The equant in India redux

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The *Almagest* equant plus epicycle model of planetary motion is arguably the crowning achievement of ancient Greek astronomy. Our understanding of ancient Greek astronomy in the centuries preceding the *Almagest* is far from complete, but apparently the equant at some point in time replaced an eccentric plus epicycle model because it gives a better account of various observed phenomena.[1] In the centuries following the *Almagest* the equant was tinkered with in technical ways, first by multiple Arabic astronomers, and later by Copernicus, in order to bring it closer to Aristotelian expectations, but it was not significantly improved upon until the discoveries of Kepler in the early 17th century.[2]

This simple linear history is not, however, the whole story of the equant. Since 2005 it has been known that the standard ancient Hindu planetary models, generally thought for many decades to be based on the eccentric plus epicycle model, in fact instead approximate the *Almagest* equant model.[3] This situation presents a dilemma of sorts because there is nothing else in ancient Hindu astronomy that suggests any connection whatsoever with the Greek astronomy that we find in the *Almagest*. In fact, the general feeling, at least among Western scholars, has always been that ancient Hindu astronomy was entirely (or nearly so, since there is also some clearly identifiable Babylonian influence) derived from pre-*Almagest* Greek astronomy, and therefore offers a view into that otherwise inaccessible time period.[4]

The goal in this paper is to analyze more thoroughly the relationship between the *Almagest* equant and the Hindu planetary models. Four models will play a role in our analysis. These are the equant plus epicycle, the eccentric plus epicycle, the concentric equant plus epicycle, and the Hindu model. The first three are geometric in nature. Each one of these has a deferent that carries an epicycle, and each one has some scheme for non-uniform motion on the deferent. Both the epicycle and the non-uniform motion result in so-called anomalies, i.e. departures from uniform, or mean, motion. The Hindu model is not directly geometric but, as we shall see,

1 James Evans, "On the function and probable origin of Ptolemy's equant," *American journal of physics,* 52 (1984), 1080-9; Noel Swerdlow, "The empirical foundations of Ptolemy's planetary theory," *Journal for the history of astronomy*, 35 (2004), 249-71; Alexander Jones, "A Route to the ancient discovery of nonuniform planetary motion," *Journal for the history of astronomy*, 35 (2004); Dennis W. Duke, "Comment on the Origin of the Equant papers by Evans, Swerdlow, and Jones," *Journal for the History of Astronomy* 36, (2005) 1-6.

2 N. M. Swerdlow and O. Neugebauer, *Mathematical Astronomy in Copernicus's De revolutionibus* (Springer-Verlag, New York and Berlin, 1984) 41-48.

3 D. W. Duke, "The equant in India: the mathematical basis of Indian planetary models," *Archive for History of Exact Sciences*, 59 (2005) 563-576.

4 O. Neugebauer, "The Transmission of planetary theories in ancient and medieval astronomy," *Scripta mathematica*, 22 (1956) , 165-192; D. Pingree, "The Recovery of Early Greek astronomy from India," *Journal for the history of astronomy*, vii (1976), 109-123; B. L. van der Waerden, "Ausgleichspunkt, 'methode der perser', und indische planetenrechnung," *Archive for history of exact sciences*, 1 (1961), 107-121.

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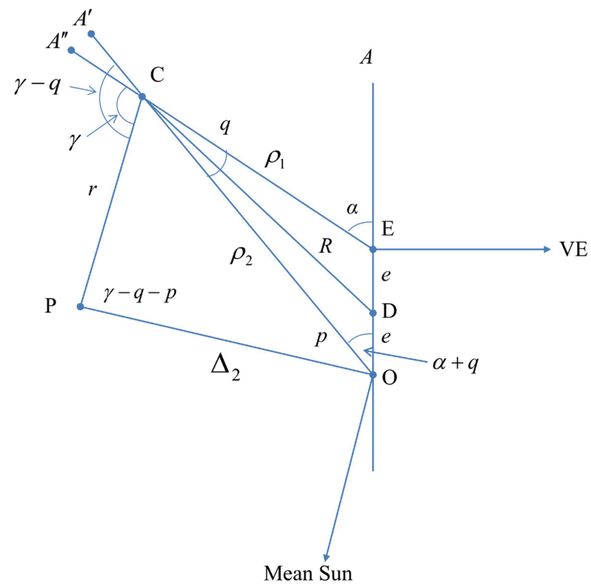

