_{1,1,t})^{n_1-1}(1-\rho_{2,2,t})^{n_2-1}\\
&\quad \times\left[(1+[n_1-1]\rho_{1,1,t})(1+[n_2-1]\rho_{2,2,t}) - \rho_{1,2,t}^2 n_1 n_2\right]\Bigg)\\
&\quad + r_t'\left(\begin{bmatrix} b_1 I_{n_1} & 0 \\ 0 & b_2 I_{n_2}\end{bmatrix} + \begin{bmatrix} c_1 J_{n_1\times n_1} & c_3 J_{n_1\times n_2} \\ c_3 J_{n_2\times n_1} & c_2 J_{n_2\times n_2}\end{bmatrix}\right) r_t\Bigg]\\
&= -\frac{1}{2}\sum_t \Bigg[n\log(2\pi) + \log\Bigg((1-\rho_{1,1,t})^{n_1-1}(1-\rho_{2,2,t})^{n_2-1}\\
&\quad \times\left[(1+[n_1-1]\rho_{1,1,t})(1+[n_2-1]\rho_{2,2,t}) - \rho_{1,2,t}^2 n_1 n_2\right]\Bigg)\\
&\quad + \left(b_1\sum_i^{n_1} r_{i,1}^2 + b_2\sum_i^{n_2} r_{i,2}^2 + c_{1,t}(\sum_i^{n_1} r_{i,1})^2 + 2c_{3,t}(\sum_i^{n_1} r_{i,1})(\sum_i^{n_2} r_{i,2}) + c_{2,t}(\sum_i^{n_2} r_{i,2})^2\right)\Bigg]
\end{aligned}
$$

The model is estimated by maximum likelihood using the parameterization of Equations 27 and 28 in likelihood Equation 31.

These results may be generalized to k-block equicorrelation matrices.[8] For higher numbers of blocks the invertibility and positive definiteness restrictions become increasingly

[8]This involves the following approach. Hypothesize the form of the inverse (this is simply the k-block analogy of Equation 30), then use the restriction that the product of the block equicorrelation matrix and

complex. One simplification that avoids cumbersome restrictions for the off-diagonal block equicorrelations is to force them to zero, yet allow dynamic block equicorrelations for all blocks on the main diagonal. For the equicorrelation matrix to be invertible in this environment the restrictions are trivial. If ρ_b is the equicorrelation for a given block on the main diagonal of R, invertibility and positive definiteness are equivalent to the condition $\rho_b \in (\frac{-1}{n_b-1}, 1)\ \forall b$.[9][10]

its inverse equals the identity matrix to identify a system of $k(k+1)/2$ equations in that many unknowns. The restrictions on invertibility are determined by the conditions dictating the existence of a solution to this system. The inequality restrictions for positive definiteness can be derived from the expression of the determinant of a partitioned matrix as done in the proof of Lemma 2.5.

[9]BDECO can accomodate LDECO as well. When using the LDECO form, equicorrelations are assumed to follow

$$\rho_{l,m,t+1} = \omega_{l,m} + \alpha_{l,m} u_{l,m,t} + \beta_{l,m}\rho_{l,m,t}$$

where $u_{i,j,t}$ represents a measurement of the equicorrelation update for the l, m block. For the blocks on the main diagonal, the update is the usual LDECO update u_t using the appropriate sub-vector $r_{l,t}$. That is, for the lth diagonal block

$$u_{l,l,t} \equiv \frac{(\sum_i r_{m,i,t})^2 - \sum_i (r^2_{m,i,t})}{(n-1)\sum_i (r^2_{m,i,t})}.$$

The off-diagonal block updates are

$$u_{l,m,t} \equiv \frac{(\sum_i r_{l,i,t})(\sum_j r_{m,j,t})}{\sqrt{n_l n_m \sum_i (r^2_{l,i,t}) \sum_j (r^2_{m,j,t})}}$$

for $l = 1, ..., k$, $m < l$, $i = 1, ..., n_l$, and $j = 1, ..., n_m$.

[10]Equivariance may also be included. For a k block model, including a separate equivariance for each block adds $3k(k-1)/2$ parameters to the model. A parsimonious alternative includes a single equivariance, which requires simply replacing R_t in Equation 31 with $S_t = \sigma^2_t R_t$.

3 Diagnostic Tests

As in all multivariate volatility models, we may write $Var_{t-1}(\tilde{r}_t) = D_t R_t D_t$. Therefore $Var_{t-1}(D_t^{-1}\tilde{r}_t) = Var_{t-1}(r_t) = R_t$, so that $Var_{t-1}(R_t^{-\frac{1}{2}} r_t) = I$. In DECO, the matrix $R_t^{-\frac{1}{2}}$ can be expressed as

$$R_t^{-\frac{1}{2}} = \left[(1-\rho_t)I + \rho_t J\right]^{-\frac{1}{2}} = a_t I + b_t J$$

where a_t and b_t depend only on the correlation ρ_t and the size of the cross section,

$$a_t = \frac{1}{\sqrt{1-\rho_t}}, \qquad b_t = \frac{-1 \pm \sqrt{1 - \frac{1\rho_t n}{(1+[n-1]\rho_t)}}}{n\sqrt{1-\rho_t}}.$$

Consequently, we can construct residuals which should be i.i.d. with an identity covariance matrix if the model is correctly specified. These are given by

$$\hat{\eta}_t = \hat{a}_t r_t + n\hat{b}_t \iota \bar{r}_t.$$

where ι is a vector of ones. The $\hat{\eta}$'s are weighted averages of the first stage volatility-adjusted residuals and an equally weighted factor. These residuals might be called correlation-adjusted residuals to distinguish them from the volatility-adjusted residuals from the first stage. Various diagnostic tests can be constructed with these adjusted residuals to assess the degree to which covariance dynamics have been removed from the data.

