Ancient & Medieval Traditions in the Exact Sciences

Essays in Memory of Wilbur Knorr

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Pappus' Notes to Euclid's *Optics*

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A question that has been much discussed is, to what extent the Renaissance theorists of linear perspective were inspired by ancient sources (Andersen (1987a)). Here I want to consider a related question, whether writers on mathematical optics in antiquity themselves studied problems of perspective. To avoid confusion, I ought to explain that I mean by a perspective problem the problem of constructing a plane figure or drawing that will present the same impression to the eye as a given configuration of objects in space. I am not concerned here, on the one hand, with problems in ancient optics that do not involve a plane projection, nor, on the other, with theorems that we can now recognize as applicable to the theory of perspective but that do not appear in optical contexts in the historical sources. This limitation cuts down the material available for consideration severely. Indeed, I am aware of just two texts that cast much light on this topic.

One of these is Ptolemy's instructions for drawing a picture of a ringed globe in Book VII of the *Geography*. Since the chapter has been the subject of excellent articles by Neugebauer (1959) and Andersen (1987b), I can limit myself to a few points. Ptolemy tells us that such pictures were something of a tradition: 'many people (he writes) have attempted such a demonstration, but they have manifestly undertaken it in a most illogical fashion.' He, on the contrary, promises to do it 'in as close agreement as possible with the optical representations.' Ptolemy's construction falls into two parts. In the first part he makes a correct central projection of four points of each ring from a suitably chosen eye point on to the plane through the poles of the globe. These points are sufficient to determine the ellipse into which each ring is projected. Ptolemy is very precise about how the rings should be drawn: 'In applying the rings, care must be taken that each passes through the four points mentioned above, in an oval shape and not one that terminates in a cusp at the points of intersection on the outermost circle, so that it does not give the illusion of a fracture, but rather it should take on a curvature even at this point that is comparable to the adjacent curvature, even if the bends that form the ends of the ellipse fall outside the circle that surrounds the figure; for this is seen to occur in real rings too.' It is worth noting also that Ptolemy contrives to have the plane of the figure simultaneously represent the plane through the eye and the poles as well as the projection plane, a device that may have been suggested by analemma constructions. The drawing of the globe itself, with the parallels and meridians, is not a valid central projection, although Ptolemy continues to use the eye-point to determine plausible arcs for the parallels. There were likely two reasons for the abandonment of the principles of optics. First, a correct projection of all the critical points would have complicated the construction considerably, and practically have required an auxiliary figure, which Ptolemy evidently wishes to avoid. Secondly, insisting on a perspective projection of the globe would have made it necessary to draw the globe far smaller than the rings, in order to keep the known part of the world visible between the rings, not to mention the resulting foreshortened view of the northern end of the map. I do not think, therefore, that the compromises that Ptolemy introduces in the latter half of his construction should be taken as evidence that he did not understand the relationship between perspective drawing and central projection, or that he did not know how to make a correct projection.

Not only in this chapter, but in all those parts of the *Geography* that deal with map projections, Ptolemy repeatedly shows his familiarity with the qualitative phenomena of perspective representation, at least so far as this concerns the circles of latitude and longitude on the globe. The rationale of his two world map projections should indeed be understood as much in terms of Ptolemy's desire to give the spectator the visual *impression* of a spherical surface as of the magnitude-preserving properties that characterize modern map projections.

The other text concerning perspective is much less well known than Ptolemy's: although both Heath's and Loria's histories of Greek mathematics discuss it, it seems to have gone unnoticed by students of optics. This is, I suspect, largely because of where it appears. Everyone knows that Pappus's *Collection* is a treasure trove of material for the historian of mathematics, but its sixth book is usually regarded as one of the more barren parts. Most of Pappus's Book VI is devoted to rather niggling commentary on a curriculum of books that were apparently read as a course in elementary astronomy in late antiquity. Pappus's comments would have been invaluable evidence for the contents of these books had any of them not survived, but alas! we have them all. This is not to say that there is nothing to learn from Book VI: it is a useful document for the textual critic and for the historian of education; and the occasional passing allusion reminds us of how much more Pappus knew about ancient mathematics than we ever shall. And when Pappus turns his attention to Euclid's *Optics*, he becomes positively interesting.