Figure 1. The Almagest equant plus epicycle for an outer planet. The Earth is at O, the equant point is at E, the center of the deferent is at D (the midpoint of OE), the center of the epicycle is on the deferent at C, and the planet is at P. Calculation of the lengths ρ_1, ρ_2, and Δ_2 in terms of e, r, and the angles α and γ, and application of the law of sines to the triangles OEC and OCP, yield the equations q and p, and form the basis of our analysis.

approximates the equant plus epicycle using individual elements found in the concentric equant plus epicycle. For all four models, however, the mathematical relationship between the true longitude λ and the mean longitude $\bar{\lambda}$ is of the form:

(1) $$\lambda = \bar{\lambda} + q + p$$

where the equation of center q is the anomaly associated with the non-uniform motion on the deferent and the equation of the epicycle p is the anomaly associated with the motion on the epicycle. To be more specific, let us consider the outer planets. Given the mean longitude $\bar{\lambda}$ of the planet, the longitude λ_A of the planet's apogee, and the mean longitude $\bar{\lambda}_S$ of the sun, let

(2) $$\alpha = \bar{\lambda} - \lambda_A = \alpha_0 + \omega_P (t - t_0)$$

and

(3) $$\gamma = \bar{\lambda}_S - \bar{\lambda} = \gamma_0 + \omega_a (t - t_0)$$

where ω_P is the mean motion in longitude of the planet and ω_a is the mean motion on the epicycle. If we let ω_S be the mean motion in longitude of the Sun, then the requirement that the radius of the epicycle points in the same direction as the mean Sun implies the relation:

(4) $$\omega_S = \omega_P + \omega_a$$

Then, with the convention $R = 1$ throughout, in terms of the static variables e and r and the dynamic variables α and γ, the calculation of q and p for the equant plus epicycle model is (see Fig. 1):

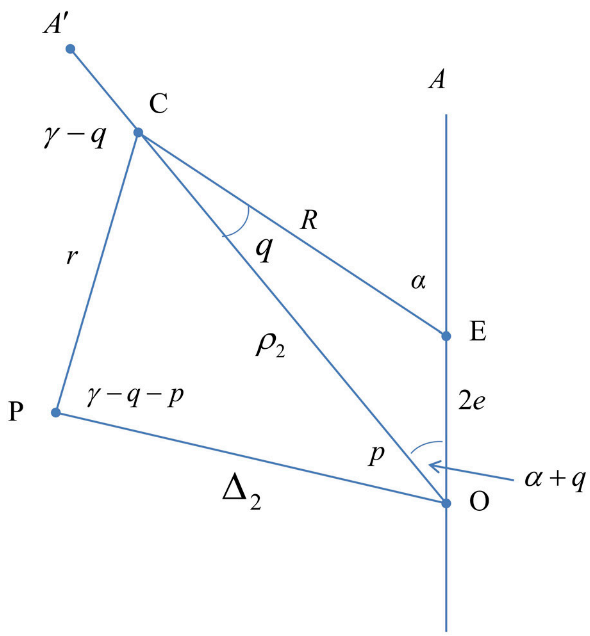

Figure 2. The eccentric plus epicycle model. As in Figure 1 except now the center of the deferent is at E.

(5) $$\rho_1 = \sqrt{1-(e\sin\alpha)^2} - e\cos\alpha$$

(6) $$\rho_2 = \sqrt{(\rho_1 + 2e\cos\alpha)^2 + (2e\sin\alpha)^2}$$

(7) $$\Delta_2 = \sqrt{(\rho_2 + r\cos(\gamma-q))^2 + (r\sin(\gamma-q))^2}$$

(8) $$\sin q(\alpha) = \frac{-2e\sin\alpha}{\rho_2} = \frac{-2e\sin(\alpha+q)}{\rho_1}$$

(9) $$\sin p(\gamma-q) = \frac{r\sin(\gamma-q)}{\Delta_2} = \frac{r\sin(\gamma-q-p)}{\rho_2}$$