3.1 Conditional Moment Tests

A conditional covariance model ideally achieves adjusted returns that have constant conditional variance of unity and constant conditional correlation of zero.[11] To formally examine the quality of a conditional covariance model we adopt the conditional moment test (CMT)

[11] One may also wish to test unconditional variances and correlations. Conditional tests tend to have have power against a wider set of alternatives and are therefore our focus here (Engle and Ding, 1996).

framework developed by Newey (1985) to construct asymptotic specification tests in the context of MLE. This approach tests relevant moment conditions while taking into account the fact that the ML parameters are estimated, and hence random variables rather than constants. It is possible to test any moment restrictions implied by the null model either individually or jointly. The test statistic is given by

$$d_T = \iota' M_T \left[M_T' \big(I - \mathcal{S}_T (\mathcal{S}_T' \mathcal{S}_T)^{-1} \mathcal{S}_T' \big) M_T \right]^{-1} M_T' \iota$$

which is distributed χ_p^2 under the null where p is the number of moment conditions being considered. Calculation of d_T requires the construction of two matrices. The first is a vector of elements that additively compose the relevant sample moments. These are time t elements of the function

$$M_T(r, \hat{\theta}_T) = \big(m_1(r, \hat{\theta}_T), \ldots, m_T(r, \hat{\theta}_T) \big)'$$

which depends on the return data matrix r and the fitted DECO parameter estimates. Each element $m_t(r, \hat{\theta}_T)$ is the period t analog of the p-dimensional moment condition being tested ($\iota' M_T / T$ is the corresponding sample moment). Also needed for d_T is the score matrix (denoted with script $\mathcal{S}$ to distinguish from the equicovariance matrix S_t) evaluated at the ML estimates,

$$\mathcal{S}_T = \big(s(r_1, \hat{\theta}_T), \ldots, s(r_T, \hat{\theta}_T) \big)'$$

where $s(r_t, \theta) = \partial \log f(r_t | \theta) / \partial \theta$. I and ι are a comformable identity matrix and vector of ones, respectively.

A test of primary interest for the DECO model is that conditional equicorrelations are zero,

$$H_0 : E_t(\rho_{t+1}) = 0, \ \forall t. \tag{32}$$

This hypothesis is tested by calculating the sample covariance between an estimate of the correlation (among $\hat{\eta}$'s) and conditioning variables. One estimate of equicorrelation in adjusted returns is $c_t = \frac{1}{n(n-1)}\sum_{i\neq j}\hat{\eta}_{i,t}\hat{\eta}_{j,t}$. To make the test conditional we interact c_t with variables in the $t-1$ information set. Interesting conditioning variables to consider are lags of c, which makes the test similar to a Ljung-Box test with the exception of an adjustment to account for parameter uncertainty, which the Ljung-Box test ignores. Under the null in 32, $E[c_t c_{t-1}] = 0$, so $m_t(r, \hat{\theta}_T) = c_t c_{t-1}$. The asymptotic test distribution in this example is χ^2_1 since only one moment is used.

3.2 Lagrange Multiplier Tests

A different testing method is to specify an alternative and derive an optimal test against it. Lagrange Multiplier (LM) tests are ideal for this approach.

The general setup assumes a model with parameter vector θ containing a subvector of parameters γ. The LM statistic is

$$LM = T^{-1}\big[\big(\sum_t s(\hat{\eta}_t, \tilde{\theta}_T)\big)\hat{\Im}(\hat{\eta}_t, \tilde{\theta}_T)^{-1}\big(\sum_t s(\hat{\eta}_t, \tilde{\theta}_T)'\big)\big] \tag{33}$$

where $\tilde{\theta}_T$ is the vector of ML estimates imposing the restriction $\gamma = 0$, and $\hat{\Im}$ is an estimate of the information matrix. LM is asymptotically distributed as χ^2_p, where p is the dimension of the restricted parameter vector, γ.

Many interesting alternatives that present dimensionality problems in other testing frameworks (such as conditional moment tests) are easily implemented with an LM test. For instance, a CMT of pairwise conditional correlations is a test of $n(n-1)/2$ moments, and calculation of the test statistic requires inverting a matrix of this dimension, which may not be feasible when the number of assets is large. By sacrificing some of the generality of that approach, a parsimonious LM test can be formulated to test if pairwise correlations differ. As an example, let the alternative correlation model be such that each off-diagonal element of R_t follows DECO plus an additional term depending on the history of that pair of assets. That is,

$$\rho_{i,j,t} = \rho_t + \gamma u_{i,j,t-1} \tag{34}$$

where

$$u_{i,j,t-1} = r_{i,t-1} r_{j,t-1}.$$

Suppose data has been generated according to this model using the basic DECO specification, that is, omitting the pair-specific effects. Using the LM approach as a diagnostic test requires re-estimating the base DECO model using the DECO-adjusted $\hat{\eta}$'s and calculating the test statistic imposing the restriction $\gamma = 0$.

4 Correlation Monte Carlos

In this section we present results from a series of Monte Carlo experiments that allow us to evaluate the performance of the DECO framework when the true data generating process is known.

4.1 Equicorrelated Processes

We begin by exploring the model's estimation ability when the generating process is DECO-DCC or LDECO. For each model, asset return data for cross section sizes n=10, 30 and 100 are simulated over 1,250 periods (corresponding to roughly five years of trading), then the correctly specified model is estimated. The simulations are repeated 1000 times and summary statistics for the maximum likelihood parameter estimates are calculated; results are reported in Table 1. We also report the root mean squared error of the fitted DECO process and the average pairwise rolling correlation estimator (using rolling window length of 30 days) versus the true DECO process. The results show that point estimates are quite accurate with the exception of a slight downward bias in β estimates, a common effect seen in GARCH models. For DECO-DCC, asymptotic standard error approximations closely match the standard deviation of coefficient estimates. This is also true of LDECO, though for smaller n the asymptotic standard error approximation for β underestimates its variability in simulations. For both DECO-DCC and LDECO the precision of estimates improves as the number of assets increases.

Our next set of analyses evaluates the performance of the DECO model when the true correlation structure is known, but the generating process is a Gaussian equicorrelation process other than DECO. We assume the true equicorrelation parameter evolves according to one of four deterministic functions shown in Figure 1:

1. Low frequency sine wave ("Slow Sine")
2. High frequency sine wave ("Fast Sine")
3. Gradually rising then falling sharply ("Climb and Drop")
4. Step function ("Step")

For each of the four generating processes, return data for a cross section of 30 assets is simulated over 1,250 periods, repeated 1000 times. Fitted equicorrelations are then calculated using the DECO-DCC, LDECO and average pairwise rolling correlation estimators. Table 2 reports root mean squared errors for fitted equicorrelation processes versus the true equicorrelation functions.

DECO-DCC is consistently more precise than LDECO and rolling correlations. The benefit of LDECO over the simple rolling correlation estimator is only evident when correlations change rapidly, as in the "Fast Sine" and "Climb and Drop" functions.

4.2 Non-Equicorrelated Processes

Proposition 2.1 highlights the ability of DECO to consistently estimate the parameters of a DCC-style process despite the violation of equicorrelation. To demonstrate the performance of the DECO model in this light we simulate time series that follow the DCC model. In DCC the correlation of any pair i, j follows

$$\rho_{i,j,t} = \frac{q_{i,j,t}}{\sqrt{q_{i,i,t} q_{j,j,t}}} \tag{35}$$

where

$$q_{i,j,t} = \omega_{i,j} + \alpha r_{i,t-1} r_{j,t-1} + \beta \rho_{i,j,t-1}. \tag{36}$$

Thus, while equicorrelation is violated, the average pairwise correlation behaves according to DECO.

