Triangular gear teeth

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Introduction

The first craftsmen to fashion metal gears did not have the advantage of specially shaped cutters. It may be that they used some ancestral version of a present day file to cut away the metal between the teeth and that they endeavored to leave square or triangular shaped teeth. This note addresses some properties of gears with triangular teeth. The serious reader will find most of what follows, and a great deal more, in the first few pages of Buckingham 1998.

Uniform rotation

When two cylinders roll against each other, without slipping, about fixed centers, their rates of rotation are inversely proportional to their radii. A cylinder of diameter *2D*, in contact with one of diameter *D*, will rotate half as fast as the other. For gears, the result is not quite true. After all, what would be meant by the diameter of a gear? As two gears turn, the point of contact moves up and down along the sides of the gear teeth so that the concept of a diameter is blurred. As the point of contact moves towards maximum radius at the tip of a tooth on one gear, it approaches the minimum radius on the other gear. For equal gears the ratio of the two diameters varies from r_{max}/r_{min} to r_{min}/r_{max} with the passage of each tooth. For a typical gear of 10 teeth, the height of the tooth is about a quarter of the outside radius. Therefore the ratio of the rates of rotation for two 10 tooth gears varies from 9/7 to 7/9. The variation occurs within the passage of each tooth. A gear with N teeth will rotate at a steady rate over rotations of any multiple of $2\pi/N$.

The effect of the varying radius can be compensated for by choosing appropriate shapes for the gear teeth. In fact, for teeth with profiles based on the involute curve the compensation is exact. Let us refer to the state of motion where both gears have constant angular velocity simply as the state of *uniform rotation*. We will meet below a necessary condition for uniform rotation. We will find that gears with triangular teeth do not satisfy this condition.

The fundamental theorem

When two gears rotate about fixed centers, they interact almost always at a single point of contact. (The “almost” is necessary because there may be isolated instants when the gears have two points of contact, more about which below). The profiles of the two interacting gear teeth will be curves with a common tangent at the point of contact. Ignoring frictional effects, the forces exerted by each gear on the other lie along the line perpendicular to the common tangent at the point of contact. Because they are in contact, the displacement s of the two gears along this normal are equal. In Figure 1 the rotations, i. e. the angular displacements, are $\mathrm{d}S/AB$ and $\mathrm{d}S/DE$. The ratio of the two rotations is:

(1) $$(\mathrm{d}S/_{AB}) : (\mathrm{d}S/_{DE}) = -DE : AB = -AC : CD$$

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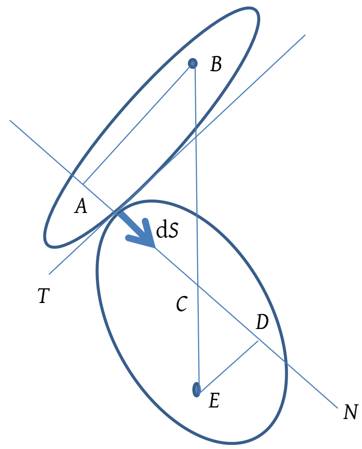

Figure 1. The two shapes *AB* and *CDE* rotate about the fixed centers *B* and *E*. *T* is the common tangent at the point of contact. *N* is perpendicular to *T*. The displacement d*S* is common to both shapes. But the rotations are d*S*/*AB* and d*S*/*ED*. The line of centers *BE* is cut by *N* to form *BC* and *CE* which have the same ratio as *AB* to *DE* . If the shapes are such that the point *C* remains fixed, then the two bodies will rotate uniformly, like two cylinders with radii *BC* and *CE*. *C* is called the pitch point.

The last equality follows from the similarity of triangles *ABC* and *CDE*, and the negative sign reminds us that the gears rotate in opposite senses. For uniform rotation, the ratio of the two rates of rotation remains constant even as the point of contact moves up and down along the gear teeth. For uniform motion, the point *C* on the line of centers is fixed. Two gears are said to have *conjugate gear tooth profiles* if they rotate uniformly. For a given tooth profile, it may be possible to construct one that is conjugate to it. The conjugate profile may be very similar to or very different from the given profile.

