Studies in the Astronomy of the Roman Period IV Solar Tables Based on a Non-Hipparchian Model

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Ptolemy's solar model was the only fully worked-out kinematic solar model from classical antiquity of which medieval and early modern astronomers had knowledge. Its features are well known. It is a simple eccentre model, entirely in the plane of the ecliptic, with a constant eccentricity in a direction that is fixed relative to the intersections of the ecliptic and celestial equator. Hence the period of the sun's anomaly is equal to the tropical year, so that the tropical year and the the astronomical seasons do not vary in duration. The specific eccentricity, apsidal line, and tropical year adopted by Ptolemy were first established by Hipparchus in works that have not come down to us (*Almagest* III 1 and 4). Moreover, since the angle between the ecliptic and the equator is also constant (*Almagest* I 12), there will be no variation in the maximum and minimum noon altitudes of the sun on summer and winter solstices, nor any variation in the extreme points of the horizon where the sun rises and sets at the solstices. To calculate the sun's ecliptic coordinates for a given date, it suffices to tabulate a single mean motion representing the sun's uniform motion along its eccentre (*Almagest* III 2), together with an anomaly table with this mean motion as its sole argument (*Almagest* III 6). The sun's longitude is obtained by adding together the mean motion, the

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correction from the anomaly table, and the longitude of the eccentre's apogee; the sun's latitude is always zero.

About a generation before Ptolemy completed the *Almagest*, the Platonic philosopher Theon of Smyrna gave an account of the Hipparchian solar model in his work *Things Useful in Mathematics for the Reading of Plato*[1]. Theon uses the model to illustrate the interchangeability of the epicyclic and eccentric models (ed. Hiller, 152–172). Between the detailed demonstration for the sun and the extension of the equivalence to the other planets, Theon interjects the following remarks (ed. Hiller, 172–173):

> These things are demonstrated also for the other planets. But the sun appears to behave in this way without deviation according to both models, because its periods of restitution, i.e. that in longitude, that in latitude, and that in "depth" and what is called anomaly, are so close to each other that most of the mathematicians believe that they are equal, each being $365^1/_4$ days.
>
> But those who inquire more precisely believe [1] that the period in longitude, in which the sun traverses the zodiacal circle from some point to the same point and returns from solstice to the same solstice and from equinox to the same equinox, is approximately the stated time, with the sun's longitude restored to the same point at the same hour after a four-year period; [2] that the period in anomaly, according to which the sun becomes furthest from the earth and consequently smallest in the appearance of its size and slowest in its motion towards the trailing signs, or contrariwise closest to the earth and consequently seeming largest in size and fastest in motion, is approximately $365^1/_2$ days, again with the sun appearing at the same point in depth at the same hour after a two-year period; and [3] that the period in latitude, in which the sun returns from the same point to the same point, being furthest north or furthest south, so that the shadows of the same gnomons are seen again equal, is approximately $365^1/_8$ days, with the sun returning to the same point in latitude after an eight-year period.

These are very odd things for Theon to say. While discussing the Hipparchian model, both before and after this passage, Theon draws attention to the specific and apparently fixed longitude of the apogee, Gemini $5^1/_2°$, which is required if the model (epicyclic or eccentric) is to reproduce the lengths of the astronomical seasons that Theon assumes earlier in his work. Yet here he asserts that the sun's period of anomaly is a quarter of a day longer than the tropical year, so that the longitude of the solar apogee must advance by close to $^1/_4°$ per year. The specification of a solar latitudinal motion is also a surprise, since in the treatment of the Hipparchian model it was assumed that the components of the model lie in the plane of the ecliptic.

Theon does, however, mention solar latitude in two other places, stating that the sun deviates from the ecliptic by as much as $^1/_2°$ either way, making a total latitudinal range of 1°. This information was derived from a source, presumably the philosopher Adrastus of Aphrodisias whom Theon often cites, that was also used by Chalcidius (4th century A.D.) in his commentary on Plato's *Timaeus*. Chalcidius' assertion (ed. Waszink, 139) that the sun diverges about half a degree to the north or to the south of the ecliptic agrees closely with Theon. The fifth-century writer Martianus Capella seems to have misunderstood Chalcidius' likening the ecliptic to a plumb-line (*libra*), so that in his version (*De Nuptiis* 867, ed. Willis, 328) the latitudinal deviation of $\pm{}^1/_2°$ occurs only in the zodiacal sign Libra. It is plausible, but not demonstrable, that Theon's solar periodicities came from the same source in which he and Chalcidius found the maximum solar latitude of $^1/_2°$.

