Background Risk and Trading in a Full-Information Rational Expectations Economy

Richard C. Stapleton[1], Marti G. Subrahmanyam[2], and Qi Zeng[3]

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[1]University of Manchester
[2]New York University
[3]Melbourne University

Abstract

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In this paper we assume that investors have the same information, but trade due to the evolution of their non-market wealth. In our formulation, investors rebalance their portfolios in response to changes in their expected non-market wealth, and hence trade. We assume an incomplete market in which risky non-market wealth is non-hedgeable and independent of the market risk and thus represents an additive background risk. Investors who experience positive shocks to their expected wealth buy more stocks from those who experience less positive shocks.

1 Introduction

It has long been a challenge for financial economists to explain trading in the context of rational expectations asset pricing models. For example, in the complete markets Arrow-Debreu model, agents choose state-contingent claims on the initial date, but do not trade at subsequent dates, since they have already purchased claims that hedge against various future outcomes; thus, there is no need for them to adjust their portfolio holdings as the state of the world is revealed. This inability to explain trading in a rational model flies in the face of evidence that there is a large volume of trading in various securities: bonds, stocks, and increasingly in various types of contingent claims, such as options and futures contracts.

Several attempts have been made in the literature in the past to explain trading by relaxing some of the assumptions of completeness of markets and information available to agents in the economy. One possibility is that when investors have asymmetric information, this gives them an incentive to trade in order to profit from that information. However, as Grossman and Stiglitz (1980) point out, the mere act of trading reveals the information possessed by a particular agent and this gets reflected in market prices. While there may be some "sand in the gears" introduced if the process of expectations formation is noisy, the central intuition that prices reflect private information still prevails, reducing the motivation to trade substantially.

This argument was taken one step further by Milgrom and Stokey (1982) who argue that when the agents begin with a Pareto optimal allocation relative to their prior beliefs, they do not trade upon receiving private information, even at equilibria that are less than fully revealing, since "the information conveyed by price changes swamps each traders private information." This surprisingly general result arises because if the initial allocation is Pareto optimal, there is no valid insurance motive for trading. The willingness of other traders to take the opposite side implies at least to one trader that his own bet is unfavorable. Hence no trade is acceptable to all traders. The Milgrom and Stokey propositions rely on two crucial assumptions: a) that it is common knowledge that when a trade occurs it is feasible and acceptable to all agents, and b) the agents beliefs are concordant, i.e., that they agree about how the information should be interpreted.

Another strand of the literature that has provided a motivation for trading is on market micro-structure, most prominently by Kyle (1985) and Glosten and Milgrom (1985). These models try to explain the bid-offer spread in markets by appealing to asymmetric information. However, a crucial assumption in such models is the existence of noise traders, who trade for liquidity reasons, and these are not explicitly modeled. Furthermore, it is unclear why in such models, investors trade for liquidity reasons in risky securities such as stocks, rather than trading bonds, unless some market imperfection is assumed. In the Milgrom and Stokey sense, it must be the case that the allocation in these models is not ex-ante Pareto optimal, and/or that the beliefs are not concordant.

The broad conclusion from the information-based literature on trading is that the Milgrom and Stokey "no-trade" result will obtain, unless there is some market imperfection, significant deviation from rational expectations equilibria or an exogenous reason to trade, such as liquidity motivations.

In this paper, we explore an alternative motivation for trading, which is the existence of non-marketable wealth. Non-marketable wealth may take many forms, but the most obvious example is wealth arising from labor income. Human capital, which is the value of future labor income, has been shown in many studies, both theoretical and empirical, to have an influence on portfolio demand. Another example is housing wealth, which is a significant component of the portfolios of households. Again, there is a extensive literature documenting how housing wealth affects portfolio choice and, in turn, feeds back on to the equilibrium prices of traded assets. The effect of non-market wealth is that it alters the agents' demand for the traded assets. An early example of this distortion is the work of Bodie, Merton and Samuelson (1992) in the context of non-stochastic, positive non-marketable wealth for an agent with constant relative risk aversion. They show that this agent acts much like another agent with a lower, but increasing relative risk aversion.

The problem gets more complex when the non-marketable wealth has stochastic properties. There is a extensive literature on background risk that studies the portfolio behaviour of agents with such non-marketable wealth, whose future cash flows are also stochastic. For most common utility functions, the existence of background risk makes agents more risk averse and hence reduces their demand for risky securities. [See, for example, Gollier and Pratt (1996), Kimball (1993) and Eekhoudt, Gollier and Schlesinger (1996).] The natural question is how the changes in the agents' portfolio decisions affect the portfolio demand and sharing rules of the marketable securities in equilibrium, a problem first analyzed by Franke, Stapleton and Subrahmanyam (1998) [FSS].