The *Optics* is itself a remarkable and under-studied book, which has come down to us in two rather mangled versions. Heiberg, the rediscoverer of one of these recensions, distinguished them as a more or less genuine Euclidean text and a later revision, which he ascribed on rather flimsy grounds to Theon of Alexandria, that favourite scapegoat for tampered-with mathematical texts. Be that as it may, there is enough doubt about how faithfully either of the extant versions represents what Euclid wrote to make any early witness to the text valuable, though this is only the beginning of Pappus's significance.

Euclid's purpose in the *Optics* was to develop a mathematical theory of how objects appear to the eye, which could be counterposed to the various philosophico-physical theories of vision that were current about 300 B.C., and especially to the atomic film images of the Epicureans. Euclid postulates that vision occurs when an οχις, or line of sight, falls upon an object, and that these lines of sight are a bundle of discrete rectilinear rays emanating cone-wise from a point in the eye. Perception of size and relative position is then merely derived from the angles between lines of sight.

The theorems of Euclid's *Optics* fall into several categories. A few use the supposition that the lines of sight are discrete and separated by gaps to explain various phenomena involving failure to see part or all of an object: for example Proposition 9 accounts in this way for the alleged phenomenon of a distant square's appearing rounded, which plays an important part in Epicurus's theory of vision. The largest group of propositions concerns the appearance of a geometrical figure, and the apparent relative sizes and positions of its components, when the eye is situated in a particular place. Thus Proposition 10 proves that more distant parts of a plane lying below the eye level are seen to be 'higher'; Proposition 6 proves that parallel lines are seen as furthest apart where they are nearest the eye; and Proposition 23 proves that the part of a sphere seen by the eye is less than a hemisphere and always appears circular. An extension of this kind of theorem concerns the locus of eye-points from which some aspect of an object's appearance is invariant, or conversely the locus of positions of the object that preserve its appearance as seen by a stationary eye. Finally there are theorems involving motion: for example Proposition 24 shows that as one approaches a sphere, the visible part of it becomes smaller and smaller, but appears bigger and bigger; and Proposition 53 shows that of objects moving parallel to each other at equal speed, the more distant from the eye appear to move more slowly. Euclid's purpose throughout is to explain what the eye sees, not to simulate it; there are no perspective problems in the *Optics*. Of course the concept of a bundle of lines of sight emanating from the eye-point contains the germ of central projection and hence of perspective drawing, but at this stage the potentiality remains undeveloped.

Six hundred years after Euclid, Pappus picks up on one particular theorem in the *Optics*, Propositions 34-35. This states that if a straight line joining the eye-point to the centre of a circle is at right angles to the circle's plane, the diameters of the circle will all appear equal; and likewise if the line is not perpendicular but is equal to the radius of the circle, or if the line makes equal angles with the diameters, the diameters will still appear equal; but if none of these conditions are met, the diameters will appear unequal. I will repeat here only the outline of part of Euclid's treatment of the last case.

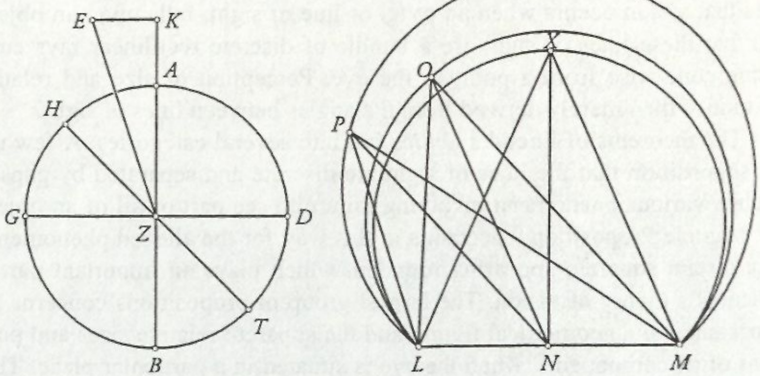
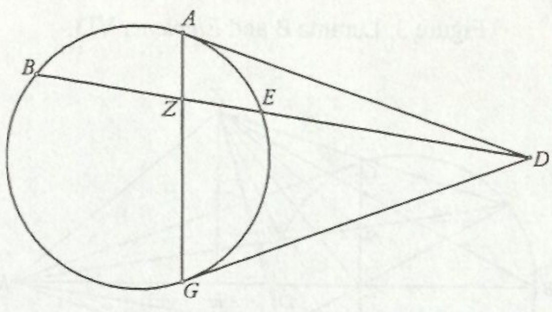