Solar latitude was a widespread concept in earlier Greek astronomy[2]. In particular, Eudoxus believed that the sun's solstitial rising and setting points on the horizon exhibited a small variation, and if we are to believe Simplicius, his homocentric sphere model for the sun involved three spheres expressly to account for this phenomenon. In his extant *Commentary on the Phaenomena of Aratus and Eudoxus* Hipparchus dismisses a solar latitudinal motion, pointing out that lunar eclipse predictions accurate to within two digits, i.e. about $^1/_{12}°$, were being made on the basis of models that assumed that the earth's shadow (and hence the sun) were centred on the ecliptic. On the face of it this argument should have been decisive. Ptolemy does not so much as mention the notion of solar latitude in the *Almagest*. As for Theon's three periods, the neat geometrical progression of the fractions has the air, as Neugebauer observed, of a numerological speculation (1975, 630–631).

Two Greco-Roman papyri put Theon's remarks on solar theory in a new light. The first is a fragment from the top of a table of sexagesimal numerals that I came across in 1996 among the papyri from Oxyrhynchus at the Ashmolean Museum, Oxford; it is now published as *P. Oxy.* LXI 4174a[3]. The table is written on the back of a document; both hands can be dated paleographically to within a few decades of A.D. 200. Parts of seven columns, separated by vertical rulings, are present:

i. Integer multiples, from zero to at least five, of a sexagesimal number ending in 16.
ii. Integer multiples, from zero to at least six, of 0;59,5,48.
iii. Integer multiples, from zero to at least fourteen, of 0;59,9,30.
(vacant column)

iv. Integers from one to at least 11.

v. Integer multiples, from one to at least thirteen, of 0;2,27,50,40.

vi. Integer multiples, from one to at least thirteen, of a sexagesimal number beginning 0;2,2x (where x is an undetermined digit).

This is easy to recognize as a mean motion table. Since $^{360}/_{365;15}$ is approximately 0;59,8, the numbers in cols. ii–iii appear to be some sort of daily mean motions of the sun. The format is as in Ptolemy's *Handy Tables*, with a mean motion of zero corresponding to day 1, so that these columns evidently give the part of the mean motion corresponding to the actual day number in a calendar month, rather than the mean motion corresponding to a given number of elapsed days as in the *Almagest*[4]. The calendar was probably the Egyptian, with thirty-day months, but there is no way to prove this.

Columns v–vi, following a vacant column and an index column, are also clearly mean motions corresponding to the number of equinoctial hours in col. iv. If we multiply the base number of col. v, 0;2,27,50,40, by 24, we obtain a daily motion of 0;59,8,16, which is likely to be the base number of col. i. Conversely, dividing the base number of col. ii, 0;59,5,48, by 24, we obtain an hourly motion of 0;2,27,44,30, which could be the base number of col. vi. Thus we have (at least) three distinct tabulated mean motions, all close to the sun's mean motion in longitude.

The periods of these motions, assuming the third fractional place of each daily motion to be precise within ± 1, are:

$$\text{(a)}\ \frac{360}{0;59,8,16} = 365;14,55,32 \pm 0;0,6,11 \text{ days}$$

$$\text{(b)}\ \frac{360}{0;59,5,48} = 365;30,10,15 \pm 0;0,6,11 \text{ days}$$

$$\text{(c)}\ \frac{360}{0;59,9,30} = 365;7,18,39 \pm 0;0,6,10 \text{ days}$$

By way of comparison, Theon's solar periods of (a) longitude, (b) anomaly, and (c) latitude are:

(a) $365^{1}/_{4}$ days$=$365;15 days

(b) $365^{1}/_{2}$ days$=$365;30 days

(c) $365^{1}/_{8}$ days$=$365;7,30 days

The correspondence is beyond doubt, so that we can securely identify cols. i and v of the papyrus as a solar mean motion in longitude, cols. ii and vi as a mean motion in anomaly, and col. iii as a mean motion in latitude. We may further remark that none of the daily motions in the papyrus is precisely what one would obtain by dividing Theon's periodicities into 360° and either rounding or truncating at the third fractional place. On the other hand it is probably not a coincidence that the difference between the papyrus's mean daily motions in longitude and anomaly, 0;0,2,28, is exactly twice the difference between its mean daily motions in longitude and latitude, 0;0,1,14. In other words, the sun's apogee is presumed to advance at twice the speed that the sun's nodes precess. This suggests that the rates of mean motion, whatever their origin, have been adjusted according to some notion of numerical tidiness. Also the mean motion in longitude is too short to be a good sidereal year, and too long to be a good tropical year in the light of Hipparchus' investigations of precession.