We extend this framework to consider a multi-period version of the FSS framework. Following the outcome of the background risk in the intermediate period, agents adjust their holdings of the marketable securities, to be in line with their new level of derived risk aversion in the presence of the updated distribution of background wealth. If the outcomes of the background risk are heterogeneous across agents, it creates a motivation for trading, as different agents may wish to adjust their portfolio holdings in opposite directions. We explore this simple intuition formally for investors with constant relative risk aversion in our analysis.

Section 2 presents the set up of the model and derives the portfolio demand for traded state-contingent claims. Section 3 describes the evolution of the background risk over time Section 4 derives optimal demand in the special case where all uncertainty of background risk is resolved at time 1. Section 5 generalizes the results using an approximation. Section 6 presents our conclusions.

2 Setup of the Model

Our model is an extension of the model in Franke, Stapleton and Subrahmanyam (1998) [FSS]. In FSS, it is assumed that the agents maximise the expected utility of wealth ,w at the end of a single period. For agent i, $w_i = x_i + e_i$, where x_i is a set of claims on a single aggregate market cash flow, X_a and e_i is an independent, zero mean background risk. Each agent solves the following maximization problem:

$$\max_x E[E_e[u(x+e)]], \text{ s.t. } E[\phi(X_a)x] = E[\phi(X_a)x_0], \quad (1)$$

given an initial endowment of x, x_0. In (1), $\phi(X_a)$ is the forward pricing kernel. The budget constraint states that the forward price of the chosen portfolio of claims has to equal the forward value of the endowed claims. In FSS, agents have utility functions $u_i(w_i)$ which belong to the HARA class, excluding the exponential function. Here, we assume essentially the same setup with

$$u_i(w_i) = \frac{w_i^{1-\gamma_i}}{1-\gamma_i}, \quad (2)$$

where $w_i = x_i + a_i + e_i$. In this formulation, a_i is a constant representing the expected value of non-market wealth. Utility for wealth is a power function, exhibiting constant relative risk aversion, but the derived utility for x_i is of the HARA form, when the background risk e_i does not exist.

Let λ_i be the Lagrangian multiplier associated with the budget constraint of investor i. Then, the first order condition of the optimization problem is:

$$E_e[(x_i + a_i + e_i)^{-\gamma_i}] = \lambda_i \phi(X_a).$$

Following Kimball (1990), we can define the precautionary premium ψ_i by the relation

$$E_e[(x_i + a_i + e_i)^{-\gamma_i}] \equiv [x_i + a_i - \psi_i]^{-\gamma_i} \quad (3)$$

Hence $(x_i + a_i - \psi_i)^{-\gamma_i}$ is the certainty equivalent of $E_e(x_i + a_i + e_i)^{-\gamma_i}$. Note that ψ_i itself will be a function of x_i and also depends on the distribution of e_i.

It follows that:

$$x_i = (\lambda_i)^{-1/\gamma_i} \phi(X_a)^{-1/\gamma_i} - a_i + \psi_i. \tag{4}$$

Using the market clearing condition $\frac{1}{I}\sum_i x_i = X$, where I is the number of agents and assuming $\gamma_i = \gamma$ for all i, we have:[1]

$$X = \lambda^{-1/\gamma} \phi^{-1/\gamma} - A + \psi, \tag{5}$$

where

$$\psi = \frac{1}{I}\sum_i \psi_i, \tag{6}$$

$$A = \frac{1}{I}\sum_i a_i, \tag{7}$$

$$\lambda^{-1/\gamma} = \frac{1}{I}\sum_i \lambda_i^{-1/\gamma}. \tag{8}$$

Note that the aggregate ψ is a function of the state indexed by X and it depends also on the distribution $\{e_i\}_{i=1,\ldots,N}$.

Solving this we find

$$\phi(X) = (X + A - \psi)^{-\gamma} \lambda^{-1}. \tag{9}$$

Now using the no-arbitrage condition that $E(\phi) = 1$ we find

$$\lambda = E[(X + A - \psi)^{-\gamma}] \tag{10}$$

$$\text{and} \quad \phi(\mathrm{X}) = \frac{(X + A - \psi)^{-\gamma}}{E[(X + A - \psi)^{-\gamma}]}. \tag{11}$$

Substituting (10) and (11) into the demand equation (4) , yields

$$\begin{aligned} x_i &= \lambda_i^{-1/\gamma} \phi(X)^{-1/\gamma} - a_i + \psi_i \\ &= \lambda_i^{-1/\gamma} \left[E(X + A - \psi)^{-\gamma}\right]^{\frac{1}{\gamma}} (X + A - \psi) - a_i + \psi_i. \end{aligned} \tag{12}$$

[1] It is convenient to define market aggregates in terms of per capita values for the 'average' agent, without loss of generality. Note $\phi(X_a)$ can be written $\phi(X)$.