Using DCC as the data generating process[12] we simulate asset return data over 1,250 periods for cross sections of size n =10, 30 and 100. This data is then used to estimate both DECO and DCC. Table 3 reports summary statistics of coefficient estimates and the RMSE of each model's fitted average correlation in relation to the true average.

DECO is able to accurately estimate DCC parameters, though estimates of β appear slightly downward biased, and for cross sections of 10 and 30 assets QMLE standard errors

[12] The simulations use correlation targeting, that is, $\omega_{i,j} = \bar{q}_{i,j}(1 - \alpha - \beta)$, where $\bar{q}_{i,j}$ is the unconditional correlation for the pair. The target matrix is non-equicorrelated; its standard deviation of off-diagonal elements equal to 0.33, demonstrating that the differences in pairwise correlations for the simulated cross sections are substantial.

substantially underestimate the true variability of point estimates for β. This is not true for α standard errors, and the shortfall for β disappears when the cross section is large.

4.3 Forecasting

To evaluate the three forecast methods outlined in Section 2.7 we simulate returns for n=10 and 100 assets over 500 (and 1250) periods under DECO-DCC, and use the first 250 (and 1000) days of data to estimate the model. The fitted parameters are used to forecast ρ for the remaining 250 periods by the simulation-based method and two analytical approximation methods. These forecasts are then compared to the true equicorrelation process and to a naive constant method that forecasts all future correlations as the most recent fitted correlation. The Monte Carlo results are shown in Figure 2a. The two analytical approximations are indistinguishable. Simulation based forecasting is only slightly more accurate for long-horizon forecasts in terms of root mean squared forecast error.

The same experiment is conducted for the LDECO model. Since the value for $\alpha + \beta$ may exceed one for this form of the model, the obvious analytical approximation, $E_t[u_{t+1}] = E_t[\rho_{t+1}]$, can lead to divergent forecasts. To avoid this we only consider simulation-based forecasting and naive constant forecasts. The results are shown in Figure 2b.

5 Equicorrelation in US Stock Returns

Our assessment of DECO has thus far relied on simulated data. To get a sense of its ability to fit actual data we apply it to daily US stock returns from January 2000 through December 2005. We consider two samples, consituents of the Dow Jones Industrial and S&P 500 indexes. To be included in our sample we require that a stock was listed for the full length of the sample and was a member of one of the indexes at any time during the sample period. This amounts to 32 stocks in the Dow Jones sample and 484 stocks in the S&P 500 sample.

The first step of the consistent two-step estimation procedure is to estimate univariate GARCH models for each asset return series. Conditional variances are modeled with asymmetric[13] GARCH(1,1) models, and these variances are used to construct volatility-standardized returns ($r_{i,t}$) which may be used to estimate DECO.[14]

The second step correlation estimation is performed for four versions of the DECO model, DECO-DCC and LDECO, both with constant (unit) equivariance and dynamic equivariance.

The Dow Jones has the attractive feature that all its constituents have actively traded stock options. This allows us to use implied volatilities from options on the Dow Jones index and each of its members to calculate implied correlations, which provide an informative comparison series for DECO. The benefit of option implied correlations is that they are a daily measure incorporating up to date information embedded in market prices.[15]

To calculate the implied correlation series we use standardized 30-day constant maturity call and put option implied volatilities from OptionMetrics, and account for four changes

[13]The form of the asymmetric term follows Glosten, Jagannathan and Runkle (1993).

[14]Other GARCH models were tried, such as factor GARCH (discussed in the context of factor DCC in Engle 2007), Student t GARCH, and other asymmetric forms. Our results are qualitatively insensitive to these variations.

[15]Implied correlation is not exactly comparable to daily DECO estimates since implied correlations are naturally market expectations of future correlations over the time to expiration of the contract, and take as inputs Black-Scholes implied volatilities. Any implied volatility distortions due to model inaccuracy will be inherited by implied correlations. Nonetheless, model-based implied correlations are a natural starting point and reasonable approximation for our purposes. For an analysis of implied correlations based on model free option implied volatilities see Driessen, Maenhout and Vilkov (2005).

in the index during our sample period.[16]

Index volatility is given by $\sigma_{index,t} = \sqrt{w_t' \Omega_t w_t}$, where w_t is the vector of index weights and Ω_t is the conditional covariance matrix of returns. The Dow Jones is a price weighted index, hence returns on the index are an equal weighted average of constituent returns, so $w_t = (1/n, ..., 1/n)$. If we use option implied volatilities for each σ, we can calculate the implied average correlation as

$$\bar{\rho}_t^{implied} = \frac{\sigma_{index,t}^2 - \frac{1}{n^2}\sum_i \sigma_{i,t}^2}{\frac{1}{n^2}\sum_{i \neq j} \sigma_{i,t}\sigma_{j,t}}.$$

Implied correlations cannot be computed for the S&P 500 sample since many of its members do not have actively traded options. As an alternative comparison series we calculate the average pairwise rolling correlation of the S&P 500 sample.

Estimation results are presented in Table 4. Panel A compares DECO-DCC and LDECO with and without dynamic equivarance for Dow Jones stocks and Panel B repeats the analysis for the larger S&P 500 sample. For each DECO-DCC specification the model is estimated twice, once using the unconditional correlation as the target matrix, and once assuming an equicorrelated target whose parameter is estimated in the maximum likelihood step (see footnote 1). Figures 3 and 4 plot the fitted equicorrelation series and Figure 5 shows fitted dynamic equivariances.

A few points of interest emerge from this estimation. First, for both data sets, the Akaike and Bayesian information criteria prefer the DECO-DCC version over LDECO as a description of the data data. Second, for all models and datasets, likelihood ratio tests and information criteria suggest that a dynamic equivariance component be included (LR

[16]The changing constituency of the Dow Jones over time demonstrates an advantage LDECO provides in accommodating non-constant cross section composition over time, which cannot be addressed with DECO-DCC.

test p-value $< .001$). Also, the persistence in equicorrelation is high for all models judging by the value of $(\alpha + \beta)$. For the S&P 500 sample, however, the α estimate is substantially higher for both models[17] than in the smaller Dow Jones sample. This indicates that the model incorporates more information from the most recent return when S&P data is used. Intuitively, this may suggest that the equicorrelation information in a given realized return vector is richer the larger the sample. Fitted equicorrelations show substantial day-to-day variation ranging from correlations of nearly zero to levels exceeding 0.60. The patterns of fitted correlations closely resemble equicorrelations implied from options data or calculated using rolling estimators. Finally, estimates for γ, η, and ϕ show that the dynamic equivariance process is much more persistent in the larger sample. This is also visible in the plots of the the Dow Jones and S&P equivariances.