Figure 1. Euclid, *Optics* prop. 35

Let *ABDG* be the circle, with centre *Z*, and let *E* be the eye-point (Fig. 1). We assume that *ZE* is greater than the circle's radius, and not at right angles to the circle's plane. We drop the perpendicular *EK* to the plane, and draw diameters *AB* (through *K*) and *GD* at right angles to each other; and we also draw an arbitrary diameter *HZT*. By a lemma supplied by Euclid, angle *EZA* is less than angle *EZH*, which in turn is less than angle *EZG*. Now to compare the angles under which the three diameters are seen from *E*, Euclid makes an auxiliary construction. *LM* is drawn equal to any of the diameters, with *N* at its midpoint. We draw *NX* equal to *ZE*, and such that angle *LNX* equals angle *GZE* (which happens to be a right angle); and similarly we draw *NO* and *NP* equal to *ZE*, and such that angle *LNO* equals angle *HZE*, and angle *LNP* equals angle *AZE*. Thus the auxiliary drawing conflates the three planes through the eye (represented by *X*, *O*, and *P*) and each of the three diameters (represented by line *LNM*). To show that angle *LPM* is less than angle *LOM*, which is in turn less than angle *LXM*, Euclid merely has us draw the circular segments through *L*, *M*, and each of *X*, *O*, and *P*, and treats it as obvious that *P* falls outside the segment through *O*, and *O* outside the segment through *X*.

The first half of Pappus's excursus is a rewriting of Euclid's theorem. Pappus dispenses with the auxiliary construction entirely. Instead, he prefixes to the theorem a series of lemmas, which help to clarify Euclid's line of argument. All this makes Pappus's version somewhat longer than the original, but in compensation he has streamlined the central proof. Moreover, the lemmas may turn out to be applicable in other contexts.

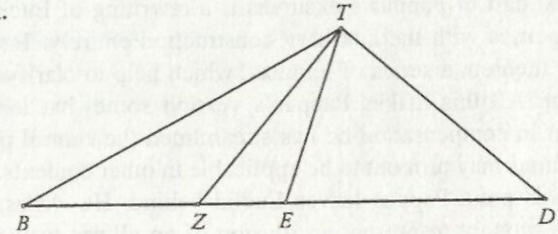
From this point Pappus leaves Euclid behind. He writes, 'Thus since the circle is thought to present an illusion of an ellipse to the eye, and its centre to be the apparent centre of the ellipse, the theorem has a not trivial difficulty. For it is possible to prove that *another* point in the circle is seen as the centre of the apparent curve.' I want to go through this theorem of Pappus's in some detail.



BZ:ZE = BD:DE

Figure 2. Lemma A.

Pappus will make use of three auxiliary theorems. The first of these, which I call Lemma **A**, is merely assumed here, but is proved in an entirely different context in Pappus's Book VII, among the lemmas to Euclid's *Porisms*. It states (Fig. 2) that if *DA* and *DG* are tangents to a circle, and the tangent points *A* and *G* are joined, then any transversal *BZED* drawn through *D* will be cut by the circle and the chord harmonically, that is, *BZ*:*ZE* = *BD*:*DE*. The second auxiliary theorem, which I call Lemma **B**, is as follows (Fig. 3). Let points *B*, *Z*, *E*, and *D* lie on a straight line such that *BZ*:*ZE* = *BD*:*DE*, and let point *T* be such that angle *BTZ* equals angle *ZTE*. Then angle *ZTD* will be right. Pappus proves this lemma here, but also assumes its converse, that if angle *ZTD* is right then angle *BTZ* equals angle *ZTE*. The third auxiliary theorem is Euclid's *Elements* VI 3 (Fig. 3): if points *B*, *Z*, and *E* lie on a straight line and angle *BTZ* equals angle *ZTE*, then *BZ*:*ZE* equals *BT*:*TE*, and conversely if the ratios are equal, the angles too are equal.



If *BZ*:*ZE* = *BD*:*DE*, then
angle *BTZ* = angle *ZTE* <=> angle *ZTD* is right

If angle *BTZ* = angle *ZTE*, then *BT*:*TE* = *BZ*:*ZE*

Figure 3. Lemma B and *Elements* VI3.

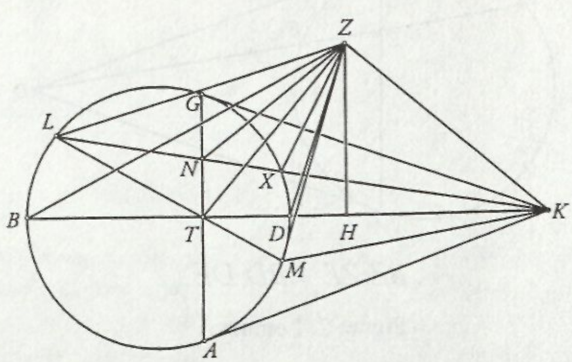


Figure 4. Pappus, *Collection* 100-101.