Similarly, for the eccentric plus epicycle model we have (see Fig. 2):

(10) $$\rho_2 = \sqrt{(1+2e\cos\alpha)^2 + (2e\sin\alpha)^2}$$

(11) $$\Delta_2 = \sqrt{(\rho_2 + r\cos(\gamma-q))^2 + (r\sin(\gamma-q))^2}$$

(12) $$\sin q(\alpha) = \frac{-2e\sin\alpha}{\rho_2} = -2e\sin(\alpha+q)$$

(13) $$\sin p(\gamma-q) = \frac{r\sin(\gamma-q)}{\Delta_2} = \frac{r\sin(\gamma-q-p)}{\rho_2}$$

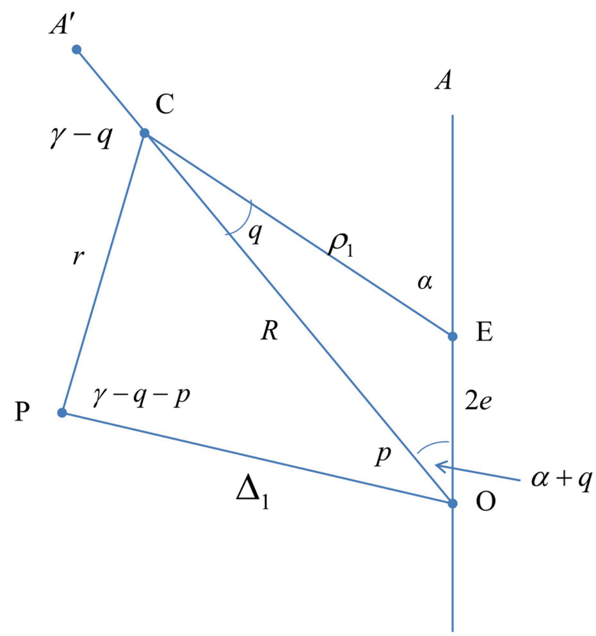

Figure 3. The concentric equant plus epicycle. As in Figure 1 except now the center of the deferent is at O.

Note that the results for these two models are formally quite similar. Both models have the property that while q is a function of just the dynamic variable α, the angle p is, through both its numerator and the denominator, a function of both dynamic variables α and γ. This means that if your goal is to compute q and p using tables and interpolation, a strategy widely used in both Greek and Hindu astronomy, then the table for q can be simple, just a column for α_i values and a column for the corresponding $q(\alpha_i)$ values. The table for the numerator of p is also just a pair of columns. However, the table for the denominator Δ_2 of p would, if built in the most direct way, have as many rows as we have γ values and as many columns as we have α values, and thus be uncomfortably large. For the eccentric plus epicycle model there are no surviving tables that tell us how this problem was solved, if indeed it ever was solved, but the *Almagest* gives a clever interpolation scheme for the equant plus epicycle that reduces the number of columns down to just a few additional auxiliary columns.[5]

Let us now consider the concentric equant plus epicycle model (see Fig. 3). The computation of q and p is:

(14) $$\rho_1 = \sqrt{1-(2e\sin\alpha)^2} - 2e\cos\alpha$$

(15) $$\Delta_1 = \sqrt{(1+r\cos(\gamma-q))^2 + (r\sin(\gamma-q))^2}$$

(16) $$\sin q(\alpha) = -2e\sin\alpha = \frac{-2e\sin(\alpha+q)}{\rho_1}$$

5 G. Van Brummelen, "Lunar and planetary interpolation tables in Ptolemy's *Almagest*," *Journal for the history of astronomy*, 25 (2004) 297-311.

(17) $$\sin p(\gamma - q) = \frac{r\sin(\gamma - q)}{\Delta_1} = r\sin(\gamma - q - p)$$

For this model the tables for the computation of *q* and *p* both require just two columns, so no special interpolation scheme is needed.