The last empirical exercise fits the two block DECO model to a sample divided into high growth and high value stocks. The first block is comprised of 25 randomly selected high value stocks (stocks in the highest book-to-market quintile according to Daniel, Grinblatt, Titman and Wermers (1997) for all months in the sample) and the second block contains 25 randomly selected high growth stocks (those in the lowest lowest book-to-market quintile). For reference, we first estimate the base DECO-DCC model without separate block equicorrelations. The second model is two block DECO-DCC with scalar coefficients, which allows each block a separate equicorrelation process, while restricting α and β to be equal for all blocks. Results are presented in Table 5 and Figure 6. Despite the fact that number of parameters has not changed, the likelihood improves using the two block structure. The fitted series reveal that high value stocks tend to have higher correlation levels on average

[17]This point refers to the models including equivariance, since specification analysis suggests this is the superior form.

than growth stocks, a fact that could prove valuable, for instance, when hedging a portfolio designed to capture the value spread.

6 Conclusion

The DECO model represents a major simplification in modeling time varying conditional covariance matrices for returns of an arbitrary number of assets. The equicorrelation assumption arises in derivatives valuation and trading for securities such as CDO's and correlation swap positions. We examine a range of model extensions accommodating data regularities that have traditionally been challenges for multivariate covariance models such as changes in portfolio composition and residual aggregate variance dynamics. We have shown that DECO is a valuable model even in the presence of non-equicorrelated series by its QML property when the true generating process DCC. DECO allows for incorporation of exogenous effects or asymmetric correlations, and the block DECO generalization provides added flexibility in the correlation structure while retaining much of the simplicity of DECO. Diagnostic tests based on conditional moment restrictions implied by the null model are a general class of tests of the efficacy of DECO specifications for removing dynamics in conditional covariances.

Finally, we take DECO to simulated data as well as return data for Dow Jones and S&P 500 stocks. Simulations reveal that DECO satisfactorially fits data even in the presence of misspecified generating processes or violations of equicorrelation. In US stock data, fitted equicorrelations show substantial day-to-day variation and closely match patterns of equicorrelations implied from options data. Lastly, we apply two block DECO to a sample divided into high growth and high value stocks. The fits demonstrate how grouping stocks can capture a degree of cross-sectional heterogeneity while maintaining the tractability of

DECO.

References

Ang, A., and J. Chen, 2002, Asymmetric Correlations of Equity Portfolios, *Journal of Financial Economics* 63, 443–494.

Bauwens, L., S. Laurent, and J.V.K. Rombouts, 2006, Multivariate GARCH Models: A Survey, *Journal of Applied Econometrics* 21, 79–109.

Bollerslev, T., R.F. Engle, and D.B. Nelson, 1994, ARCH Models, *Handbook of Econometrics* 4, 295–3038.

Cappiello, L., R.F. Engle, and K. Sheppard, 2006, Asymmetric Dynamics in the Correlations of Global Equity and Bond Returns, *Journal of Financial Econometrics* 4, 537.

Daniel, K., M. Grinblatt, S. Titman, and R. Wermers, 1997, Measuring Mutual Fund Performance with Characteristic-Based Benchmarks, *Journal of Finance* 52, 1035–1058.

Ding, Z., and R.F. Engle, 2001, Large Scale Conditional Covariance Matrix Modeling, Estimation and Testing, *Academia Economic Papers*.

Driessen, J., P.J. Maenhout, and G. Vilkov, 2006, Option-Implied Correlations and the Price of Correlation Risk, *University of Amsterdam and INSEAD Working Paper*.

Elton, E.J., and M.J. Gruber, 1973, Estimating the Dependence Structure of Share Prices–Implications for Portfolio Selection, *Journal of Finance* 28, 1203–1232.

Engle, R.F., 2002, Dynamic Conditional Correlation: A Simple Class of Multivariate Generalized Autoregressive Conditional Heteroskedasticity Models, *Journal of Business and Economic Statistics* 20, 339–350.

———, 2007, High Dimension Dynamic Correlations, *NYU Working Paper*.

———, and K. Sheppard, 2001, Theoretical and Empirical Properties of Dynamic Conditional Correlation Multivariate GARCH, *NBER Working Paper*.

Glosten, L.R., R. Jagannathan, and D.E. Runkle, 1993, On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks, *Journal of Finance* 48, 1779–1801.

Graybill, F.A., 1983, Matrices with Applications in Statistics, *Wadsworth, Inc.*

Morck, R., B. Yeung, and W. Yu, 2000, The Information Content of Stock Markets: Why Do Emerging Markets Have Synchronous Stock Price Movements?, *Journal of Financial Economics* 58, 215–260.

Nelson, D.B., 1991, Conditional Heteroskedasticity in Asset Returns: A New Approach, *Econometrica* 59, 347–370.

Newey, W.K., 1985, Maximum Likelihood Specification Testing and Conditional Moment Tests, *Econometrica* 53, 1047–1070.

Ribeiro, R., and P. Veronesi, 2002, The Excess Comovement of International Stock Markets in Bad Times: A Rational Expectations Equilibrium Model, *University of Chicago Working Paper*.

Veldkamp, L.L., 2006, Information Markets and the Comovement of Asset Prices, *Review of Economic Studies* 73, 823–845.

White, H., 1994, Estimation, Inference and Specification Analysis, *Cambridge University Press*.

Figure 1. Alternative Equicorrelation Processes for Simulations.
The figure presents functions used for simulations of non-DECO equicorrelated Gaussian variables corresponding to the results in Table 2. Clockwise, the graphs represent the functions “Slow Sine”, “Fast Sine”, “Step” and “Climb and Drop”.