Unfortunately the papyrus does not contain evidence of the epoch values, so that we do not know where the solar apogee and nodes were supposed to be for any particular date. Moreover, the existence of a mean motion table implies an accompanying anomaly table, and probably also a table giving latitude as a function of the mean motion in latitude, but the papyrus gives us no information about their nature.

Our second papyrus may take us a step towards filling these gaps. This fragment, in the collection of the Istituto Papirologico "G. Vitelli" (Florence), has to date been published only in a provisional edition by M. Manfredi (1966), without inventory or publication number, although subsequent references to it identify it as PSI inv. 515 and as PSI XV 1490, to appear in the long-awaited fifteenth volume of the *Papiri della Società Italiana*[5]. According to Manfredi, the hand of the astronomical text belongs to the second century A.D., and more probably to the first half of that century; documentary notes in the margins and a document written on the back were added about the end of the second or the beginning of the third century[6]. We give below a translation of Manfredi's provisional text, enclosing in brackets the breaks in the construable text as well as hypothetical restorations. The superscript numerals indicate the corresponding line numbers of the Greek text[7].

> [... in longitude] $^{(1)}$ 156;28,[xx],23; in depth $^{(2)}$ 51;14$^{?}$,7,44; in latitude 238$^{?}$;[xx,xx,xx]. $^{(3)}$ Again setting these out [in the] $^{(4)}$ second row of the [... thir]$^{(5)}$ty-day interval, we will extend downwards [the re$^{(6)}$maining] 29 rows and [...]$^{(7)}$ of one four-year interval [...]$^{(8)}$ adding on [...$^{(9)}$...]

> course [... in longitude?] (10) 44;[x]x,x,1?1?, [in depth? (11) x?]38;48,x[x?,xx? ...(12)...(13)...] in position? [...(14) in latitude?] 294;35,[xx,xx ...] (15) It is evident? that also [...(16)...] (17) procedure? outside? [...](18) of anomaly [...](19) we [shall set out] according to the [...](20) way. The sun in both mo(21)dels? [...](22) increases? and diminishes? [its] (23) motion by two degrees [24? minutes?] in a quad(24)rant? whether it travels situated on an [eccentric circle] (25) or [on an epi](26)cycle. For this has been [demonstrated?] by us in the (discussion) concerning (27) the sun's anomaly. (28) So since by two degrees 24? (minutes) [in] a quad(29)rant? the difference is [...](30) of the motion, I? added? [...](31) from? the quadrant [...](32) three degrees. For not [...(33)...] two are [...(34)... (35)...] (36) as? I? demonstrated? [...] (37) travel? [to the] mean? [...(38) semi?]circle [...(39)... (40)...] (41) approximately two fifths? [...(42) it is evident?] that if [...(43) from?] the two fifths [...(44)] will obtain [...(45)] of the 2;24? degrees? [...(46)] to be [...](47) the increment of ano[maly...]

As the translation makes evident, there are numerous breaks in the text, and indeed not a single line is complete in the papyrus. The gaps are, however, for the most part small. If Manfredi's arrangement of the three pieces of papyrus constituting the manuscript is correct, we have a vertical strip containing part of a single column of text from the papyrus roll, comprising forty-seven lines from top margin to bottom margin. (Manfredi expresses some uncertainty about a join of fragments at line 12.) Except in lines 1–5, in which the left edge of the column is preserved, the beginnings and ends of all lines are missing. The width of the column can, however, be estimated as roughly twenty-five letters on the basis of Manfredi's plausible restorations in lines 1–5, 19–20, and 24–28. Hence about half the text of the column is actually legible, and this is enough to give us some hope of making out the drift of the argument.

Lines 1–2 set out three sexagesimal numbers. The second and third are identified as "of depth" and "of latitude." The first number, which was the beginning of the list (it is preceded by the series-initiating particle *μὲν*), was without doubt "of longitude." Unless a numeral has broken off at the end of line 1, which is possible but I think unlikely, the mean motions are approximately as follows:

longitude: 156;28°
anomaly: 51;14°
latitude: 238°[8]

According to lines 3–6 these numbers are to be written in the second row of a table covering a thirty-day interval, and the "remaining twenty-nine days" are to be filled in. In other words, we are constructing a mean motion table representing the daily motion of a heavenly body. Because the text specifies the second line, and marks the operation as a repetition ("again"=πάλιν), it looks as if the lost preceding text gave mean motions to be entered for day 1 of the table, and then described how to obtain the numbers for day 2 by adding appropriate increments. The rest of the table would be produced by repeating this step as many times as needed.