Having solved $\phi(X)$ from the market clearing condition, we next solve for λ_i. Substitute the solution of x_i in (12) above back into the individual budget constraint using the expression for $\phi(X)$:

$$E[\phi(X)x_i] = w_{0,i}r,$$

where $w_{0,i}$ is wealth at time 0 and r is the gross risk-free rate. It then follows [again using $E(\phi) = 1$] that:

$$\begin{aligned} w_{0,i}r &= E[\phi(\lambda_i^{-1/\gamma}\phi^{-1/\gamma} - a_i + \psi_i)] \\ &= \lambda_i^{-1/\gamma}E(\phi^{1-1/\gamma}) - a_i + E(\phi\psi_i). \end{aligned}$$

Then, λ_i is:

$$\begin{aligned} \lambda_i &= (E\phi^{1-1/\gamma})^{\gamma}[w_{0,i}r + a_i - E(\phi\psi_i)]^{-\gamma} \\ &= [E(X + A - \psi)^{1-\gamma}]^{\gamma}[E(X + A - \psi)^{-\gamma}]^{1-\gamma}[w_{0,i}r + a_i - E(\phi\psi_i)]^{-\gamma}. \end{aligned} \tag{13}$$

Hence, the optimal individual investor demand is:

$$\begin{aligned} x_i &= \frac{E(X + A - \psi)^{-\gamma}}{E(X + A - \psi)^{1-\gamma}}(X + A - \psi)[w_{0,i}r + a_i - E(\phi\psi_i)] - a_i + \psi_i, \\ &= \frac{E[(X + A - \psi)^{-\gamma}(x_{0i} + a_i - \psi_i)]}{E[(X + A - \psi)^{1-\gamma}]}(X + A - \psi) - a_i + \psi_i. \end{aligned} \tag{14}$$

The expression for the demand for contingent claims in (14) is complex. If there would be no background risk for all investors, ψ would be zero and x_i linear in X. However, in general both ψ and ψ_i are convex functions implying a non-linear demand function. Also, the optimal demand is implicit since ψ_i is a function of x_i for each i.

3 The Evolution of Background Risk Over Time

So far, we have assumed that agents face a background risk e_i which is resolved at the end of a single period. As in FSS, e_i has a zero mean and is independent of the market cash flow, X. We now introduce a multiperiod model in which the risk, e_i, evolves over time. This is required to study trading volume in the following sections, since trading is essentially an intertemporal issue.

We begin by assuming for simplicity that $w_i = x_i + e_i$ with $a_i = 0$. Also, $E_0(e_i) = 0$ and e_i is independent of X for all i. We now model the background risk e_i as the sum of two independent, zero-mean random variables:

$$e_i = \xi_i + \eta_i, \tag{15}$$

where the outcome of ξ_i is determined at $t = 1$ and the outcome of η_i is determined at $t = 2$. In this case the outcome of ξ_i is the conditional expectation of e_i at $t = 1$. For example, in the following analysis we assume a simple binomial distribution for ξ_i, where $\xi_i = \pm a$ with equal probability. In this case, if $\xi_i = +a$, then $w_i = x_i + a + \eta_i$. Alternatively, if $\xi_i = -a$, then $w_i = x_i - a + \eta_i$. The investor then solves the maximization problem at $t = 1$, given the outcome of ξ_i.

The trading at $t = 1$ depends on the outcome of ξ_i for different investors. Without loss of generality, assume there are just two groups of investors indexed by M and N. For each group we again suppose that $\xi_i = \pm a$. Now, if it happens that the outcome ξ_i is the same for both groups of investors there will be no trade. However, if for one group $\xi_i = +a$ and for the other group $\xi_i = -a$, then there will be trade in general.

In the multiperiod model, we need to distinguish the pricing kernel and the precautionary premium at $t = 0$ and $t = 1$ respectively. Let $\psi_{i,t}$, $t = 0, 1$ be the precautionary premium for investor i and ψ_t, $t = 0, 1$ be the average precautionary premium across investors. Also, let ϕ_t, $t = 0, 1$ be the pricing kernel for valuation at those dates. We proceed by first considering a special case where the precautionary premia at $t = 1$ are zero for all investors.

4 A Special Case: Full Resolution of Background Risk at Time 1

In this section, we investigate the case where all the uncertainty of e_i is resolved at $t = 1$. As discussed above, in the general case the demand for contingent claims is an implicit function. This is due to the fact that the precautionary premium is a function of the demand itself. However, in the special case where all the uncertainty of e_i is resolved at $t = 1$, the precautionary premium is zero at time 1. So in this case there is an explicit solution for the optimal demand at time 1.

We again assume that there are two sets of investors in the economy indexed by M and N. At time 0, all the investors are identical, and only differ in the resolution of the uncertainty of e_i. Since the investors are identical at $t = 1$ and since e_i has the same distribution for all i, they must hold the same portfolios at $t = 0$. That implies that the initial demand $x_i = X$, since X is the average alloction of claims across investors. Now we assume again that ξ_i has two outcomes $\xi_i = \pm a$.