The construction in Pappus's theorem begins like Euclid's (Fig. 4). We draw the circle *ABGD*, and choose an eye-point *Z*, not in the circle's plane. We drop perpendicular *ZH* to the plane, and draw diameter *BD* passing through *H*. We then find *T* on this diameter, such that angle *BZT* equals angle *TZD*. And we draw *ATG* at right angles to *BD*, and the tangents at *A* and *G*, which intersect at *K* on *BD* produced. Pappus promises to show that when the eye is at *Z*, the circle will be seen as an ellipse with *T* as centre, and *AG* and *BD* as major and minor axes, and apparently exhibiting all the properties of the conic section.

T is, by construction, the apparent midpoint of *BD*, and by symmetry also the apparent midpoint of *AG*. It remains to show that it also appears to bisect an arbitrary chord *LTM*. We join *MK* and *LK*, which meets *AG* in *N* and the circle again in *X*. Now because *AG* joins two tangent points whose tangents meet at *K*, by Lemma **A** the transversal *BK* is divided so that *BT*:*TD* = *BK*:*KD*. But angles *BZT* and *TZD* are equal, so that by Lemma **B** angle *TZK* is right. It easily follows that *KZ* is perpendicular to the plane through *AG* and *Z*, so that angle *NZK* too is right. But by Lemma **A**, again, *LN*:*NX* = *LK*:*KX*, and hence by the converse of Lemma **B** angles *LZN* and *NZX* are equal. By *Elements* VI 3, *LZ*:*ZX* (or by symmetry *LZ*:*ZM*) equals *LN*:*NX* (or by parallels *LT*:*TM*). Thus by *Elements* VI 3 again, angle *LZT* equals angle *TZM*, which was to be proved.

But now comes the most curious part of Pappus's theorem. He is not content to have shown that *T* is the apparent centre of the apparent ellipse. He also maintains that lines drawn parallel to the apparent major axis *AG* will also *appear* parallel, but that the lines that will appear parallel to *BD* must actually be drawn through *K*. Thus Pappus knew the principle of the vanishing point in perspective. But let us see how he proves these assertions.

As it turns out, he doesn't bother to show that the lines parallel to *AG* appear as parallel: he takes this for granted, as well as that these lines are seen as perpendicular to *BD*. Perhaps he considered these things obvious from symmetry. For the other direction, however, he argues as follows.

We continue with the figure as drawn before (Fig. 5), but now we want to prove that an arbitrary line *LX* through *K* appears as parallel to *BD*. We drop *LO* perpendicular to *BD* and produce it to *R*. We have already shown that *LK*:*KX* = *LN*:*NX*, while by *Elements* VI 3 *LN*:*NX* = *LZ*:*ZX*, and by similar triangles *LK*:*KX* = *LR*:*XM*. Hence *LZ*:*ZX* = *LR*:*XM*. But by symmetry *LZ* equals *ZR* and *XZ* equals *ZM*, so that *LR*:*XM* = *ZR*:*ZM*, and thus triangle *LZR* is similar to triangle *XZM*. Consequently angles *LZR* and *XZM* are equal, and their halves too, angles *LZO* and *XZP*, are equal. And hence transversal lines *LO* and *XP*, which are seen to be parallel, are also seen to be equal, so that we have proved that *LXK* and *BDK* are seen as parallels.

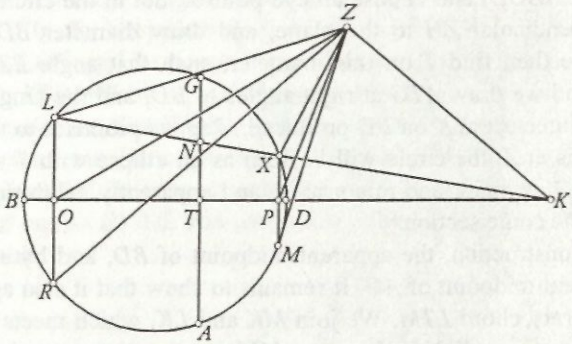


Figure 5. Pappus, *Collection* 102.