The larger structure of the table can be inferred from the combination of a thirty-day table with the allusion to a four-year interval in line 7. In mean motion tables like Ptolemy's, which employ the old Egyptian calendar with its uniform years of 365 days, the calendrical units by which the tables are organized are equinoctial hours, days, Egyptian months of thirty days, Egyptian years, and larger cycles of years. A four-year periodicity is, however, present in the Alexandrian (reformed Egyptian) calendar of the Roman period, in which every fourth year ended with six extra "epagomenal" days instead of the usual five epagomenals. Thus in all likelihood the mean motions of line 2 pertain to the second day of the first month of the first year of a cycle of four Alexandrian years.

Now these mean motions are obviously not just the increments relative to an arbitrary epoch, as they would be in Ptolemy's mean motion tables. No heavenly body could possibly be imagined to travel 156° in longitude in a day or two! The table being constructed in the papyrus has to be of a different kind, containing the actual mean positions for specific dates in specific years. The numbers in lines 1–2 would pertain to the second day, Thoth 2, of an Alexandrian year beginning a four-year intercalation cycle. Since the first such cycle began with the regnal year Augustus 5 (26/25 B.C.), the year in question would be Augustus 5 + 4*n* for some integer *n*.

Manfredi quite reasonably supposed that the mean motions in this section of the papyrus were the moon's; and with this hypothesis Neugebauer concurred[9]. The moon is, after all, the only heavenly body having an apsidal line and nodes that move comparatively rapidly, justifying the separate tabulation of mean motions in longitude, anomaly, and latitude. Accepting this assumption, I attempted to find dates within a historically plausible range for which the mean motions of the papyrus approximately matched the lunar mean motions according to the theory of the *Almagest*. My search was fruitless. It was only after the discovery and identification of *P. Oxy.* LXI 4174a that it occurred to me that the mean motions might belong to the sun.

My guess turned out to be easy to verify. Since the sun's longitudinal period is very close to the mean length of the Alexandrian calendar year, the sun's mean longitude on a given calendar day of a particular year in the four-year cycle shifts only a small amount after each cycle. Using Ptolemy's tables, we find the mean tropical longitude of the sun at noon, Thoth 2, Augustus 5 as 156;11°; the papyrus has 156;28°. The agreement would be almost perfect for noon, Thoth 2, Augustus 85 (A.D. 55/56), or for 6 P.M., Thoth 2, Augustus 5. In other words, the mean position in longitude in line 1 of the papyrus is within a few minutes of the sun's mean position according to Ptolemy's model–how few minutes depends on the epoch time and date employed by the author of the papyrus, which we do not know. We may conclude, not only that the papyrus is giving solar mean motions, but also that its longitudes are tropical rather than sidereal. As is well known, Ptolemy's tropical longitudes are systematically too large by about 0;24° in 26 B.C., with the error increasing by about 0;0,15° per year[10]. The longitudes of the papyrus seem to exhibit a similar error, which could be equal to Ptolemy's error or a bit smaller.

The longitude of the sun's apogee, according to the papyrus, is approximately:

$$\lambda_A = 156;28° - 51;14° = 105;14°$$

This is about 37° higher than the actual longitude of the solar apogee for about A.D. 0; by contrast, the Ptolemy-Hipparchus apogee, 65;30°, is only about 3° too high for that date[11]. It seems reasonable to suppose that the solar theory behind this papyrus was essentially the same as underlay the mean motions of Theon and *P. Oxy.* LXI 4174a (the alternative being that *two* kinematic models for the sun with solar latitude were the basis of popular tables). Assuming the daily progress of 0;0,2,28° for the apsidal line that we deduced from *P. Oxy.* LXI 4174a, it would require about 159 years for the apogee to shift from Hipparchus' 65;30° (determined probably some time before 135 B.C.) to 105;14°. Hence in this respect the model could be held to be more or less consistent with Hipparchus' work.

The starting point for measuring (lunar) mean motion in latitude in Ptolemy's works is the northern limit of the orbit; one could also imagine using the ascending node as the starting point. According to the papyrus, the longitude of the starting point, whichever it is, was:

$$\lambda_B = 156;28° - 238° \approx 278°$$

This is fairly close to the winter tropic point. Assuming with *P. Oxy.* LXI 4174a that the nodes precess 0;0,1,14° per day, 159 years earlier the longitude of this point would have been about 298°.