At $t = 1$, there will be two situations depending on the realization of $\xi_i, (i = M, N)$ as discussed above:

- Homogeneous case: $(\xi_M, \xi_N) = (\pm a, \pm a)$. In this case all investors are in the same situation at $t = 1$. The average per capita realization of background risk is $A = \frac{1}{2}(\xi_M + \xi_N) = (\pm a)$. Again, because of symmetry, each set of agents will hold the same amount of contingent claims $x_{1i} = X, i = M, N$.

  Since all background risk is resolved at $t = 1$, the precautionary premia, $\psi_{i,0}$ and ψ_0 are zero. It follows that the pricing kernel at $t = 1$ is:

$$\phi_1(X) = \frac{[(X \pm a)^{-\gamma}]}{E_1[(X \pm a)^{-\gamma}]}.$$

  and there is no trading volume.

- Heterogeneous case: In this case, the outcomes of ξ_i differ between the two sets of agents. We assume $(\xi_M, \xi_N) = (\pm a, \mp a)$. The average per capita realization of background risk is $A = \frac{1}{2}(\xi_M + \xi_N) = 0$. Again $\psi_1 = 0$.

  Let us consider the case when $(\xi_M, \xi_N) = (a, -a)$. The case where $(\xi_M, \xi_N) = (+a, a)$ is similar.

  According to the general demand equation(14), the demands of two types of investors at $t = 1$ are:

$$x^*_{1M} = \frac{E_1[X^{-\gamma}(X+a)]}{E_1[X^{1-\gamma}]}X - a = X + \frac{aE_1[X^{-\gamma}]}{E_1[X^{1-\gamma}]}X - a, \tag{16}$$

$$x^*_{1N} = \frac{E_1[X^{-\gamma}(X-a)]}{E_1[X^{1-\gamma}]}X + a = X - \frac{aE_1[X^{-\gamma}]}{E_1[X^{1-\gamma}]}X + a. \tag{17}$$

  It follows that the trading volume is then:

$$|\Delta| = \left| \frac{aE_1[X^{-\gamma}]}{E_1[X^{1-\gamma}]}X - a \right|$$

  The above equation has an intuitive interpretation. If there exists a stock which pays of average per capital endowment X, then with the revelation of the background risk a, the average trading volume will be $aE_1(X^{-\gamma})/E_1(X^{1-\gamma})$ shares. The additional $-a$ term is because investors cannot use the revealed expected income a to buy the shares. So instead the investor has to trade a dollars worth of risk-free bond to offset the trading of shares.

To confirm this, note the pricing kernel at time $t = 1$ is:

$$\phi_1 = \frac{X^{-\gamma}}{E_1(X^{-\gamma})}.$$

So the total value of $aE_1(X^{-\gamma})/E_1(X^{1-\gamma})$ shares is:

$$\begin{aligned} E_1\left(\phi_1 \frac{aE_1[X^{-\gamma}]}{E_1[X^{1-\gamma}]} X\right) &= \frac{aE_1[X^{-\gamma}]}{E_1[X^{1-\gamma}]} E_1(\phi_1 X) \\ &= \frac{aE_1[X^{-\gamma}]}{E_1[X^{1-\gamma}]} E_1\left(\frac{X^{-\gamma}}{E_1(X^{-\gamma})} X\right) \\ &= a. \end{aligned}$$

The trading volume in the heterogeneous case is a linear function of the average per capita endowment. A straightforward extension of the above analysis shows that what is really needed is $A = 0$. In other words, as long as there is no aggregate growth in background risk, the trading volume will be a linear function of the average per capita endowment regardless of the number of states.

In the above setup, there are just two extreme outcomes: either there is aggregate (positive or negative) increase in expected non-market wealth and a homogeneous background risk shock, or there is no aggregate increase in expected non-market wealth and a heterogeneous shock. In general (with multiple states), there will exist situations that there are both aggregate changes in background wealth and agents are heterogeneous. In this case, the trading volume may not be a linear function of average per capital endowment.

5 The General Case

As we saw earlier, in the general case where there is unresolved background risk at time 1 the optimal demand cannot be solved analytically. The demand x_i depends on the precautionary premium, which in turn depends on the demand. We solved the problem above by considering a special case. We now analyse the general case, but with the use of an approximation.

We start with an approximation for ψ_i. From equation (3)

$$E_e(x_i + a_i + e_i)^{-\gamma} \equiv (x_i + a_i - \psi_i)^{-\gamma}$$

Taking the Taylor expansion for the left hand side we have

$$\begin{aligned} E_e(x_i + a_i + e_i)^{-\gamma} &= (x_i + a_i)^{-\gamma} E_e \left(1 + \frac{e_i}{x_i + a_i}\right)^{-\gamma} \\ &\approx (x_i + a_i)^{-\gamma} E_e \left[1 - \gamma \frac{e_i}{x_i + a_i} + \frac{\gamma(\gamma+1)}{2} \left(\frac{e_i}{x_i + a_i}\right)^2\right] \\ &= (x_i + a_i)^{-\gamma} \left[1 + \frac{\gamma(\gamma+1)\sigma_{e_i}^2}{2(x_i + a_i)^2}\right]. \end{aligned}$$

where, in the last step, we use the assumption that $Ee_i = 0$.