Or have we? I am afraid that Pappus is guilty of a blunder here. He has transferred a valid criterion for real parallel lines to the domain of appearances, where it no longer applies. After all, Euclid had proved in the *Optics* that the distance between parallel lines does not appear constant, but diminishes with distance from the eye. It is easy to see that the equality of tranversals *only* applies to the special pairs constructed in Pappus's figure. The truth is, of course, that one can only say that two lines are seen as parallel with reference to the plane that the eye assumes them to lie in.

In defense of Pappus, however, it has to be admitted that the apparent ellipse provides such a plane of reference, although Pappus does not make any explicit allusion to this fact. Pappus is tacitly assuming that the eye will interpret the circle and its chords as if they lay in a plane at right angles to line *ZT*, and under this assumption *K* does indeed serve as a vanishing point for lines perceived as parallel to *BD*.

This situation suggests interesting conclusions. Pappus shows himself aware of the concept of a vanishing point, and knows how to construct it. Yet his attempt to prove its property is a failure. Did Pappus accidentally hit upon the vanishing point, and was he merely lucky that his faulty proof was in support of a truth? I cannot help suspecting that he had some general knowledge of this and other rules of linear perspective, and that he was attempting to prove for a special problem something he knew was correct in general. If this was Pappus's discovery, it was a more acute insight than recent historians have been willing to accredit him with.

Lastly, Pappus sets himself the problem of finding the locus of points from which the circle will appear as an ellipse with a given point as its apparent centre. This is not difficult. In the preceding theorem, the only relationships that were determined between the eye-point *Z* and the construction in the circle's plane were that *Z* lies in the perpendicular plane through *BD*, and that angle *TZK* is right. Pappus therefore has no trouble showing that the locus of eye-points from which *T* is seen as the apparent centre is the circle in the perpendicular plane with *TK* as diameter.

I want to conclude with two remarks. First, there is the question of Pappus's originality. It is fashionable these days to describe Pappus as an utterly derivative mathematical magpie, collecting other people's work and straying from his sources only in order to make original mistakes. According to this view, the theorems on the perspective view of a circle would have to be due to some unknown earlier Greek geometer. Now Pappus was himself conscious that he lived in a debased period for mathematics, but he also insists that he tried harder than his contemporaries to achieve some modest advancement. I am inclined to take him at his word. The handful of occasions when he explicitly takes credit for something, he shows himself to be, not to be sure a second Archimedes, but a competent workman, familiar with the resources of his trade and able to pose and to solve nontrivial problems. There is nothing in the perspective theorems, seen as mathematics, that surpasses what we know Pappus could do; and although I do believe there was some background of studies of perspective that inspired Pappus's choice of problems, I am not unwilling to let him keep the credit for their execution.

And did Pappus have any influence on the perspective theorists of the Renaissance? Any early direct knowledge of the *Collection* is unlikely, since the rediscovery and first copying of the unique medieval manuscript seem to have occurred little before 1500. Greekless readers, moreover, had to wait for the publication of Commandino's Latin translation in 1589. Yet Pappus came very close to being a 'father of modern perspective' by an indirect route. Undoubtedly the most influential treatise on optics produced by the Latin Middle Ages was the *Perspectiva* of Witelo, written late in the 13th century. Now Witelo repeats several theorems from Pappus in the first book of the *Perspectiva*, a fact first pointed out by Witelo's editor Risner, and since rediscovered by Unguru (1974).What is interesting is that the borrowed propositions are all from Pappus's preliminary lemmas to the revised version of Euclid's *Optics* 35. Witelo almost certainly got this material from William of Moerbeke, who we know had access to the manuscript of Pappus (Jones (1986)). Having almost direct contact with the *Collection*, there was nothing to stop Witelo from incorporating the whole of Pappus's optical excursus in his own book. For some reason he did not do so, and the preliminary lemmas remain unapplied in the rest of the *Perspectiva*. Thus those parts of Pappus's theorems that would have been of the greatest interest to the Renaissance theorists just missed being transmitted to them in a form they could have read.

Note. This paper was read at the annual meeting of the Canadian Society for the History and Philosophy of Mathematics in 1989, and I subsequently gave a copy to Wilbur Knorr. In an astonishingly short time he sent me drafts of his two weighty articles 'On the principle of linear perspective in Euclid's *Optics*' (*Centaurus* 1991: 34, 193–210) and 'When circles don't look like circles' (*Archive for History of Exact Sciences* 1992: 44, 287–329), which were to some extent written under the stimulus of my bagatelle. Now that I am publishing it in his memory, I thought it best not to entangle the cross-references by revising the piece in the light of his contributions.

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