Finally, let us consider the ancient 4-step Hindu planetary model. For the time being we will consider a specific variant (henceforth 4Ha, the 'a' distinguishing it from other variants, 4Hb and 4Hc, that will appear later) found in the *Aryabhatiya* of Aryabhata, written around 500 CE.[6] As mentioned earlier, the text is not specifically geometrical, but instead refers to two functions, *manda* and *sighra*, the same as the functions *q* and *p* of the concentric equant plus epicycle, and in terms of the basic variables $\alpha = \bar{\lambda} - \lambda_A$ and $\gamma = \lambda_S - \bar{\lambda}$ these functions are computed as

(18) $$\sin q(\alpha) = -2e\sin\alpha$$

and

(19) $$\sin p(\gamma) = \frac{r\sin\gamma}{\sqrt{(1 + r\cos\gamma)^2 + (r\sin\gamma)^2}}$$

In terms of these functions the text leads us through the following four steps:

with argument $\alpha = \bar{\lambda} - \lambda_A$ compute $v_1 = \bar{\lambda} + ½q(\alpha)$

with argument $\gamma = \lambda_S - v_1$ compute $v_2 = v_1 + ½p(\gamma)$

with argument $\alpha' = v_2 - \lambda_A$ compute $v_3 = v_2 + q(\alpha')$

with argument $\gamma' - \lambda_S - v_3$ compute $\lambda = v_3 + p(\gamma')$

It is straightforward to verify that the result of the four steps is equivalent to the following sequence of computations:

(20) $$\alpha' = \alpha + \tfrac{1}{2}q(\alpha) + \tfrac{1}{2}p(\gamma - \tfrac{1}{2}q(\alpha))$$

(21) $$\sin q' = -2e\sin(\alpha')$$

(22) $$\sin p' = \frac{r\sin(\gamma - q')}{\sqrt{(1 + r\cos(\gamma - q'))^2 + (r\sin(\gamma - q'))^2}}$$

(23) $$\lambda = \bar{\lambda} + q' + p'$$

Note that at every step of this calculation we are using only the simple single variable functions *q* and *p* that characterize the concentric equant plus epicycle model, albeit with arguments of varying complexity, so we can use simple, single variable tables and avoid completely the coupled variable problem that occurs in the equant plus epicycle and eccentric plus epicycle models. What is perhaps less obvious is that the specific series of 4Ha steps leads to values of $q' + p'$ that approximate very well, at least for small to moderate values of *e* and *r*, $q + p$ for the equant

6 D. Pingree, "History of Mathematical astronomy in India," *Dictionary of Scientific Biography*, 15 (1978), 533-633.

plus epicycle model, but not $q + p$ for the eccentric plus epicycle model. Our task now is to understand why this happens and how that might have been realized by some ancient mathematician.

Taking into account that the agreement between the equant plus epicycle model and the 4Ha model is best for small to moderate values of e and r suggests expanding $q + p$ for the equant and $q' + p'$ for the 4Ha model in power series in e and r and keeping the first and second order terms. This is something we can do just to see if we are on the right track, but of course this has nothing to do with how some ancient mathematician understood things. The results for the 4Ha model are:

(24) $$q' = -2e\sin\alpha + 2e^2\sin\alpha\cos\alpha - er\cos\alpha\sin\gamma + O(e^3, e^2r, er^2, r^3)$$

(25) $$p' = r\sin\gamma - r^2\sin\gamma\cos\gamma + 2er\sin\alpha\cos\gamma + O(e^3, e^2r, er^2, r^3)$$

and for the equant plus epicycle model:

(26) $$q = -2e\sin\alpha + 2e^2\sin\alpha\cos\alpha + O(e^3)$$

(27) $$p = r\sin\gamma - r^2\sin\gamma\cos\gamma - er\cos\alpha\sin\gamma + 2er\sin\alpha\cos\gamma + O(e^3, e^2r, er^2, r^3)$$

Therefore, $q' + p'$ and $q + p$ agree exactly through second order terms and differ only for third and higher order terms, and these terms are small unless either e or r (or both) get too large. Note that this agreement occurs even though q and q', and likewise p and p', differ already in the second order terms. This explains why we see such good agreement for small to moderate e and r. We now need to investigate how the ancient astronomer understood that such agreement could arise.