Reversing the two sides, we can write:

$$(x_i + a_i - \psi_i)^{-\gamma} = (x_i + a_i)^{-\gamma} \left[1 + \frac{\gamma(\gamma+1)\sigma_{e_i}^2}{2(x_i + a_i)^2}\right]$$

and hence

$$\begin{aligned} x_i + a_i - \psi_i &= (x_i + a_i) \left[1 + \frac{\gamma(\gamma+1)\sigma_{e_i}^2}{2(x_i + a_i)^2}\right]^{-1/\gamma} \\ &\approx (x_i + a_i) \left[1 - \frac{\gamma(\gamma+1)\sigma_{e_i}^2}{2(x_i + a_i)^2}\right] \\ &= x_i + a_i - \frac{(1+\gamma)\sigma_{\epsilon_i}^2}{2(x_i + a_i)}. \end{aligned}$$

Which yields the approximate result for ψ_i:

$$\psi \approx \frac{(1+\gamma)\sigma_{\epsilon_i}^2}{2(x_i + a_i)}.$$

Thus we have an approximate solution for ψ_i as a function of x_i. As we can see, it satisfies all the properties for the precautionary premium, ψ_i, as stated in FSS:

$$\psi_i > 0, \quad \frac{\partial \psi_i}{\partial x} < 0, \quad \frac{\partial^2 \psi_i}{\partial x^2} > 0, \tag{18}$$

$$\frac{\partial \psi_i}{\partial \sigma} < 0, \quad \frac{\partial^2 \psi_i}{\partial \sigma \partial x} < 0, \quad \frac{\partial^3 \psi_i}{\partial \sigma \partial x^2} > 0. \tag{19}$$

Also the approximation has additional implications with respect to the constant mean change in a_i:

$$\frac{\partial \psi_i}{\partial a} < 0, \quad \frac{\partial^2 \psi_i}{\partial a^2} > 0 \tag{20}$$

Finally, the cross relationships on σ and a_i are similar to those on σ and x_i.

Using the approximation for ψ_i, we now proceed to derive the optimal demand. Assuming that the the agent i has wealth $w_{0,i}$:

$$\begin{aligned} x_i &= (E\phi^{1-1/\gamma})^{-1}\phi^{-1/\gamma}[w_{0,i}R + a_i - E(\phi\psi_i)] - a_i + \psi_i && (21) \\ &= \beta\phi^{-1/\gamma}w_{0,i}R + (\beta\phi^{-1/\gamma} - 1)a_i + \frac{(1+\gamma)\sigma_{e_i}^2}{2}\left(\frac{1}{x_i + a_i} - \beta\phi^{-1/\gamma}E\frac{\phi}{x_i + a_i}\right) && (22) \end{aligned}$$

where

$$\beta \equiv (E\phi^{1-1/\gamma})^{-1}. \tag{23}$$

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Another way of writing the above demand function is:

$$x_i + a_i = \beta\phi^{-1/\gamma}(w_{0,i}r + a_i) + \frac{(1+\gamma)\sigma_{e_i}^2}{2}\left[\frac{1}{x_i + a_i} - \beta\phi^{-1/\gamma}E\frac{\phi}{x_i + a_i}\right] \tag{24}$$

Note that the expectation is here taken only with respect to X because ψ_i takes into account of the expectation with respect to e_i.

As in FSS, the optimal demand can be decomposed into three parts. The first term indicates that as W_i increases, agents increase their holdings for all contingent claims proportionally. This is consistent with prior literature. When agents are initially endowed with more wealth, agents become less risk-averse and invest more in the contingent claims. The second term is the demand due to the existence of the income a_i. Again the investors will regard the income as if given, then subtract the sure amount from each contingent claim. The third term is the non-linear demand from the background risk.

The general case with unresolved background risk η_i cannot be solved analytically. Thus we will use the approximation formula derived earlier. The basic idea is to approximate to the order of σ_η^2 and ignore all the higher order terms.

