On the distances of the sun and moon according to Hipparchus

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Introduction

Among the Ancient astronomers that tried to measure the distance of the sun and moon, three stand out: Aristarchus of Samos, Hipparchus and Ptolemy. But, while we know with certainty what was the procedure followed by Aristarchus and Ptolemy, because the works in which they made the calculations survived, we cannot but conjecture what was Hipparchus's method, because the book in which he probably made the calculations, titled *On Sizes and Distances* or *On Sizes and Distances on the Sun and Moon*[1], did not come down to us. Nevertheless, we have two important references to the content of the book: the first from Ptolemy and the second from Pappus, who, commenting on Ptolemy's passage, added important information, including some values for the distances.

After discussing the difficulties for obtaining the lunar distance, Ptolemy (Almagest V, 11; Toomer 1998:243-244) says:

> Now Hipparchus used the sun as the main basis of his examination of this problem. For, since it follows from certain other characteristics of the sun and moon (which we shall discuss subsequently) that, given the distance to one of the luminaries, the distance to the other is also given, Hipparchus tries to demonstrate the moon's distance by guessing at the sun's. First he supposes that the sun has the least perceptible parallax, in order to find its distance, and then he uses the solar eclipse which he adduces; at one time he assumes that the sun has no perceptible parallax, at another that it has a parallax big enough [to be observed]. As a result the ratio of the moon's distance came out different for him for each of the hypotheses he put forward; for it is altogether uncertain in the case of the sun, not only how great its parallax is, but even whether it has any parallax at all.

Pappus, commenting on this passage asserts:[2]

> Now, Hipparchus made such an examination principally from the sun, [and] not accurately. For since the moon in the syzygies and near the greatest distance appears equal to the sun, and since the size of the diameters of the sun and moon is given (of which a study will be made bellow), it follows that if the distance of one of the two luminaries is given, the distance of the other is also given, as in Theorem 12, if the distance of the moon is given and the diameters of the sun and moon, the distance of the sun is given. Hipparchus tries by conjecturing the parallax and the distance of the sun to demonstrate the distance of the moon, [but] with respect to the sun, not only the amount of this parallax, but also whether it shows any parallax at all is altogether doubtful. For in this way Hipparchus was in doubt about the

1 See Toomer 1974: 127, note 1 for a discussion of the title.

2 The translation is taken from Swerdlow 1969: 18-19, except for a short paragraph around the end of the text that Swerdlow didn't translate. The translation of this paragraph was taken from Toomer 1974: 127.

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> sun, not only about the amount of its parallax but also about whether it shows any parallax at all. In the first book "On the Sizes and Distances" it is assumed that the earth has the ratio of a point and center of the [sphere of the] sun. And by means of the eclipse adduced by him first it is assumed that the sun shows the smallest parallax, then a greater parallax. And thus there arose the different ratios of the distances of the moon. For, in book 1 of "On Sizes and Distances" he takes the following observation: an eclipse of the sun, which in the regions round the Hellespont was an exact eclipse of the whole solar disc, such that no part of it was visible, but at Alexandria by Egypt approximately four-fifths of it was eclipsed. By means of this he shows in Book 1 that, in units of which the radius of the earth is one, the least distance of the moon is 71, and the greatest 83. Hence the mean is 77. Having shown the foregoing, at the end of the book he says: "In this work we have carried our demonstrations up to this point. But do not suppose that the question of the moon's distance has been thoroughly examined yet. For there remains some matter of investigation in this subject too, by means of which the moon's distance will be shown to be less than what we have just computed." Thus Hipparchus himself also admits that he cannot be altogether sure concerning the parallaxes. Then, again, he himself in Book 2 of "On Sizes and Distances" shows from many considerations that, in units of which the radius of the earth is one, the least distance of the moon is 62, the mean 67⅓, and the sun's distance 490. It is clear that the greatest distance of the moon will be 72⅔.