Conjugate teeth

Consider the situation where the gears rotate uniformly like cylinders with radii R_1 and R_2. Take the origin of rectangular coordinates (x, y) at the pitch point on the line of centers and take the y-axis along that line. The given data for a tooth profile has the form of a table with specified y values ranging from somewhat below zero to somewhat above. For each y value there is given an

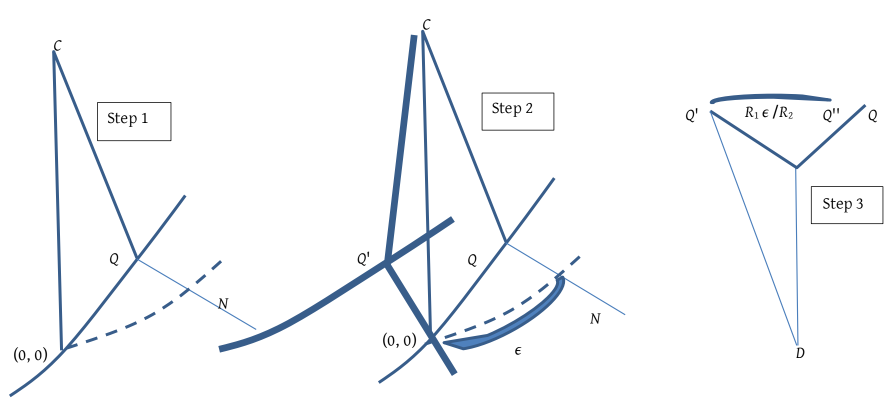

Figure 2. Steps in the construction of the point *Q''* (on the gear *D*) which is conjugate to *Q* (on gear *C*).

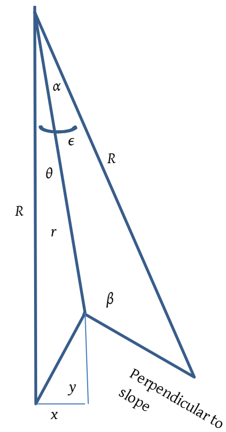

Figure 3. Angles and sides used to determine ϵ.

x value. When referring to a tooth profile, or comparing two profiles, we always mean the set of *x*, *y* values when the profile includes the origin $x = 0$, $y = 0$. We will say such a profile is in *standard position.*

We will also use polar coordinates *r*, *θ*, centered at the fixed center of the R_1 gear. For each *θ* there is an *r*. Adding a constant angle to all values of *θ* has no effect on the tooth shapes. The construction of the conjugate profile proceeds one point at a time.

Step 1. With the given profile in standard position, choose a point $Q = (x, y)$ on the profile. Define the radius vector (r, θ) from $C = (x = 0, y = R_1)$ to *Q*. Draw the line *N* perpendicular to the profile at *Q*.

Step 2. Rotate the line *CQN* clockwise about *C* until it passes through the origin. Record the angle of rotation ϵ. This moves *Q* to a point *Q*' on the line of contact.

Step 3. Locate the center of the conjugate gear at $D = (0,-R_2)$. Rotate the vector *DQ*' clockwise about *D* through an angle $R_1\,\epsilon/R_2$. *Q* now occupies a point *Q*'' on the conjugate gear . *Q*'' is not the same as *Q*. Even if the radii of the two gears are the same, $R_1 = R_2$, the distances *CQ* and *DQ*' are not equal. Only the pitch point $(x = 0, y = 0)$ maps onto itself.

Repeat steps 1-3.

The angle ϵ is easy to determine graphically. Some consideration of Figure 3 also produces ϵ:

(2) $\theta = \text{atan}\left(x / R - y\right)$

(3) $r = \sqrt{[x^2 + (R - y)^2]}$

(4) $\zeta = \pi/2 + \text{atan}(dy/dx)$

(5) $\beta = \pi/2 + \theta + \zeta$

(6) $\sin \lambda = (r \sin \beta)/R$

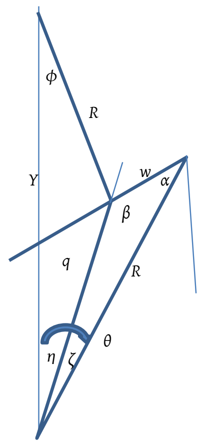

Figure 4. Quantities used to determine the rotation of two gears with triangular teeth. Y is the line joining the centers of the gears. R is the radius vector to the tip of a tooth on the upper gear.

(7) $\alpha = \pi - \beta - \lambda,\ \varepsilon = \theta + \alpha$

This procedure works for any given profile $r(\theta)$. We will consider next the special case of triangular teeth.