Recall the optimal demand of an agent, say M, at $t = 1$ is:

$$x_{1M} = \frac{E_1[(X - \psi_1)^{-\gamma}(X + a - \psi_{1M})]}{E_1[(X - \psi_1)^{1-\gamma}]}(X - \psi_1) - a + \psi_{1M}, \tag{25}$$

where

$$\psi_{1M} = \frac{(1+\gamma)\sigma_\eta^2}{2(x_{1M} + a)}.$$

For convenience, define:

$$Y \equiv x_{1M} + a \tag{26}$$
$$Z \equiv x_{1N} - a. \tag{27}$$

We further define:

$$Y^* \equiv x^*_{1M} + a = \frac{E_1[X^{-\gamma}(X+a)]}{E_1[X^{1-\gamma}]} X \tag{28}$$
$$Z^* \equiv x^*_{1N} - a = \frac{E_1[X^{-\gamma}(X-a)]}{E_1[X^{1-\gamma}]} X, \tag{29}$$

where x^*_{1M}, x^*_{1N} are the optimal demand in the special case discussed above.

Then the above optimal demand function becomes:

$$Y = \frac{E_1[(X-\psi_1)^{-\gamma}(X+a-\psi_{1M})]}{E_1[(X-\psi_1)^{1-\gamma}]}(X-\psi_1) + \psi_{1M},$$

where

$$\psi_{1M} = \frac{(1+\gamma)\sigma^2_\eta}{2Y}.$$

Furthermore, the average ψ is:

$$\psi_1 = \frac{1}{2}(\psi_{1M} + \psi_{1N}) = \frac{(1+\gamma)\sigma^2_\eta X}{2YZ}$$

First, we look at the terms in the pricing kernel using the above definition of Y and Z:

$$\begin{aligned} (X-\psi_1)^{-\gamma} &= \left[X - \tfrac{(1+\gamma)\sigma^2_\eta}{2}\tfrac{X}{YZ}\right]^{-\gamma} \\ &= X^{-\gamma}\left[1 - \tfrac{(1+\gamma)\sigma^2_\eta}{2YZ}\right]^{-\gamma} \\ &\approx X^{-\gamma}\left[1 + \tfrac{\gamma(\gamma+1)\sigma^2_\eta}{2YZ}\right], \end{aligned}$$

where the last step we use the approximation that $\sigma^2_\eta/(YZ)$ is small.

Similarly we obtain the approximation:

$$(X-\psi_1)^{1-\gamma} \approx X^{1-\gamma}\left[1 - \frac{(1-\gamma^2)\sigma^2_\eta}{2YZ}\right].$$

Thus:

$$\frac{1}{E_1\left\{X^{1-\gamma}\left[1-\frac{(1-\gamma^2)\sigma_\eta^2}{2YZ}\right]\right\}} \approx \frac{1}{E_1(X^{1-\gamma})}\left\{1+\frac{E_1\left[\frac{(1-\gamma^2)\sigma_\eta^2 X^{1-\gamma}}{2YZ}\right]}{E_1(X^{1-\gamma})}\right\}$$

Substituting these into the optimal demand function, it follows:

$$Y \quad = Y^* + \frac{(1+\gamma)\sigma_\eta^2}{2}\left(B_{1Y}X + B_{2Y}\frac{1}{X}\right) \tag{30}$$

$$Z \quad = Z^* + \frac{(1+\gamma)\sigma_\eta^2}{2}\left(B_{1Z}X + B_{2Z}\frac{1}{X}\right), \tag{31}$$

where $B_{1Y}, B_{2Y}, B_{1Z}, B_{2Z}$ are constants. Equations (30) and (31) and the associated constants are derived in the appendix.

The effect of remaining unresolved uncertainty in background risk involves trading in both stock, bond and derivatives. The extent of the trade in derivatives depends crucially on the size of the coefficient $B_{2,Y}$ (and $B_{2,Z}$).

6 Conclusion

There is an extensive literature on background risk, which arises from stochastic cash flows generating non-marketable wealth. Since this risk cannot be directly hedged, it affects the derived risk aversion of the individual agent. Generally speaking, as documented by several researchers and synthesized by Gollier (2001), in the presence of background risk, agents generally become more risk-averse in their derived utility functions, and thus, behave like a more risk-averse agent would, in the absence of such a risk. This, in turn, influences the demand for insurance.

There has been rather less attention devoted to the pricing of securities and sharing rules in equilibrium, when agents in the economy face background risk. A notable early paper is by FSS, who analyze the equilibrium in such an economy, and derive the portfolio demand of individual agents in this equilibrium. The agents take into account their non-marketable background risk in optimally determining their demand for the marketable assets. Specifically, FSS show that agents with background risk depart from the linear sharing rule that characterizes behavior in complete markets, and may buy or sell non-linear contingent claims such as options.

In this paper, we take the presence of background risk and its influence on risk taking in a different direction. We explore how the prices of assets are determined in equilibrium by the interplay of portfolio demands across agents in the economy, which take into account the background risks they face. If the agents face different background risks, it is reasonable to expect that their portfolio demands will differ: this is the argument first made by FSS. We extend this argument to the multi-period setting and derive the changes in the portfolio demand of different agents as the background risk is revealed over time. To the extent that these changes differ across agents, it establishes a motive for trading, even in the presence of symmetric (full) information across agents.