These texts are not too clear, they are not totally consistent with each other, and one surely would like Ptolemy and Pappus to be more explicit. Nevertheless, at least a couple of things seem clear: Hipparchus conjectured the solar distance for obtaining lunar distances. In the first book he obtained the set 71—77—83 for the lunar distance and the solar distance is not mentioned. In this calculation, Ptolemy used the solar eclipse described in the text. In book two, however, he made a new calculation and obtained a new set of values: 62—67⅓—72⅔. In this case, a solar distance is mentioned: 490, but there are no details about the method used to obtain these distances. Pappus says that Hipparchus obtained the values "from many considerations."

The fact that in each case Pappus mentioned three distances, the minimum, the mean and the maximum is easily explained. As Toomer 1967 has shown, the three distances of each set are consistent with assuming one of the two Hipparchian attested proportions between the radii of the epicycle and deferent: $r/R = 247.5/3122.5$. So, probably Hipparchus obtained just one value for the lunar distance conjecturing the solar distances, and calculated the other two using the proportion r/R.

Three important steps have been taken up to now in recovering the methods used by Hipparchus. The first, a step now understood to have been in the wrong direction, is due to Friedrich Hultsch in 1900, the second helped to understand the method used in book two and was made by Swerdlow in 1969 and the third one was made by Toomer in 1974, helping to understand the method used in book one. No more significant steps have been taken since then. In this paper we will show that both Swerldow's and Toomer's steps are in the right direction, but that one more step could be taken that would render Hipparchus procedure even more consistent and smarter.

Swerdlow and Toomer

In his *Mathematics useful for Reading Plato*, when he is explaining the nature of eclipses, Theon of Smyrna mentions that Hipparchus found the sun to be 1880 times the size of the earth and the earth 27 times the size of the moon (Martin 1849: 320-321). Hultsch (1900) conjectured that The-

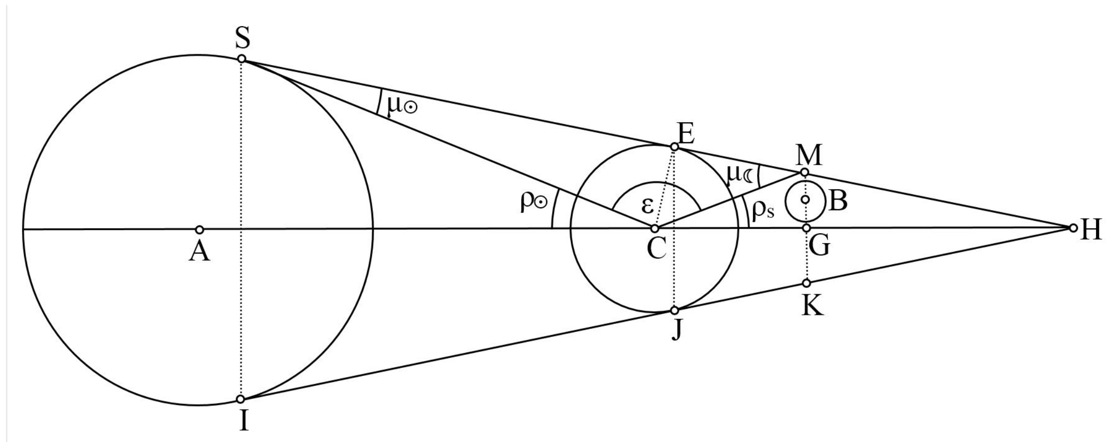


Figure 1. Hipparchus's method for calculating the lunar distance using a lunar eclipse. The center of the sun is at *A*,

on is talking about volumes and, therefore, the diameters are proportional to the cube root of these values. Accordingly, the sun is around 12⅓ times the earth and the earth 3 times the moon. Consequently, the sun is 12⅓ · 3 = 37 times bigger than the moon. But, because both luminaries have the same apparent size, this proportion expresses also the proportion between the distances. Now 37 times the mean distance (67⅓) is 2,491⅓ which could be rounded to 2,490. Hence, Hultsch (1900: 190-191) concluded that the correct number in Pappus' text must be 2490 (͵βυϙ) but the initial β disappeared leaving υϙ = 490 in the manuscripts. This modification was generally accepted and even incorporated in the text in A. Rome's edition of Pappus commentaries of books 5 and 6 of Ptolemy's *Almagest* (Rome 1931).