Resuming the reading of the papyrus, we find in lines 10–14 several rather poorly preserved sexagesimal numbers. These may be further mean motions (not necessarily of the sun), but I can make no sense of them.

Lines 18–47 describe the construction of an anomaly table for the sun, which makes good sense following the construction of the solar mean motion table in lines 1–7. This table is ostensibly based on a theoretical treatment of epicyclic and eccentric solar models given elsewhere by the author of the text (this is the clearest indication that we are dealing with part of a treatise rather than an isolated procedure text). The most noteworthy element here is the maximum equation, which is 2;24° (line 45, imperfectly legible also in lines 23 and 28)[12]. In *Almagest* III 4, Ptolemy derives from Hipparchus' solar eccentricity ($^1/_{24}$ of the radius of the solar eccentre) a maximum equation of 2;23°. This is not significantly different from the 2;24° of the papyrus; indeed a discrepancy of 0;1° could easily arise from the use of a table of chords tabulated for fewer angles than Ptolemy's. It follows that the solar model of the papyrus incorporated the Hipparchian eccentricity, while attributing a rotation to the apsidal line.

It is difficult to tell what the anomaly table was like. I rather get the impression from the disjointed instructions in lines 28–47 that the equations were not computed trigonometrically as in Ptolemy's equation tables, but interpolated linearly between zero at the apogee and perigee and the maximum, 2;24°, at the quadrants.

One of the points of interest about *P. Oxy.* LXI 4174a and the tables described in PSI inv. 515 is that, unlike many of the astronomical tables preserved on papyrus, they are truly kinematic: like Ptolemy's tables, they "exhibit the uniform circular motion" of a model. P. Oxy. LXI 4174a is late enough so that its format could conceivably have been influenced by the example of the *Handy Tables*. We can be practically certain, however, that the treatise of which PSI inv. 515 is a fragment was composed before the publication of the *Almagest* (about A.D. 150). The mean motion table and its companion, the equation table, were inventions that antedated Ptolemy.

I conclude with some remarks on the solar model with which I believe that the two papyri discussed above are concerned. Of the two novel aspects of this model, the shifting apogee is the easier one to understand. Hipparchus established his familiar solar model from two observed time intervals, the number of days between spring equinox and summer solstice and the number of days between summer solstice and autumnal equinox. Three centuries later, Ptolemy claimed to have observed exactly the same season lengths, so that the eccentricity and apsidal line of the model must have remained stationary. But what would happen if one observed slightly different time intervals? One possibility, if the total interval from summer solstice to autumnal equinox appeared to remain the same, but the interval from spring equinox to summer solstice appeared to become shorter, would lead to an eccentricity fairly close to Hipparchus', but an apogee with a greater longitude. There is very good reason to believe that Hipparchus himself carried out such a second determination of the solar model a few years after the one reported in the *Almagest*, obtaining from season lengths $94^1/_4$ and $92^1/_2$ days an eccentricity of approximately $^{2;20}/_{60}$ (instead of $^{2;30}/_{60}$) in the direction of longitude 67° (instead of 65;30°)[13]. One can well imagine how someone might have interpreted such data as evidence for a shifting apogee. Eventually, of course, such a model would lead to predictions that ought to have been in conflict with reasonably careful observations of the dates of the equinoxes and sol-

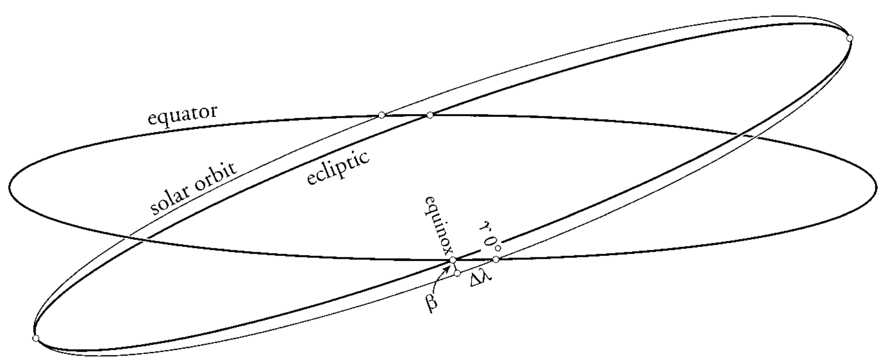

Fig. 1. Maximum displacement of longitude of vernal equinoctial point corresponding to solar latitude (β) when solar nodes coincide with tropical points.

stices; but the general acceptance of Ptolemy's solar observations during the later Roman period shows that such checking was not a matter of routine.