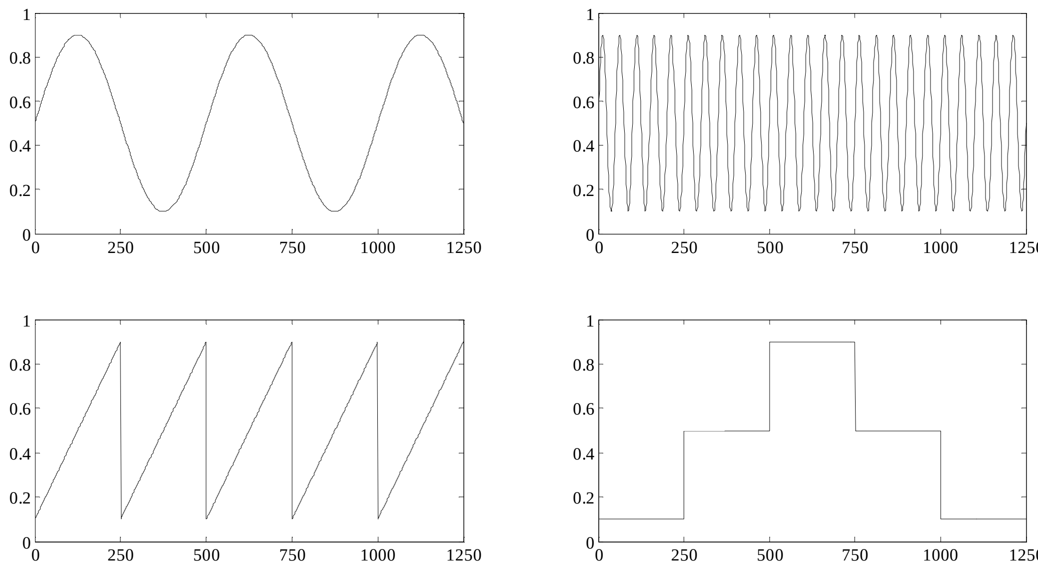

Figure 2a. DECO-DCC Multi-Step Forecasting Monte Carlo.
The figure shows root mean squared forecast error using three forecasting methods under the DECO-DCC specification. Asset return data for cross section size n=10 are simulated over 500 (and 1,250) periods, then the correctly specified model is estimated using the first 250 (and 1000) periods of data. The fitted model is used to contruct two analytical forecast approximations (dotted line), which are based on the simplifications $E_t[r_{t+1}r_{t+1}'] \approx Q_t$ and $E_t[R_{t+1}'] \approx E_t[Q_{t+1}']$, as well as a simulation-based forecast (solid line). The construction of the analytical approximation and simulation-based forecast (using iteration count Z=500) is described in Section 2.6. The experiment is repeated 1000 times and the mean squared forecast error for each method is averaged over the simulations. The two analytical approximations are essentially indistinguishable, thus they are represented by a single line in for each n,T combination.

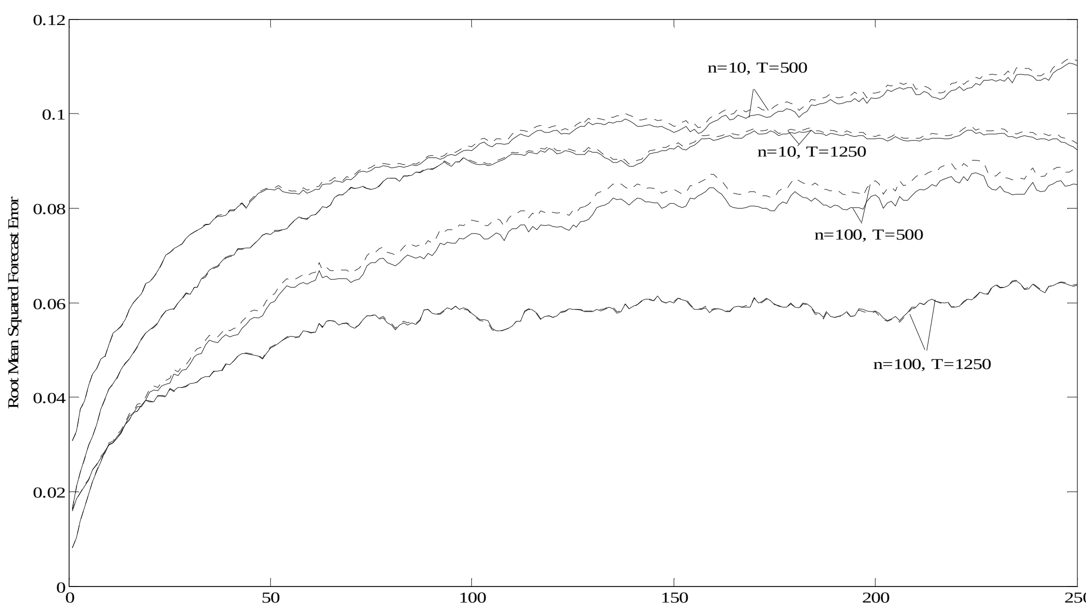

Figure 2b. LDECO Multi-Step Forecasting Monte Carlo.

The figure repeats the experiment described in the header of Figure 2a for the LDECO model, reporting results for simulation-based forecasts.

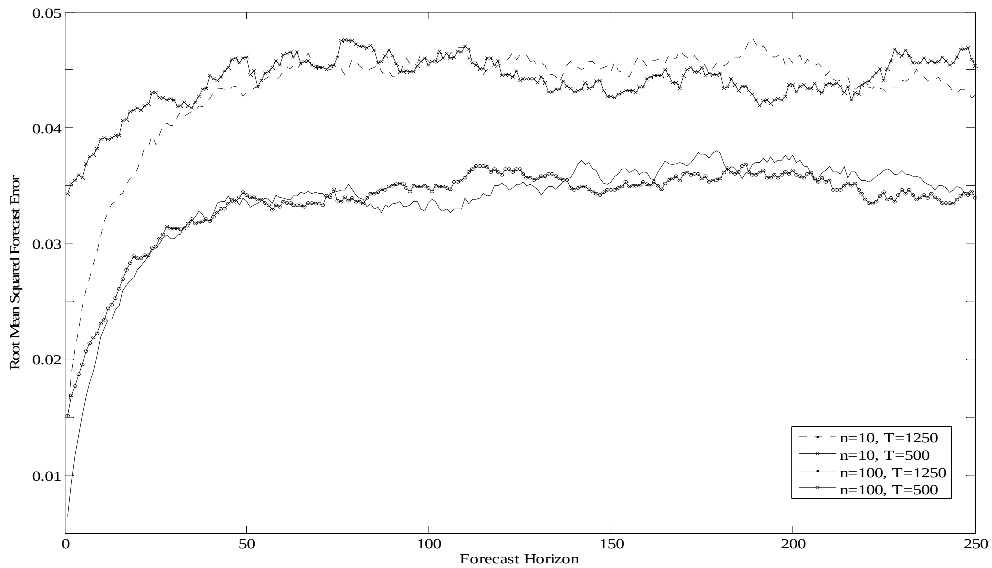

Figure 3. Fitted Dynamic Equicorrelation for Dow Jones Stocks, 2000-2005.