Swerdlow's reconstruction: the values of the second book

In 1969, however, Swerdlow proved Hultsch to be wrong. In the *Almagest*, V, 15 (Toomer 1998: 255-257), Ptolemy obtains the solar distance assuming a certain lunar distance. The procedure is perfectly reversible; you can obtain a lunar distance assuming a solar distance. The method used has its origin in Aristarchus and it is known as the eclipse diagram method. Because there is also a method that follows from the solar eclipse used by Hipparchus, I will call Aristarchus's method the *lunar eclipse method* and the method based on the solar eclipse, the *solar eclipse method*. Swerdlow shows that, if the data that Ptolemy says that Hipparchus used is used as input data, the solar distance that follows from the lunar mean distance 67⅓ is very close to 490. Thus, at the same time, Swerdlow showed that the value as it is in the manuscripts is correct (490), and identified the method used by Hipparchus.

Let me explain briefly the lunar eclipse method and how Hipparchus used it. Figure 1 represents a lunar eclipse: the center of the sun is at *A*, the center of the earth at *C* and the center of the moon at *B*. The triangle *SHI* represents the light cone of the sun, and therefore, the triangle *EHJ*, the part of the cone that represents the earth's shadow produced by the lunar eclipse. Angles $\mu_{\odot}$ and $\mu_{☾}$ represent the horizontal parallax of the sun and moon, respectively; angle $\rho_{\odot}$ represents the apparent radius of the sun (and consequently, also of the moon), while ρ_s represents the apparent radius of the earth's shadow at the lunar distance. If you look at triangle *DBF*, it is easy to see that $\mu_{\odot} + \mu_{☾} = 180 - \varepsilon$ but, also $\rho_{\odot} + \rho_s = 180 - \varepsilon$. Therefore,

1 $$\rho_{☾} + \rho_s = \mu_{\odot} + \mu_{☾}$$

i.e., the addition of the horizontal parallaxes is equal to the addition of the apparent radii of the moon and shadow. But we also know that:

$$CS = \frac{1}{\sin \mu_{\odot}} \qquad (2)$$

and

$$CM = \frac{1}{\sin \mu_{☾}} \qquad (3)$$

CS is approximately the sun-earth distance and CM the earth-lunar distance. Therefore, combining the three equations, we have that:

$$CM = \frac{1}{\sin\left(\rho_{☾} + \rho_s - \sin^{-1}\frac{1}{CS}\right)} \qquad (4)$$

So, knowing $\rho_{\odot}$, ρ_s and one of the distances, one can know the other. Fortunately, Ptolemy in *Almagest* IV,9 (Toomer 1998: 205) mentions that Hipparchus assumed that the diameter of the moon "goes approximately 650 times into its own orbit, and 2.5 times into [the diameter] of the earth's shadow, when it is at mean distance in the syzygies". Therefore, for Hipparchus, $\rho_{\odot}$ = 360/(650·2) and ρ_s = (360·2.5)/(650·2). If we assume these values and a lunar distance of 67⅓, the corresponding solar distance is 484.44 which could be rounded to 490. This is more than enough for showing that Hipparchus applied the lunar eclipse method. Swerdlow, however, adds some textual evidence: on the one hand, Pappus's text mentions "theorem 12," which is Pappus's way of referring to the lunar eclipse method; on the other, Ptolemy himself said that the method he proposes was used previously by Hipparchus (Almagest, V,14; Toomer 1998: 254).