To approach this question, consider first a simpler but related problem. Suppose that for some reason we are using a simple concentric equant model, without epicycle, perhaps for the Sun or for the Moon at syzygy (and this is exactly what does happen in the ancient Hindu texts). Then we would likely have tables for the concentric equant equation of center:

(28) $$\sin q_{ce}(\alpha) = -2e\sin(\alpha)$$

so that we could compute, using interpolation, $q_{ce}(\alpha)$ for any value of the argument α, which need not be simply the angle α itself. Now suppose further that for some reason we decide to use instead an eccentric model, for which the equation of center is:

(29) $$\sin q_{ec}(\alpha) = \frac{-2e\sin(\alpha)}{\rho_2} = -2e\sin(\alpha + q_{ec})$$

But rather than compute a new table for q_{ec} we wonder whether we can compute q_{ec} using instead the table we already have for q_{ce}? The answer is yes, and here is how. Using only the first and third terms in the above relation we can compute q_{ec} iteratively as follows:

(30) $$\begin{aligned} q_{ec}^{(0)} &= \arcsin(-2e\sin\alpha) \\ q_{ec}^{(1)} &= \arcsin(-2e\sin(\alpha + q_{ec}^{(0)})) \\ q_{ec}^{(2)} &= \arcsin(-2e\sin(\alpha + q_{ec}^{(1)})) \\ &\text{etc.} \end{aligned}$$

But notice that all the table lookups for the right-hand sides come from the concentric equant table, so there is no need for a new table involving the more complicated middle term with

$$(31)\qquad \rho_2 = \sqrt{(1+2e\cos\alpha)^2 + (2e\sin\alpha)^2}$$

in the denominator. The convergence of the iterations is quite fast, and the computation of q_{ec} from a table of q_{ce} is therefore as exact as you need it to be. In fact, for small the moderate values of e/R you can stop after the first iteration step.

The immediate lesson from this exercise is that it is possible to eliminate a denominator in terms of a shifted function argument. The lesson will be applied repeatedly in the following.

Note that this scheme is similar to the first and third steps of the 4Ha model if we ignore the epicycle terms, apart from the fact that here we are shifting α by an amount q, while in the 4Ha model we are shifting by an amount ½q.

To get a hint about how this factor of ½ arises, let us set aside for the moment the fact that

$$(32)\qquad -2e\sin\alpha = -2e\sin(\alpha + q_{ec})\cdot\rho_2$$

is an exact consequence of the law of sines and instead try to understand it as a consequence of small eccentricity e/R. We might notice that

$$(33)\qquad \rho_2 = \sqrt{(1+2e\cos\alpha)^2 + (2e\sin\alpha)^2} \simeq 1 + 2e\cos\alpha + O(e^2)$$

So that

$$(34)\qquad \frac{-2e\sin\alpha}{\rho_2} \simeq \frac{-2e\sin\alpha}{1+2e\cos\alpha} \simeq -2e\sin\alpha(1-2e\cos\alpha) \simeq -2e\sin\alpha\cos q - 2e\cos\alpha\sin q = -2e\sin(\alpha+q)$$

where the second line follows from the approximation $1/(1 + x) \simeq 1 - x + O(x^2)$, the third line follows from the approximation $\cos q \simeq 1 + O(q^2)$, and the final line is the sine addition theorem. All of these are elementary results that can reasonably be presumed to be understood at any time following, say, Archimedes.[7]

Coming now to the factor of ½q that appears in the 4Ha model, let us consider the law of sines result for the equant:

$$(35)\qquad \sin q(\alpha) = \frac{-2e\sin\alpha}{\rho_2} = \frac{-2e\sin(\alpha+q)}{\rho_1}$$

Using the first equality and the fact that $\rho_2 \simeq 1 + e\cos\alpha + O(e^2)$, we have:

$$(36)\qquad \frac{-2e\sin\alpha}{\rho_2} = \frac{-2e\sin\alpha}{1+e\cos\alpha} \simeq -2e\sin\alpha - 2e\cos\alpha\cdot(-e\sin\alpha) = -2e\sin\alpha\cos\tfrac{1}{2}q - 2e\cos\alpha\sin\tfrac{1}{2}q = -2e\sin(\alpha+\tfrac{1}{2}q)$$

7 It is perhaps worth mentioning that Ptolemy uses the law of sines in *Almagest* IX 10 without even mentioning it. See *Ptolemy's Almagest*, transl. by G. J. Toomer (London, 1984), 7 n. 10 and 462 n. 96.