The interaction between triangular gear teeth

Figure 4 shows the contact between two gears with triangular teeth. The tip of a tooth on the upper gear bears against the face of a tooth on the lower gear. The line labeled R is the maximum radius of the upper gear, reaching from the center of the gear at the top of the figure to the tip of a tooth near the center of the figure. As that gear rotates counter clockwise, the angle ϕ increases, the tip of the tooth marked by R slides along the face of the lower gear, marked by w, causing the angle θ to increase as the lower gear rotates clockwise. We first calculate $\theta(\phi)$ and from this obtain $d\theta/d\phi$.

Application of the law of cosines and law of sines to the two triangles having sides YRq and Rwq produces:

(8) $q^2 = R^2 + Y^2 - 2RY\cos\varphi$

(9) $\sin\eta = {}^{R\sin\varphi}/_{q}$

(10) $\beta = \pi - \operatorname{asin}\left({}^{R\sin\alpha}/_{q}\right)$

(11) $\zeta = \pi - \alpha - \beta$

(12) $w = {}^{q\sin\zeta}/_{\sin\alpha}$

(13) $\theta = \eta + \zeta$

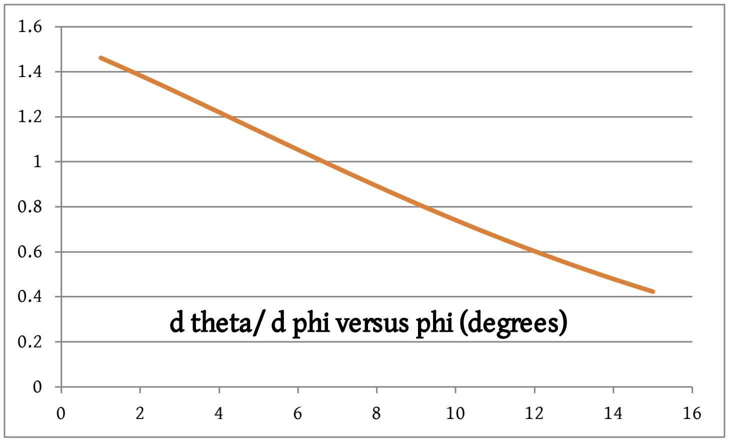

Figure 5. Showing the variation in dθ/dϕ as a single tip slides over a single face.

The awkward expression for β arises from the inverse sine function having its output between -π/2 and π/2. We regard Y, R, and α as fixed parameters, and eliminate β, η, and ζ to get:

(14) $\theta(\varphi) = \mathrm{asin}\left({}^{R\sin\varphi}/_{q}\right) + \mathrm{asin}\left({}^{R\sin\alpha}/_{q}\right) - \alpha$

(15) $q = \sqrt{(R^2 + Y^2 - 2RY\cos\varphi)}$

The preceding equations were used to calculate $\theta(\phi)$. The ratio of the rotation rates, $d\theta/d\phi$ was found by taking finite differences. For the range of values of $0 < \phi < 15$ degrees, the ratio of the rates of rotation of the two gears varies by a factor of about for the configuration $R_1 = R_2 = 10$, $R_{min} = 9$, $R_{max} = 11$. As the tip of the driving tooth slides along the face of the driven tooth, it may reach a point where the motion of the tip, as the gear rotates, is parallel to the face. The tip vector itself is perpendicular to the face. Then $d\theta$ vanishes. For continuous motion of the driven gear, the next teeth on the two gears must engage before this happens.

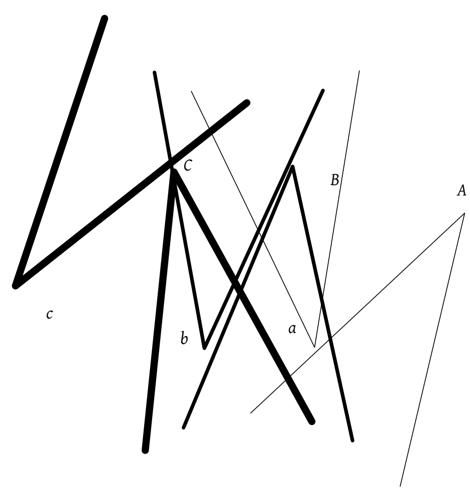

Figure 6. The discontinuity in rotation rates occurs as the gears pass through configuration *Bb*. The line of contact is *abBCa*. It comprises two circular arcs, *ab* and *BC*, and two discontinuous jumps, *bB* and *Ca*.