According to Swerdlow's reconstruction, Hipparchus started from 490 as the solar distance and, applying the lunar eclipse method, obtained a lunar mean distance of 67⅓. Why did Hipparchus assume that the solar distance is 490 earth radii (*e.r.*)? Swerdlow noted that it is almost exactly the distance that corresponds to a parallax of 7 minutes. So, probably, Hipparchus considered this parallax the "least perceptible parallax," calculated the solar distance and, applying the eclipse method, found the lunar distance. Swerdlow's contribution was a huge step in the direction of understanding Hipparchus's procedure.

Among his concluding remarks, Swerdlow asserts that "there remains to be explained the first set of lunar distances, the set presumably derived from the solar eclipse. The second set, as we have seen, has nothing to do with a solar eclipse, but was derived, as Pappus states, 'from many considerations.'" As I will show later, it is not totally true that the second set has nothing to do with the solar eclipse, but Swerdlow's request will be fulfilled by Toomer, who five years later suggested a plausible explanation for obtaining the first set of values using the solar eclipse method.

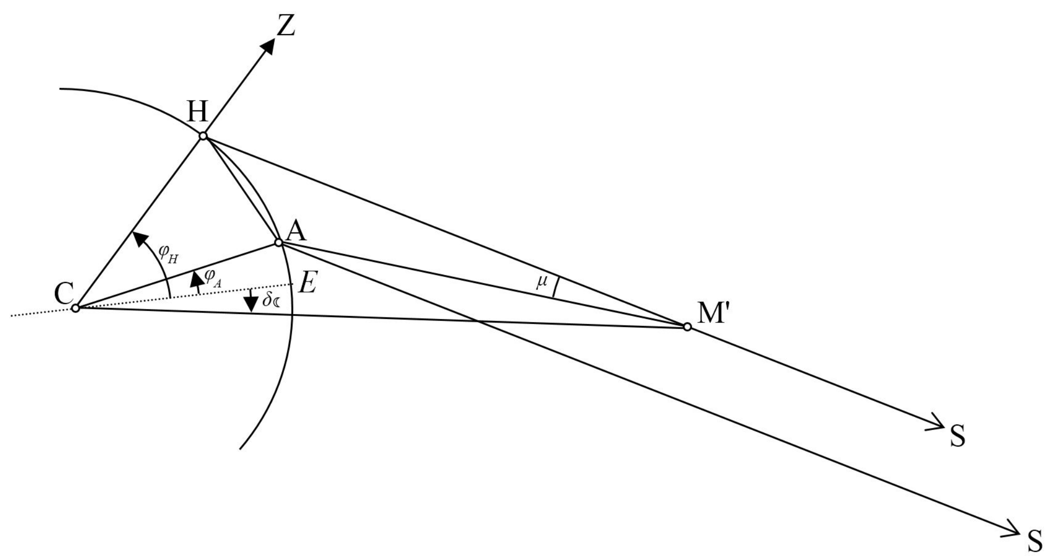


Figure 2. Hipparchus's method for calculating the Moon's distance using a solar eclipse. Similar to figure 2 of Toomer 1974: 132, the center of the earth at *C* and the center of the moon at *B*.

Toomer's reconstruction: the values of the first book.

Toomer (1974) assumed Swerdlow's contribution, but restated the demonstration in a way that he considered closer to what Hipparchus actually did. According to Toomer, when Hipparchus assumes the solar distance of 490 and calculates the lunar distance, he is looking at a maximum lunar distance. Actually, as I will show with more detail later, in the lunar eclipse method the distances are inversely proportional. If one assumes the least possible solar distance, one will find the greatest possible lunar distance. Therefore, the set of values of the second book (62,67⅓,72⅔) must be understood as the upper limit of the lunar distance.