The empirical motivation of a theory of solar latitude is much more problematic, especially since, as Hipparchus noted, it flies in the face of accurate eclipse theory. The presumed nodal motion of $^1/_8$° per year is so slow that one may wonder what kind of observations of solstitial altitudes or rising and setting points could have been adduced to detect it. One noteworthy effect of solar latitude, if the entire solar model was assumed to lie in the oblique plane, is that the intersections of the plane of the sun's orbit with the equator should oscillate around the intersections of the ecliptic with the equator (cf. fig. 1). This would lead to a trepidation of the equinoctial and solstitial points, with a period equal to the period of solar latitude (about 2900 years). With a maximum latitude of 0;30°, the amplitude of the trepidation would be about 2;16° along the ecliptic. Naturally this back-and-forth motion of the solstitial and equinoctial points would interfere with any attempt to determine a solar model by means of observed dates of equinoxes and solstices. Solar latitude may thus have had something to do with the origins of the concept of the trepidation of the equinoxes.

CLASSICAL TEXTS

Chalcidius.
Timaeus a Calcidio translatus commentarioque instructus, ed. J. H. Waszink. Plato Latinus 4. London, 1962.
Martianus Capella.
Martianus Capella, ed. J. Willis. Leipzig, 1983.
Hipparchus.
Hipparchi in Arati et Eudoxi Phaenomena commentariorum libri tres, ed. K. Manitius. Leipzig, 1894.
Ptolemy.
Ptolemy's Almagest, trans. G. J. Toomer. London: Duckworth, 1984.
Theon of Smyrna.
Theonis Smyrnaei philosophi platonici expositio rerum mathematicarum ad legendum Platonem utilium, ed. E. Hiller. Leipzig, 1878.

BIBLIOGRAPHY OF MODERN WORKS

Baccani, D.
1992: *Oroscopi greci. Documentazione papirologica*, Ricerca papirologica 1, Messina: Sicania.

Britton, J.
1992: *Models and Precision: The Quality of Ptolemy's Observations and Parameters*, New York: Garland Publishing.
Jones, A.
1999: *Astronomical Papyri from Oxyrhynchus*, 2 vols. in 1, Memoirs of the American Philosophical Society 233, Philadelphia: American Philosophical Society.
Manfredi, M.
1966: "Presentazione di un testo astronomico e discussione di un documento di Antinoe," *Atti dell'XI Congresso Internazionale di Papirologia Milano 2 – 8 Settembre 1965*, Milano. 237–248.
Neugebauer, O.
1975: *A History of Ancient Mathematical Astronomy*, 3 vols., Berlin: Springer-Verlag.
Rawlins, D.
1991: "Hipparchos' Ultimate Solar Orbit & the Babylonian Tropical Year," *Dio* 1.1, 49–66.
Thurston, H.
1995: "Three solar longitudes in the *Almagest* due to Hipparchus," *Journal for the History of Astronomy* 26, 164.

NOTES

1. For Theon's date see Neugebauer (1975) 949.
2. Neugebauer (1975) 629–631.
3. Text, translation, and commentary in Jones (1999) v. 1, 170–171 and v. 2, 164–167.
4. On the *Handy Tables* (of which no satisfactory edition exists) see Neugebauer (1975) 969–1028; the structure of the solar and lunar mean motion tables is described on pp. 983 and 986–987.
5. Neugebauer (1975) 945; Baccani (1992) 36. I have so far not succeeded in obtaining a photograph or more recent information about the papyrus.
6. Baccani (1992) 36 dates the papyrus to the first/second century, i.e. presumably c. A.D. 100.
7. In the transcription in Manfredi (1966) 240–241, the line numbers from 40 on are off by one.
8. In some contexts in Greek astronomy the mean motion in latitude is expressed in units called *bathmoi* ("steps") such that 1 *bathmos*=15°. Here the units are clearly degrees, since a mean motion cannot exceed 24 *bathmoi*.
9. Neugebauer (1975) 945.
10. Britton (1992) 41–42.
11. Britton (1992) 44.
12. In line 28 Manfredi reads what seems to be the fractional part as 21 (ka), but with doubt about the alpha.
13. Rawlins (1991), Thurston (1995).