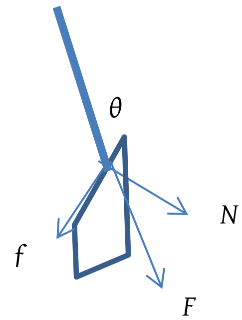

Figure 7. Friction.

Another way to measure the variation in the ratio of the rotation rates is to simulate the actual gears using computer generated gears that can be rotated independently. Begin with the tip of a tooth on gear 1 just touching the face of a tooth on gear 2. Rotate gear 1 through a small angle dϕ. Then advance gear 2 in small steps to find the angle dθ that just maintains the contact of the tip to the face. Repeat for several intervals dϕ within the passage of one tooth. Figure 6 shows the results of one such measurement.

A, *B*, and *C* mark the tip of a single tooth at three times. The positions of the tip of the mating tooth on a second identical gear, at the same three times, are marked *a*, *b*, and *c*. At time *A*, the tip – *a* – of the tooth on the upper gear makes contact with the face of the tooth on the lower gear. At time *B*, the teeth meet face to face. At *C*, the tip of the lower gear meets the face of the upper gear. The rotation of each gear during the time intervals *AB* and *BC* is measured by the arc traced out by the tips of the two teeth. Thus the ratio of the rotation of the upper gear to the lower gear is *ab* : *AB* for the first time interval and *bc* : *BC* for the second. These ratios are about 3:4 and 4:3. This demonstrates that the gears must rotate at different rates in order to maintain contact.

When the tip of a tooth of one gear slides along the surface of the tooth of the other gear, friction will oppose the motion. The force *F* that the tip exerts on the face can be resolved into a normal part *N* and a friction part *f*.

(16) $f = F \cos\theta,\ N = F \sin\theta$

Sliding will occur if $f > \lambda N$ where λ is the coefficient of static friction. Thus $\tan\theta < 1/\lambda$ is a necessary condition for sliding to be possible. If θ is bigger than atan $(1/\lambda)$ the gears jam. In Figure 8, the lower gear drives the upper. At the instant shown contact is being lost at *A* and transferred

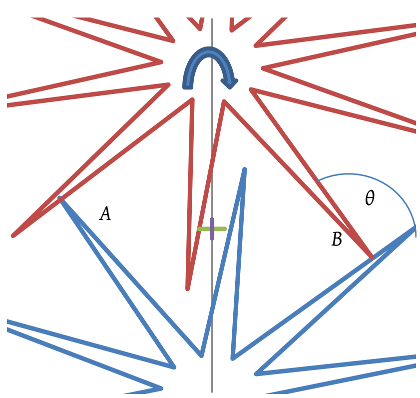

Figure 8. The gears jam when the pressure angle θ exceeds atan($1/\lambda$).

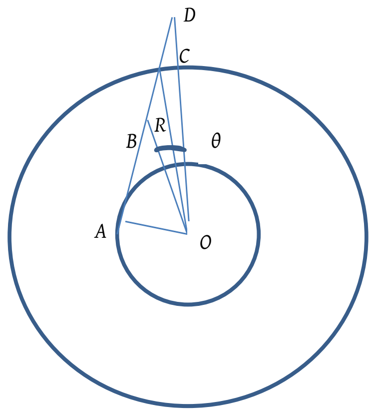

Figure 9. The sides of a triangular tooth are tangent to a circle *OA* < *OC*.

to *B*. However the pressure angle θ is too large to satisfy the sliding condition, and the gears are jammed. Pushing harder won't help.

Let us now construct the tooth profile conjugate to the triangle. We know already it will not be a triangle, because gears with triangular teeth do not transmit uniform rotations. We begin by constructing the line of contact using the condition that the normal to the profile must pass through a fixed point,—the fundamental theorem.