The figure plots fitted dynamic equicorrelation series from DECO-DCC and LDECO models along with average pairwise rolling correlation using daily return data on Dow Jones Industrial index constituents from January 2000 through December 2005. Both models are specified to include dynamic equivariance, and the DECO-DCC model uses the unconditional correlation as the correlation target. The series shown correspond to parameter estimates reported in Table 4.

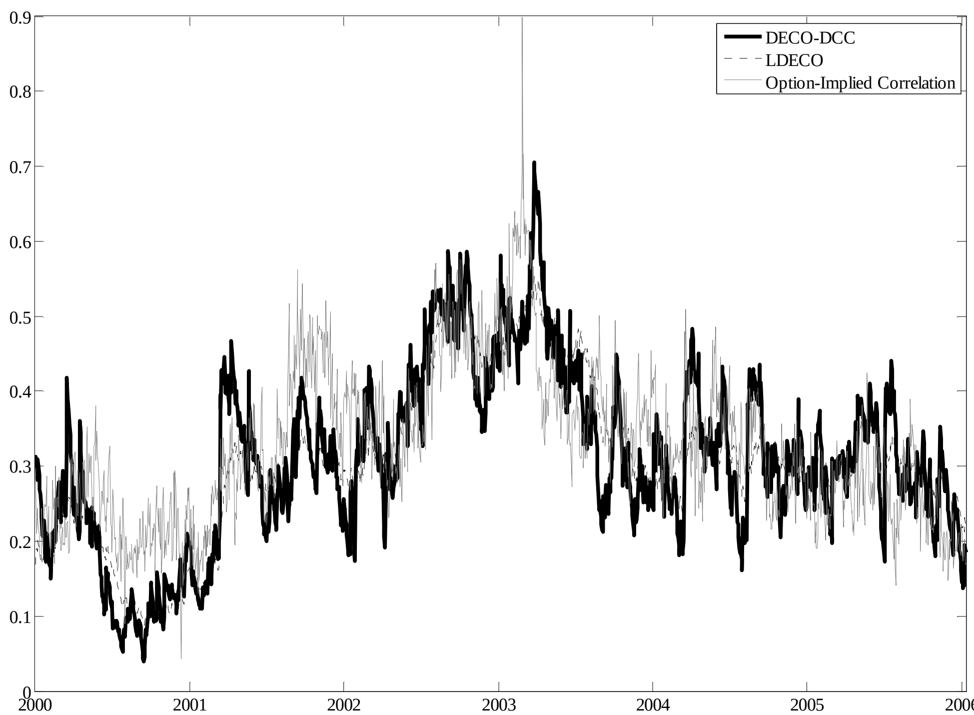

Figure 4. Fitted Dynamic Equicorrelation for S&P 500 Stocks, 2000-2005.
The figure plots fitted dynamic equicorrelation series from DECO-DCC and LDECO models along with average pairwise rolling correlation using daily return data on S&P 500 index constituents from January 2000 through December 2005. Both models are specified to include dynamic equivariance, and the DECO-DCC model uses the unconditional correlation as the correlation target. The series shown correspond to the parameter estimates reported in Table 4.

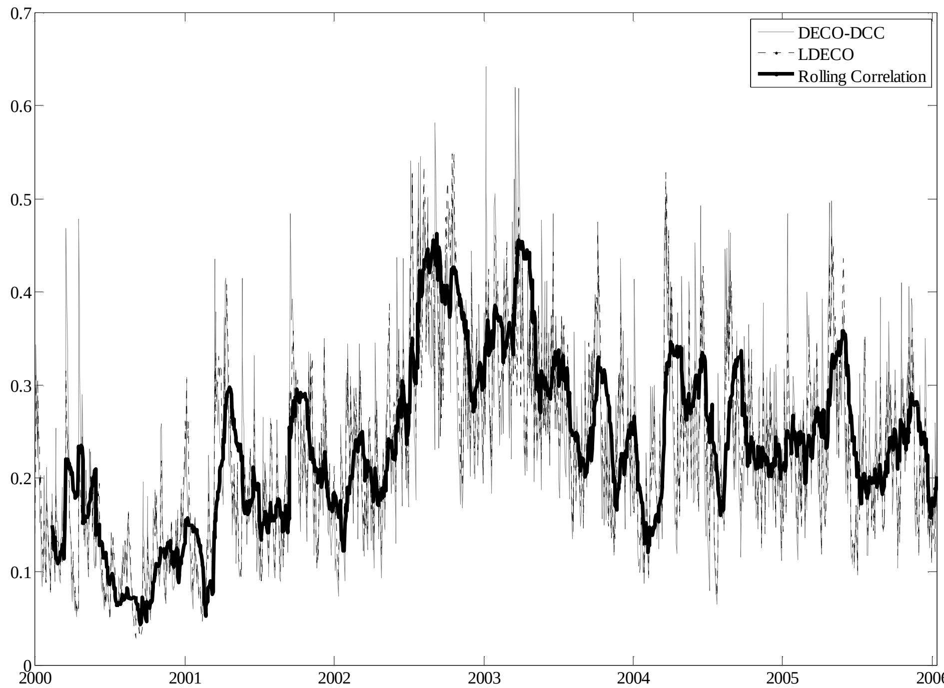

Figure 5. Fitted Dynamic Equivariance for Dow Jones and S&P 500 Stocks, 2000-2005.
The figure compares fitted dynamic equivariance series from the DECO-DCC model using daily return data on Dow Jones and S&P 500 index constituents from January 2000 through December 2005. The series shown correspond to the parameter estimates reported in Table 4.