Toomer also noted, however, that one can go further, for, in addition, the lunar eclipse method implies a lower limit for the lunar distance that corresponds to assuming the sun at an infinite distance. I will show the details later, but Toomer found that the lunar minimum distance is 59.12 *e.r.* He remarked that 59 *e.r.* is the value found by Ptolemy for the mean distance at syzygies. Toomer thinks that Ptolemy probably borrowed it from Hipparchus.

So, just as the procedure in book two must be understood as an attempt to find an upper limit for the lunar distance (assuming the sun is as close as possible), the procedure in book one could be understood as an attempt to find the lower limit for the lunar distance, assuming that the sun is as far as possible, i.e., at an infinite distance. For, he says, "in this case (i.e., if the sun is at infinite distance), the method of book 2 is not applicable". (Toomer 1974:131). So, Hipparchus proposed in book 1 a new method, based on the analysis of the solar eclipse that was total at the Hellespont but partial at Alexandria.

In Figure 2 the center of the earth is at *C*, Hellespont region at *H*, Alexandria at *A* and the dotted line *CE* represents the equator. Therefore, the angles *HCE* (φ_h) and *ACE* (φ_a) represent the latitude of Hellespont and Alexandria respectively. The moon is at *M*' and the sun, *S*, is at infinite distance, so that lines *HS* and *AS* are parallel. The moon and the sun are aligned from *H*, i.e., *H*, *M*' and *S* are in the same line. Angle *ECM*' ($\delta_☾$) is the declination of the moon. Angle *HM*'*A* (μ) represents the lunar parallax seen from *H* and from *A*. Because the sun is at an infinite distance and therefore its parallax is 0, the difference in the position of the moon with respect to the sun as seen from *H* and *A* is due entirely to the lunar parallax and, therefore, μ is ⅕ of the apparent size of the sun and moon, i.e. ⅕ of 360/650. Toomer argues convincingly that for Hipparchus the latitude of Alexandria should be around 31° (φ_a =31°) and the latitude of the Hellespont region around 41° (φ_h =41°).

Now, we want to obtain the distance of the moon, i.e., line *CM*'. It is easy to calculate line *AM*' and we know that, approximately, *CM*' is *AM*'+1. Applying the law of sines to the triangle *HAM*', we know that:

5 $$AM' = \frac{AH \cdot \sin M'HA}{\sin \mu}$$

Now, *AH* is the chord of angle *HCA*, which is the difference of the latitudes of Hellespont and Alexandria. The sides *CH* and *CA* are, of course, 1 *e.r.* each. Therefore angles *CHA* and *HAC* are equal to each other and:

6 $$CHA = HAC = 90 - \frac{(\varphi_h - \varphi_a)}{2}$$

So, one can say that:

7 $$AH = \frac{\sin HCA}{\sin CHA} = 2 \cdot \sin\left(\frac{\varphi_H - \varphi_A}{2}\right)$$

Now, angle *M*'*HA* is equal to *ZHA* minus *ZHM*'. According to Toomer, because HMC is so small, one can assume that *ZHM*' is equal to *ZCM*'. One knows that

8 $$ZCM' = \varphi_h - \delta_{☾}$$

And *ZHA* is equal to 180° minus *CHA*, that we already have in eq.(6). Therefore:

9 $$ZHA = 180 - CHA = 90 + \frac{(\varphi_h - \varphi_a)}{2}$$

So, one can express *M*'*HA*

10 $$M'HA = ZHA - ZHM' = 90 - \frac{(\varphi_h + \varphi_a)}{2} + \delta_{☾}$$

Finally, one can go back to eq.(5) and find *AM*' and therefore, *CM*'. *CM*' depends on the constants φ_a, φ_h and μ; the declination of the moon ($\delta_{☾}$) is the only variable which, in turn, depends on the solar declination and the lunar latitude.