The correspondence between points on the triangular gear tooth and points on the line of contact

Take polar coordinates with origin at the center of the gear. Measure angles positive clockwise from the line joining the centers of the two gears. Then functions of the form $r(\theta)$ can be used to describe the shape of the gear tooth profile and of the line of contact. We will use the notation $r(\theta)$ for the gear tooth profile and $r_p(\theta)$ for the line of contact. There is a one to one correspondence between the two, so our task can be stated as: Given a point r, θ on the tooth profile, find the corresponding point on the line of contact. We will find that this latter point has the same radial coordinate, so our strategy is to pick r, find θ such that r, θ is on the tooth profile, and then find θ_p such that r, θ_p is on the line of contact.

Define the triangular gear tooth with the aid of a circle of radius A as shown in Fig. 9. Here *OA* has length A, *OC* has length R, *OB* has length r and angle θ from the vertical. The circle *OC* is the pitch circle. The line *ABCD* defines one side of the triangular tooth profile. The arbitrary point *B* has coordinates r, θ. We note *DOB* = θ.

The following sequence of calculations takes us from a point on the tooth profile to a point on the line of contact.

(17) $\psi = \mathrm{asin}(A/r)$

(18) $\theta = \psi - \mathrm{asin}(A/R)$

(19) $\varphi = \mathrm{acos}(r \cos \psi/R)$

In rectangular coordinates with origin at the pitch point,

(20) $x_p = r \sin(\psi - \varphi)$

(21) $y_p = R - r\cos(\psi - \varphi)$

Or, if you like,

(22) $x_p = {}^1/_R\, \sqrt{(r^2 - A^2)}\, [A - \sqrt{(A^2 + R^2 - r^2)}]$

(23) $y_p = {}^1/_R\, [(A^2 + R^2 - r^2) - A\sqrt{(A^2 + R^2 - r^2)}]$

if we take R to be the unit of length. We obtain the dimensionless equations:

(24) $x_p = \sqrt{(r^2 - A^2)}\, [A - \sqrt{(1 + A^2 - r^2)}]$

(25) $y_p = (1 + A^2 - r^2) - A\sqrt{(1 + A^2 - r^2)}]$

—which show that the line of contact depends only on the ratio A/R. This procedure can be reversed: given points x_p, y_p on the line of contact, find corresponding points r, θ on the tooth profile. And finally, the same procedure can be used to take as given the point on the line of contact, and to find the corresponding point on the tooth profile of a gear conjugate to the initial gear with the triangular teeth. In this case:

(26) $r_2 = \sqrt{[(R_2 + y_p)^2 + x_p^{\,2}]}$

(27) $\psi_2 = \mathrm{acos}(R_2 \cos\varphi / r_2)$

(28) $\varepsilon_2 = {}^{-R_1}/_{R_2}\, \varepsilon_1$

(29) $\theta_2 = \psi_2 - \varphi - \varepsilon_2$

The cut-away-everything-that-doesn't-look-like-a-gear method

Consider a gear blank with as yet undefined teeth. If both gears are to have the same number of teeth, then the blank gear will complete one clockwise revolution in the same time that the

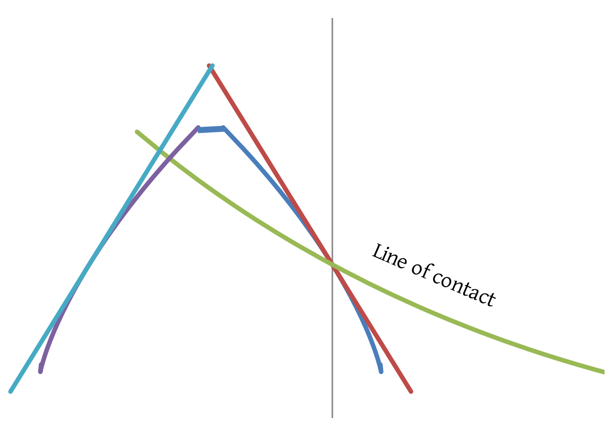

Figure 10. The triangular tooth profile, the line of contact, and the profile that is conjugate to the triangle.