Or alternatively, using the second equality and

$$(37) \qquad \rho_1 = \sqrt{1-(e\sin\alpha)^2} - e\cos\alpha \simeq 1 - e\cos\alpha + O(e^2)$$

we have:

$$(38) \qquad \begin{aligned} \frac{-2e\sin(\alpha+q)}{\rho_1} &= \frac{-2e\sin(\alpha+q)}{1-e\cos\alpha} \\ &\simeq -2e\sin\alpha - 2e\cos\alpha\cdot(-2e\sin\alpha) - 2e\cos\alpha\sin\alpha \\ &= -2e\sin\alpha + 2e\cos\alpha\cdot e\sin\alpha \\ &= -2e\sin\left(\alpha+\tfrac{1}{2}q\right) \end{aligned}$$

Coming now to the full equant plus epicycle model and its approximation with the 4Ha model, our only remaining task is to explain the origin of the additional term ½*p* in the shifted argument $\alpha' = \alpha + ½q + ½p$. Let us begin with the expression for $q + p$ in the equant plus epicycle model, and proceed through a series of steps closely related to those we just finished above to show that once again $q + p = q' + p'$. In the process we will need one new result related to the epicycle, namely the approximate factorization

$$(39) \qquad \begin{aligned} \Delta_2 &= \sqrt{(\rho_2 + r\cos(\gamma-q))^2 + (r\sin(\gamma-q))^2} \\ &\simeq 1 + e\cos\alpha + r\cos\gamma \\ &\simeq (1+e\cos\alpha)(1+r\cos\gamma) \\ &\simeq \rho_2\Delta_1 \end{aligned}$$

Proceeding along the same lines as above, we find

$$(40) \qquad \begin{aligned} q+p &\simeq \sin q + \sin p \\ &= \frac{-2e\sin\alpha}{\rho_2} + \frac{r\sin(\gamma-q)}{\Delta_2} \\ &= -2e\sin(\alpha+\tfrac{1}{2}q) + \frac{r\sin(\gamma-q)}{(1+e\cos\alpha)\Delta_1} \\ &= -2e\sin(\alpha+\tfrac{1}{2}q)\cos\tfrac{1}{2}p + \frac{r\sin(\gamma-q)}{\Delta_1} - er\sin\gamma\cos\alpha \\ &= -2e\sin(\alpha+\tfrac{1}{2}q)\cos\tfrac{1}{2}p + \frac{r\sin(\gamma-q)}{\Delta_1} - 2e\cos(\alpha+\tfrac{1}{2}q)\sin\tfrac{1}{2}p \\ &\simeq -2e\sin(\alpha+\tfrac{1}{2}q+\tfrac{1}{2}p) + \frac{r\sin(\gamma-q)}{\Delta_1} \\ &= q' + p' \end{aligned}$$

Some Hindu texts mention that the agreement between $q + p$ and $q' + p'$ can be improved by iteration to convergence, just as explained above for q_{ec}.

The 4Ha model is not the only model found in the ancient Hindu texts. A similar variant that we might call 4Hb is equivalent to the following sequence of computations:

(41) $$\alpha' = \alpha + \tfrac{1}{2}q(\alpha + \tfrac{1}{2}\mathrm{p}(\gamma)) + \tfrac{1}{2}p(\gamma)$$

(42) $$\sin q' = -2e\sin(\alpha')$$

(43) $$\sin p' = \frac{r\sin(\gamma - q')}{\sqrt{(1 + r\cos(\gamma - q'))^2 + (r\sin(\gamma - q'))^2}}$$

(44) $$\lambda = \bar{\lambda} + q' + p'$$

Once again it is straightforward to verify that just as we found for the 4Ha model, $q' + p'$ for the 4Hb model is the same as $q + p$ for the equant model up to corrections of $O(e^3, r^3, e^2r, er^2)$.