Eclipse n.	*Date*	$\delta_{\odot}$	$\beta_{☾}$	$\delta_{☾}$	*CM'*
1	-309 August 15	+16°	+⅓°	+16⅓°	85¾
2	-281 August 6	+18½°	+⅓°	+18⅚°	87
3	-216 February 11	-15⅓°	+⅓°	-15°	57⅔
4	-189 March 14	-4°	+1	-3°	71
5	-173 October 10	-5½°	+½°	-5°	69
6	-128 November 20	-19⅓°	+⅚°	-18½°	53⅓

Table 1. This table reproduces relevant data of tables 1 and 2 of Toomer 1974.

Toomer analyzed all solar eclipses between the foundation of Alexandria and Hipparchus's time that were total seen from close to the Hellespont region but a bit less than total at Alexandria and, calculating their solar declination and lunar latitude, obtained the lunar distance corresponding to each one. Table 1 reproduces relevant data of tables 1 and 2 of Toomer's paper. Among the six possible solar eclipses, he found that a lunar distance consistent with Pappus's values follows only from the eclipse of -189 March 14. Actually, he found for this eclipse a solar declination of -4° and a lunar latitude of 1°; so, $\delta_{☾} = -3$. This implies a lunar distance of 71.07 *e.r.*, almost exactly the value attributed to Hipparchus by Pappus for the lunar minimum distance. He also shows that during this eclipse the moon was close to its minimum distance[3]. Therefore, this must be the eclipse used by Hipparchus.[4]

So, Toomer's step was as important as Swerdlow's, helping us to understand the calculation that is behind the first set of values. He finishes his paper summarizing Hipparchus's procedure (Toomer 1974: 139-140):

> Starting from the fact that there is no observable solar parallax, in "On Sizes and Distances", book 1, he took the extreme situation, assuming that the solar parallax was zero, that is that the sun was (for practical purposes) infinitely distant. Then using the data from the eclipse of -189, March 14, he derived a *minimum* distance of the moon (71 earth-radii at least distance). However, he was well aware of the unreliability of his premises... In Book 2 he assumed that the solar parallax was the maximum possible, namely 7', and hence computed the sun's *minimum* distance and the corresponding maximum distance of the moon (using the method elucidated by Swerdlow), the latter being 67⅓ in the mean. He then showed that as the sun's distance increased, the moon's decreased towards a limit of 59 earth-radii, and was thus able to establish the moon's distance between quite close limits. This procedure, if I have reconstructed it correctly, is very remarkable... What is astonishing is the sophistication of approaching the problem by two quite different methods, and also the complete

3 According to Ptolemy's tables, the lunar anomaly was 228° during the solar eclipse. Toomer (1974: 136, n. 36) argued that, probably, Hipparchus's value was around 17° less, i.e., 211°, but he later (Toomer 1998: 224, n. 14) realized that this was wrong and that Hipparchus had a pretty accurate epoch of lunar anomaly. This means that at the moment of the eclipse the moon wasn't exactly at its minimum distance. And, therefore, that the 71 *e.r.* calculated doesn't correspond to the minimum distance, but to a distance between the minimum and the mean. But, because Hipparchus is looking for an upper limit –as I will show later–, it would keep himself on the safe side if he considers that the 71 *e.r.* corresponds to the minimum distance. With a lunar anomaly of 228°, the difference would be around two earth radii. For more details about the scheme and epoch used by Hipparchus for calculating lunar anomaly see Jones 1983, especially pp. 23-27.

4 Toomer's reconstruction assumed that the eclipse took place at the meridian which is not true for the eclipse of -189. See appendix for the consequences of this assumption.

> honesty with which Hipparchus reveals his discrepant results (his "maximum distance" in book 2 turns out to be smaller than his "minimum distance" in book 1).

In what follows, I will try to go one step forward in our comprehension of Hipparchus procedure.