The equilibrium we obtain turns out to be fairly complex, since portfolio demands depend on the changed derived risk aversion of agents in the presence of background risk, which in turn, depends on the portfolio holdings. We break this circularity by considering special cases of the evolution of background risk, as well as by using some approximations. We confirm these results by numerical computations.

We have thus been able to derive a theory of trading in the presence of full information, without running afoul of the powerful no-trade results of Grossman and Stiglitz (1980) and Milgrom and Stokey (1982) in the context of asymmetric information models. We believe our theory can be extended in several directions to separate the trading in linear (stocks and bonds) versus non-linear (options) claims. Potentially, our theory is testable, if one can quantify the influences of background risks such as human and housing wealth. This could be of interest to researchers in asset pricing, where the focus is mainly on returns, but could also be related to the aspects of trading analyzed in this paper.

7 Appendix: Derivation of Demand Equations: The General Case

In this appendix, we derive the demand equations in the general case [equations (30) and (31)].

Substituting the approximations:

$$
\begin{aligned}
(X-\psi_1)^{-\gamma} &\approx X^{-\gamma}\left(1+\frac{\gamma(\gamma+1)\sigma_\eta^2}{2YZ}\right), \\
(X-\psi_1)^{1-\gamma} &\approx X^{1-\gamma}\left(1-\frac{(1-\gamma^2)\sigma_\eta^2}{2YZ}\right), \\
\frac{1}{E_1\left[X^{1-\gamma}\left(1-\frac{(1-\gamma^2)\sigma_\eta^2}{2YZ}\right)\right]} &\approx \frac{1}{E_1[X^{1-\gamma}]}\left\{1+\frac{E_1\left(\frac{(1-\gamma^2)\sigma_\eta^2 X^{1-\gamma}}{2YZ}\right)}{E_1[X^{1-\gamma}]}\right\}
\end{aligned}
$$

in the demand equation (25) and defining $Y \equiv x_{2,M}+a$ and $Z \equiv x_{2,N}-a$ we have

$$
\begin{aligned}
Y &\approx E_1\left[X^{-\gamma}\left(1+\frac{\gamma(\gamma+1)\sigma_\eta^2}{2YZ}\right)\left(X+a-\frac{(1+\gamma)\sigma_\eta^2}{2Y}\right)\right]\frac{1}{E_1[X^{1-\gamma}]}\left[1+\frac{E_1\left(\frac{(1-\gamma^2)\sigma_\eta^2 X^{1-\gamma}}{2YZ}\right)}{E_1[X^{1-\gamma}]}\right] \\
&\cdot \left(X-\frac{(1+\gamma)\sigma_\eta^2 X}{2YZ}\right)+\frac{(1+\gamma)\sigma_\eta^2}{2Y}. \qquad (32)
\end{aligned}
$$

Then, assuming terms in $\sigma^4/Y^2 \to 0$ we have

$$
\begin{aligned}
Y &\approx E_1\left[X^{-\gamma}\left(X+a+\frac{\gamma(1+\gamma)\sigma_\eta^2(X+a)}{2YZ}-\frac{(\gamma+1)\sigma_\eta^2}{2Y}\right)\right]\frac{1}{E(X^{1-\gamma})}X \\
&\cdot \left[1-\frac{(1+\gamma)\sigma_\eta^2}{2YZ}+\frac{E_1\left(\frac{(1-\gamma^2)\sigma_\eta^2 X^{1-\gamma}}{2YZ}\right)}{E_1[X^{1-\gamma}]}\right]+\frac{(1+\gamma)\sigma_\eta^2}{2Y} \\
&= \left\{\frac{E_1[X^{-\gamma}(X+a)]}{E_1[X^{1-\gamma}]}X+\frac{(1+\gamma)\sigma_\eta^2}{2E_1[X^{1-\gamma}]}X\left[E_1\left(\frac{\gamma X^{-\gamma}(X+a)}{YZ}\right)-E_1\left(\frac{X^{-\gamma}}{Y}\right)\right]\right\}
\end{aligned}
$$

$$\cdot \quad \left[1 - \frac{(1+\gamma)\sigma_\eta^2}{2YZ} + \frac{(1+\gamma)\sigma_\eta^2}{2} \frac{E_1\left(\frac{(1-\gamma)X^{1-\gamma}}{YZ}\right)}{E_1[X^{1-\gamma}]}\right] + \frac{(1+\gamma)\sigma_\eta^2}{2Y} \tag{33}$$