A third variant, that we will call 4Hc, is according to the texts applicable only to the inner planets Mercury and Venus, and is equivalent to the following:

(45) $$\alpha' = \alpha + \tfrac{1}{2}p(\gamma)$$

(46) $$\sin q' = -2e\sin(\alpha')$$

(47) $$\sin p' = \frac{r\sin(\gamma - q')}{\sqrt{(1 + r\cos(\gamma - q'))^2 + (r\sin(\gamma - q'))^2}}$$

(48) $$\lambda = \bar{\lambda} + q' + p'$$

From the analysis above we know that the omission of the argument shift ½q in α' is equivalent to the omission of the denominator $\rho_2 \simeq 1 + e\cos\alpha$ in the expression

(49) $$\sin q = \frac{-2e\sin\alpha}{\rho_2}$$

that follows from the law of sines for the equant model. For Venus, and to a lesser extent Mercury, this omission might be justified by the fact that e/R is fairly small and so ρ_2 never varies from unity by more than a few percent. On the other hand, the inclusion of the shift ½p maintains the full expression for the equation of the epicycle. Even so, the epicycle of Venus is so large ($r/R \simeq 0.7$) that terms of order r^3 that are neglected in $q' + p'$ are not entirely negligible (or, to put it in terms that our ancient mathematician would use, the factorization $\Delta_2 \simeq \rho_2\Delta_1$ is no longer a good approximation). All in all, it seems that the ancient analyst was scrambling to some degree in the effort to account for Mercury and Venus in his model.

Note that the above development of an approximation to the equant in terms of two simple single variable functions uses only two essential results. The first involves elimination of the denominator ρ_2 in favor of a shifted function argument α + ½q, and the second involves the factorization $\Delta_2 = \rho_2\Delta_1$ and the elimination of the denominator ρ_2 in favor of an additional shift of ½p in the argument. Both of these maneuvers are approximations, but it is easy to check, for the values of interest for e/R and r/R, that they are usually good approximations. While the motivation for these developments is reasonably clear—a desire to evaluate the equant plus epicycle using only tables based on the concentric equant plus epicycle—we can never know with certainty what was the inspiration for this particular solution to the problem. However, if the ancient analyst ever considered the eccentric plus epicycle model, he would see immediately from the law of sines for the equation of center that the denominator ρ_2 can be eliminated with an argument shift, and in this case that shift is exact:

(50) $$\sin q(\alpha) = \frac{-2e\sin\alpha}{\rho_2} = -2e\sin(\alpha+q)$$

And this might inspire him to use the factorization $\Delta_2 = \rho_2\Delta_1$ in the law of sines for the epicycle and follow a similar path to trading the elimination of ρ_2 for an additional argument shift as follows:

(51) $$\begin{aligned} q+p &\simeq -2e\sin(\alpha+q) + \frac{r\sin(\gamma-q)}{\rho_2\Delta_1} \\ &\simeq -2e\sin(\alpha+q) + \frac{r\sin(\gamma-q)}{(1+2e\cos\alpha)\Delta_1} \\ &\simeq -2\sin(\alpha+q) + \frac{r\sin(\gamma-q)}{\Delta_1} - 2e\cos\alpha\cdot r\sin\gamma \\ &\simeq -2\sin(\alpha+q)\cos p + \frac{r\sin(\gamma-q)}{\Delta_1} - 2e\cos\alpha\sin p \\ &= -2\sin(\alpha+q+p) + \frac{r\sin(\gamma-q)}{\Delta_1} \\ &= q'+p' \end{aligned}$$

Alternatively, he might have just stared at the two law of sines results

(52) $$\sin q(\alpha) = \frac{-2e\sin\alpha}{\rho_2} = \frac{-2e\sin(\alpha+q)}{\rho_1}$$

(53) $$\sin p(\gamma-q) = \frac{r\sin(\gamma-q)}{\Delta_2} = \frac{r\sin(\gamma-q-p)}{\rho_2}$$

where the shifted arguments and eliminated denominators are more or less staring us in the face, and somehow found the inspiration to guess the answer, or made a few guesses and checks and finally stumbled upon the right answer.[8] Either way, whether it was analysis or serendipity or some combination, the ancient texts make it fairly clear that a solution was found.

8 This stare and wait for inspiration strategy is more common that you might think. For a recent example, see http://www.preposterousuniverse.com/blog/2013/10/03/guest-post-lance-dixon-on-calculating-amplitudes/.