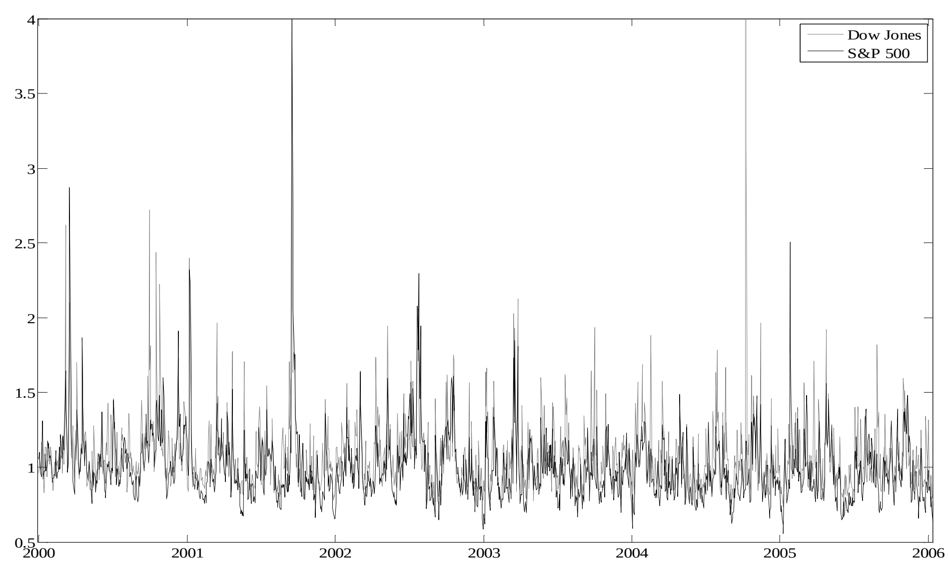

Figure 6. Fitted Block Dynamic Equicorrelation for Growth and Value Stocks, 2000-2005.
The figure shows fitted correlation series corresponding to the two-block DECO-DCC model parameters reported in Table 5. The sample consists of daily stock returns for US high growth and high value stocks. The first block is comprised of 25 randomly selected high value stocks (stocks in the highest book-to-market quintile according to Daniel, Grinblatt, Titman and Wermers (1997) for all months in the sample) and the second block contains 25 randomly selected high growth stocks (those in the lowest lowest book-to-market quintile).

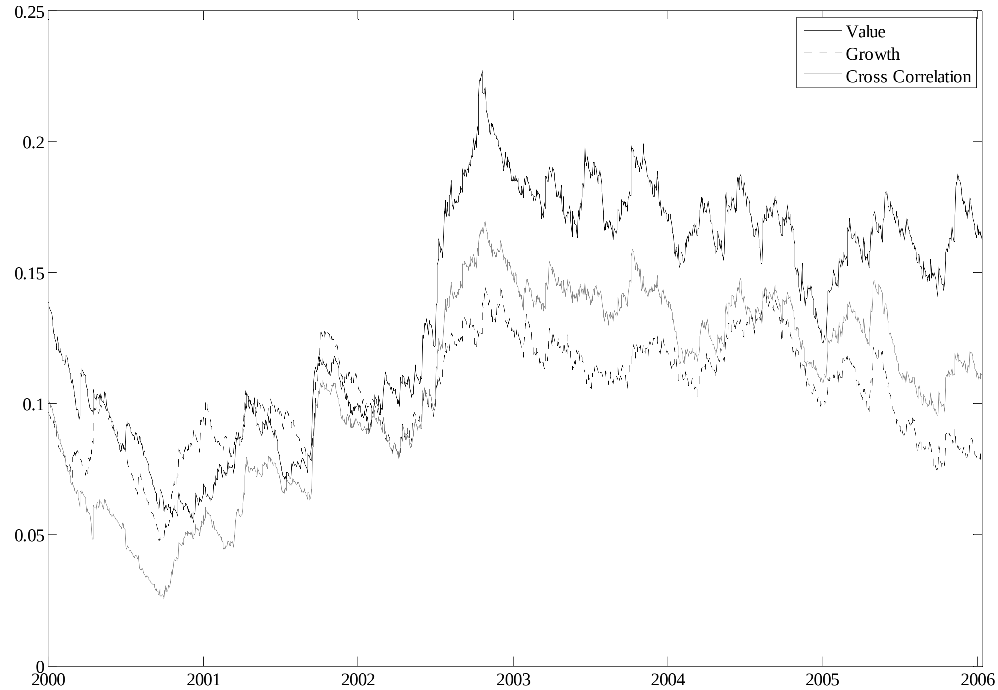

Table I. DECO Correct Specification Monte Carlo Results.

The table reports summary statistics for point estimates of the DECO model. Using model forms DECO-DCC and LDECO, asset return data for cross section sizes n=10, 30 and 100 are simulated over 1,250 periods (corresponding to roughly five years of trading), then the correctly specified model is estimated. The simulations are repeated 1000 times and summary statistics for the maximum likelihood parameter estimates are calculated. Asymptotic standard errors are calculated using the Hessian; the reported value is the mean asymptotic standard error over all simulations. For the DECO-DCC version, correlation targeting is used. In this case, we report an intercept calculated as the average of off-diagonal elements of the sample unconditional correlation matrix, thus the ML asymptotic standard error is not calculated for this coefficient. We also report the root mean squared error for the true DECO process versus the fitted DECO process and the average pairwise rolling correlation estimator (using rolling window length of 30 days). The reported value is the average RMSE over all simulations.

	DECO-DCC						**LDECO**				
				RMSE						RMSE	
	Int.	α	β	DECO	Rolling		ω	α	β	DECO	Rolling
True Value	0.200	0.040	0.950			True Value	0.200	0.040	0.950		
n=10											
Mean	0.195	0.040	0.945	0.014	0.040	Mean	0.007	0.040	0.940	0.016	0.054
Median	0.192	0.040	0.947			Median	0.006	0.039	0.947		
Std. Dev.	0.041	0.009	0.016			Std. Dev.	0.007	0.014	0.033		
Asymp. SE	-	0.009	0.015			Asymp. SE	0.004	0.013	0.025		
n=30											
Mean	0.191	0.039	0.947	0.009	0.031	Mean	0.007	0.041	0.943	0.011	0.045
Median	0.188	0.039	0.948			Median	0.006	0.040	0.946		
Std. Dev.	0.031	0.007	0.012			Std. Dev.	0.005	0.010	0.022		
Asymp. SE	-	0.007	0.012			Asymp. SE	0.003	0.010	0.017		
n=100											
Mean	0.194	0.040	0.949	0.008	0.028	Mean	0.006	0.041	0.945	0.008	0.042
Median	0.188	0.040	0.949			Median	0.006	0.041	0.945		
Std. Dev.	0.031	0.005	0.007			Std. Dev.	0.002	0.005	0.010		
Asymp. SE	-	0.005	0.007			Asymp. SE	0.002	0.006	0.010		

Table 2. Alternative Equicorrelation Generating Process Monte Carlo Results.

The table reports root mean squared errors for fitted equicorrelation processes versus the Gaussian equicorrelation processes shown in Figure 1. Fitted equicorrelations are calculated using the DECO-DCC, LDECO and average pairwise rolling correlation (using rolling window length of 30 days) estimators. For each of the four generating processes, asset return data for a cross section of size *n*=30 are simulated over 1,250 periods (corresponding to roughly five years of trading), repeated 1000 times. The reported value is the average RMSE over all simulations.