Multiplying out the brackets and with $\sigma^4/Y^2 \to 0$ and $\sigma^4/YZ \to 0$:

$$\begin{aligned} Y \quad &\approx \quad \frac{E_1[X^{-\gamma}(X+a)]}{E_1[X^{1-\gamma}]}X + \frac{(1+\gamma)\sigma_\eta^2}{2E_1[X^{1-\gamma}]}X\left[E_1\left(\frac{\gamma X^{-\gamma}(X+a)}{YZ}\right) - E_1\left(\frac{X^{-\gamma}}{Y}\right)\right] \\ &- \quad \frac{(1+\gamma)\sigma_\eta^2}{2YZ}\frac{E_1[X^{-\gamma}(X+a)]}{E_1[X^{1-\gamma}]}X + \frac{(1+\gamma)\sigma_\eta^2}{2}\frac{E_1\left(\frac{(1-\gamma)X^{1-\gamma}}{YZ}\right)}{E_1[X^{1-\gamma}]}\frac{E_1[X^{-\gamma}(X+a)]}{E_1[X^{1-\gamma}]}X \\ &+ \quad \frac{(1+\gamma)\sigma_\eta^2}{2Y} \qquad (34) \\ &= \quad \frac{E_1[X^{-\gamma}(X+a)]}{E_1[X^{1-\gamma}]}X + \frac{(1+\gamma)\sigma_\eta^2}{2E_1[X^{1-\gamma}]}\left\{\left[E_1\left(\frac{\gamma X^{-\gamma}(X+a)}{YZ}\right) - E_1\left(\frac{X^{-\gamma}}{Y}\right) + \right.\right. \\ &\quad \left.\left. E_1\left[X^{-\gamma}(X+a)\right]\frac{E_1\left(\frac{(1-\gamma)X^{1-\gamma}}{YZ}\right)}{E_1[X^{1-\gamma}]}\right]X - E_1[X^{-\gamma}(X+a)]\frac{X}{YZ} + \frac{E_1(X^{1-\gamma})}{Y}\right\} \qquad (35) \end{aligned}$$

Finally, the approximate explicit solution is found by substituting $Y = Y^*$, $Z = Z^*$ to obtain

$$\begin{aligned} Y \quad &\approx \quad Y^* + \frac{(1+\gamma)\sigma_\eta^2}{2E_1[X^{1-\gamma}]}\left\{\left[E_1\left(\frac{\gamma X^{-\gamma}(X+a)}{Y^*Z^*}\right) - E_1\left(\frac{X^{-\gamma}}{Y^*}\right)\right.\right. \\ &+ \quad \left.\left. E_1\left[X^{-\gamma}(X+a)\right]\frac{E_1\left(\frac{(1-\gamma)X^{1-\gamma}}{Y^*Z^*}\right)}{E_1[X^{1-\gamma}]}\right]X - E_1[X^{-\gamma}(X+a)]\frac{X}{Y^*Z^*} + \frac{E_1(X^{1-\gamma})}{Y^*}\right\} \\ &= \quad Y^* + \frac{(1+\gamma)\sigma_\eta^2}{2}\left\{B_{1Y}X - \frac{E_1[X^{-\gamma}(X+a)]X}{E_1(X^{1-\gamma})Y^*Z^*} + \frac{1}{Y^*}\right\} \\ &= \quad Y^* + \frac{(1+\gamma)\sigma_\eta^2}{2}\left[B_{1Y}X + B_{2Y}\frac{1}{X}\right], \end{aligned}$$

where

$$\begin{aligned} B_{1Y} \quad &= \quad \frac{1}{E_1[X^{1-\gamma}]}\left[E_1\left(\frac{\gamma X^{-\gamma}(X+a)}{Y^*Z^*}\right) - E_1\left(\frac{X^{-\gamma}}{Y^*}\right) + E_1\left[X^{-\gamma}(X+a)\right]\frac{E_1\left(\frac{(1-\gamma)X^{1-\gamma}}{Y^*Z^*}\right)}{E_1[X^{1-\gamma}]}\right], \\ B_{2Y} \quad &= \quad \frac{E_1[X^{1-\gamma}]}{E_1[X^{-\gamma}(X+a)]} - \frac{E_1[X^{1-\gamma}]}{E_1[X^{-\gamma}(X-a)]}. \end{aligned}$$

Similarly,

$$Z = Z^* + \frac{(1+\gamma)\sigma_\eta^2}{2}\left[B_{1Z}X + B_{2Z}\frac{1}{X}\right],$$

where

$$B_{1Z} = \frac{1}{E_1[X^{1-\gamma}]}\left[E_1\left(\frac{\gamma X^{-\gamma}(X-a)}{Y^*Z^*}\right) - E_1\left(\frac{X^{-\gamma}}{Z^*}\right) + E_1\left[X^{-\gamma}(X-a)\right]\frac{E_1\left(\frac{(1-\gamma)X^{1-\gamma}}{Y^*Z^*}\right)}{E_1[X^{1-\gamma}]}\right]$$

$$B_{2Z} = \frac{E_1[X^{1-\gamma}]}{E_1[X^{-\gamma}(X-a)]} - \frac{E_1[X^{1-\gamma}]}{E_1[X^{-\gamma}(X+a)]}.$$

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