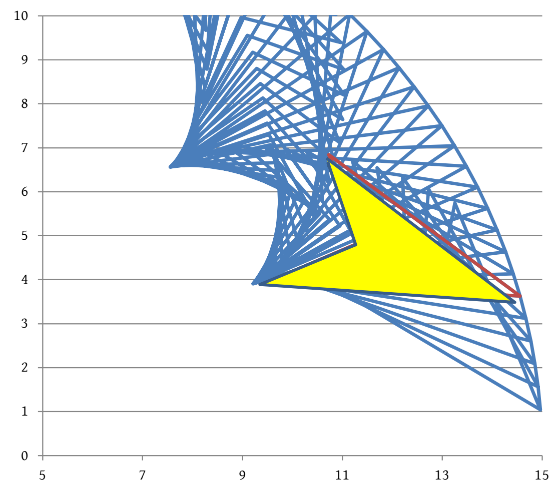

Figure 11. A gear with triangular teeth, only a slice of which is seen here, is rolled around a fixed circular gear blank, cutting away, as it goes, any material that gets in its way. What remains is the conjugate tooth shape conjugate to the triangle. Compare to the conjugate shape determined by analysis in figure 10.

other completes one counter clock wise revolution. If one gear rotates at the rate ω, the other will rotate at the rate $-\omega$. If now we consider the situation from the view point of an observer for whom the first gear is at rest, then, for him, the second gear rotates at the rate -2ω about its center, and rolls around the circumference of the stationary gear. If we imagine the second gear as cutting away at the blank gear every place they overlap, what remains is the shape of the conjugate gear.

Summary

1. Mating gears with triangular teeth do not rotate uniformly. The driven gear speeds up and slows down during the passage of each tooth. For larger rotations, the motion is uniform.

2. The ratio of the fastest to the slowest speed is roughly $(r_{max}/r_{min})^2$ where r_{max} is measured to the tip of a tooth and r_{min} to the gap between teeth. The ratio is within a few percent of unity for gears with many teeth . But it can be any times unity for small gears.

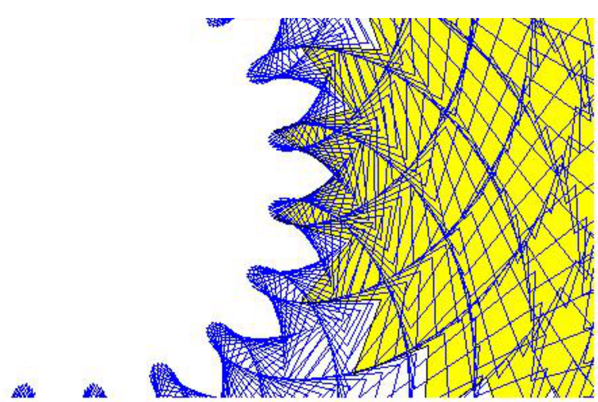

Figure 12. A gear with triangular teeth and its conjugate.

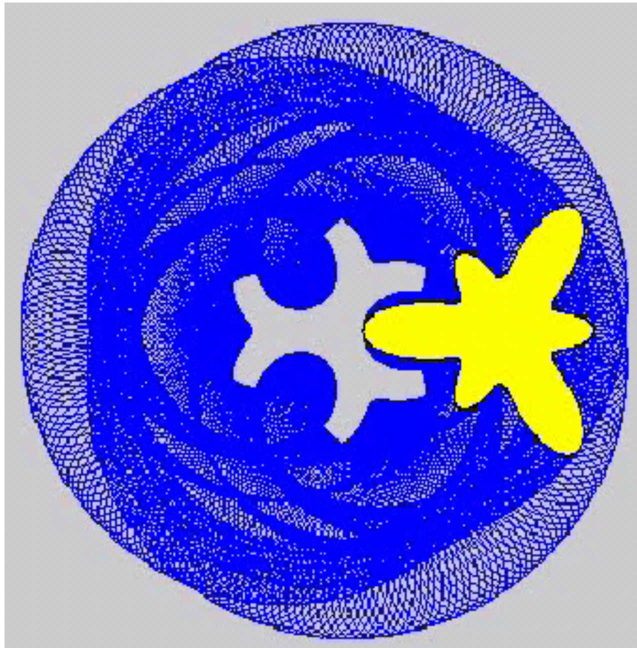

Figure 13. A conjugate pair.

3. Tooth profiles conjugate to the triangle are nearly triangular. The conjugate profiles resemble slightly rounded triangles. So it may be that makers of early gears were able to smooth out the motion of the gears by slightly rounding off the triangular teeth.

4. Finally, Christián Carlos Carman is to blame for these notes being prepared and offered up to honor our dear colleague, Jim Evans.

References

Buckingham, E. 1998. Analytical mechanics of gears. New York: Dover.