	DECO-DCC	**LDECO**	**Rolling**
Slow Sine	0.070	0.107	0.072
Fast Sine	0.157	0.188	0.354
Climb and Drop	0.079	0.080	0.090
Step	0.083	0.101	0.087

Table 3. Monte Carlo Performance of DECO Under DCC Misspecification.

The table compares estimates from DECO and DCC when DCC is the data generating process. Asset return data for cross sections of size *n*=10, 30 and 100 are simulated over 1,250 periods (corresponding to roughly five years of trading). The simulation is repeated 1000 times. For the DECO model, asymptotic standard errors are calculated using QMLE standard errors. For DCC, asymptotic standard errors are calculated using the Hessian. We also report the root mean squared error for the true average pairwise DCC process versus the fitted DECO and DCC counterparts. The reported value is the average RMSE over all simulations. The model uses correlation targeting, thus the estimated intercept term is the same for both models and not reported.

	DECO		**DCC**	
	α	β	α	β
True Value	0.03	0.96	0.03	0.96
n=10				
Mean	0.031	0.945	0.032	0.962
Median	0.030	0.955	0.032	0.962
Std. Dev.	0.013	0.096	0.001	0.002
Asymp. SE	0.012	0.025	0.001	0.001
RMSE	0.016		0.011	
n=30				
Mean	0.033	0.939	0.031	0.968
Median	0.031	0.955	0.031	0.968
Std. Dev.	0.021	0.079	0.000	0.001
Asymp. SE	0.017	0.036	0.000	0.000
RMSE	0.013		0.011	
n=100				
Mean	0.034	0.947	0.029	0.970
Median	0.033	0.951	0.029	0.970
Std. Dev.	0.009	0.020	0.000	0.000
Asymp. SE	0.018	0.027	0.000	0.000
RMSE	0.010		0.008	

Table 4. DECO Parameter Estimates for Dow Jones and S&P 500 Stocks, 2000-2005.
The table reports parameter estimates from the DECO-DCC and LDECO models for two samples of daily returns on US stocks. Panel A shows results for the sample of Dow Jones constituent stocks and Panel B gives results for S&P 500 constituent stocks. The DECO-DCC model is estimated once imposing constant unit equivariance and one including dynamic equivariance. For each of these DECO-DCC specifications the estimates are calculated first using correlation targeting (the left two columns under each sub-heading) and next estimating an equicorrelated intercept matrix in the ML step (the right two columns under each sub-heading). The LDECO model is estimated once imposing constant unit equivariance and one including dynamic equivariance, and the intercepts are estimated in both cases in the ML step. *p*-values for parameter estimates are calculated using QMLE standard errors. LL denotes the log likelihood and the *p*-value adjacent to likelihood values corresponds to a likelihood ratio test of the dynamic equivariance version of the model versus constant equivariance. Also reported are the Akaike and Bayesian information criteria. The DECO-DCC models using the unconditional correlation matrix as a target have $n(n-1)/2-1$ parameters in additional to the counterpart that estimates the intercept matrix, where n=32 and 484 for the Dow Jones and S&P 500, respectively.

	DECO-DCC								**LDECO**			
	Unit Equivariance				**Dynamic Equivariance**				**Unit Equivariance**		**Dynamic Equivariance**	
	Coeff.	*p-value*	Coeff.	*p-value*	Coeff.	*p-value*	Coeff.	*p-value*	Coeff.	*p-value*	Coeff.	*p-value*
Panel A: Dow Jones												
Intercept / ω	0.298	-	0.120	*0.036*	0.298	-	0.382	*0.001*	0.000	*<.001*	0.000	*<0.001*
α	0.038	*<.001*	0.036	*<.001*	0.096	*0.006*	0.102	*0.014*	0.044	*<.001*	0.040	*<0.001*
β	0.958	*<.001*	0.960	*<.001*	0.897	*<0.001*	0.890	*<0.001*	0.965	*<.001*	0.971	*<0.001*
γ					0.438	*<0.001*	0.443	*<0.001*			0.500	*<0.001*
η					0.268	*<0.001*	0.273	*<0.001*			0.276	*<0.001*
φ					0.335	*0.003*	0.332	*0.004*			0.265	*0.006*
LL	-18522		-18502		-18141	*<0.001*		*<0.001*	-18523		-18194	*<0.001*
AIC	38039		37010		37284				37052		36399	
BIC	40711		37026		39971				37068		36432	
Panel B: S&P 500												
Intercept / ω	0.217	-	0.030	*0.771*	0.217	-	0.483	*<0.001*	0.002	*0.118*	0.002	*0.945*
α	0.032	*<.001*	0.047	*<.001*	0.368	*<0.001*	0.380	*<0.001*	0.026	*<.001*	0.275	*0.002*
β	0.968	*<.001*	0.953	*<.001*	0.596	*<0.001*	0.575	*<0.001*	0.983	*<.001*	0.794	*<0.001*
γ					0.164	*<0.001*	0.209	*<0.001*			0.150	*<0.001*
η					0.318	*<0.001*	0.337	*<0.001*			0.252	*<0.001*
φ					0.529	*<0.001*	0.499	*<0.001*			0.599	*<0.001*
LL	-264860		-264829		-260271	*<0.001*	-260221	*<0.001*	-264369		-260763	*<0.001*
AIC	763496		529664		754325		520453		528745		521538	
BIC	1390622		529680		1381466		520486		528761		521570	

Table 5. Block DECO Estimates for Growth and Value Sample, 2000-2005.
The table reports estimates of the two-block DECO model for a sample of daily returns on US high growth and high value stocks. The first block is comprised of 25 randomly selected high value stocks (stocks in the highest book-to-market quintile according to Daniel, Grinblatt, Titman and Wermers (1997) for all months in the sample) and the second block contains 25 randomly selected high growth stocks (those in the lowest lowest book-to-market quintile). The first set of estimates are for the base DECO-DCC model without separate block equicorrelations. The second set of estimates uses the two block DECO-DCC model with scalar coefficients (ie. each block has a separate equicorrelation process, but the α and β parameters are restricted to be equal for all blocks). The reported intercepts are the average of elements in the unconditional correlation matrix (the target matrix) corresponding the each block. All reported model parameters are significant at the 0.1% level using QMLE standard errors.

	DECO	**BLOCK DECO**
$Intercept_{1,1}$	0.115	0.116
$Intercept_{2,2}$	-	0.117
$Intercept_{1,2}$	-	0.110
α	0.028	0.014
β	0.972	0.986
γ	0.231	0.228
η	0.105	0.103
φ	0.630	0.635
LL	-28999	-28933