Text analysis

In order to make a new step forward, it is very convenient to take a close look to Ptolemy's and Pappus's texts and see what can be inferred from them about Hipparchus's calculations. This is not an easy task, for, as we already mentioned, the texts are really obscure and probably not totally consistent with each other. Let me start with Ptolemy's. I present the text again, but now with some index letters and some Greek words between brackets for convenience:

> [0] Now Hipparchus used the sun as the main basis of his examination of this problem. For, since it follows from certain other characteristics of the sun and moon (which we shall discuss subsequently) that, given the distance to one of the luminaries, the distance to the other is also given, Hipparchus tries to demonstrate the moon's distance by guessing at the sun's. [A] First [τὸ μὲν πρῶτον] he supposes that the sun has the least perceptible parallax, in order to find its [αὐτοῦ] distance, [B] and then [μετὰ δὲ ταῦτα] he uses the solar eclipse which he adduces; [1] at one time [ποτὲ μὲν] he assumes that the sun has no perceptible parallax, [2] at another [ποτὲ δὲ] that it has an adequate/sufficient [ἱκανὸν] parallax. [C] As a result the ratio of the moon's distance came out different for him for each of the hypotheses he put forward; for it is altogether uncertain in the case of the sun, not only how great its parallax is, but even whether it has any parallax at all.

From [0] we know that Hipparchus used the distance of the sun for calculating the distance of the moon, in the exactly opposite direction taken by Ptolemy, who uses the lunar distance for calculating the solar distance. Even if it is not certain from this paragraph, it seems that Ptolemy is referring to the lunar eclipse method, which he will discuss subsequently, i.e., four sections later, in the same book of the *Almagest*.

In [A] the English *its* of *its distance* is ambiguous, but the Greek αὐτοῦ could only refer to the sun (masculine), for the moon is feminine. Therefore, [A] asserts that Hipparchus supposes the least perceptible parallax in order to find the *solar* distance. [B] comes undoubtedly after [A]. So, he first calculates the minimum solar distance [A] and then he uses the solar eclipse [B]. He uses the eclipse with two different assumptions [1] and [2]. In [1] he assumes that the sun has no perceptible parallax and in [2] an ἱκανὸν parallax. ἱκανὸν means sufficient, big enough, adequate, significant. Toomer (1988: 244), in his translation of the *Almagest*, translated it by *big enough* and adds "[to be perceptible]". In the paper (1974: 126), he translates it by *significant* (and adds the word in Greek). Swerdlow (1969:287) translates it by *sufficient* and adds "[i.e. sufficient to be perceptible]". I think that the only safe thing that we can conclude is that the parallax was significant, probably observable, but there is no reason to think that it is a limiting parallax, i.e., that Ptolemy is talking about the least perceptible parallax mentioned in [A].

Now, let us take a close look to Pappus's text (the text in italics is taken textually from Ptolemy's text):

> [1] *Now, Hipparchus made such an examination principally from the sun*, [and] not accurately. For since the moon in the syzygies and near the greatest distance appears equal to the sun, and since the size of the diameters of the sun and moon is given (*of which a study will be made bel-*

Ptolemy		Pappus	
[A]	First [τὸ μὲν πρῶτον] he supposes that the sun has the least perceptible parallax, in order to find its [αὐτοῦ] distance	[2.a]	In the first book "On the Sizes and Distances" it is assumed that the earth has the ratio of a point and center of the [sphere of the] sun
[B]	and then [μετὰ δὲ ταῦτα] he uses the solar eclipse which he adduces	[2.b]	And, first, [καί ποτε μὲν] by means of [διὰ] the eclipse adduced by him
[1]	at one time [ποτὲ μὲν] he assumes that the sun has no perceptible parallax	[2.b.1]	it is assumed that the sun shows the smallest parallax
[2]	at another [ποτὲ δὲ] that it has an adequate/sufficient [ἱκανὸν] parallax.	[2.b.2]	then [ποτὲ δὲ] a greater parallax [μεῖζον]
[C]	As a result the ratio of the moon's distance came out different for him for each	[2.c]	And thus there arose the different ratios of the distances of the moon

Table 2. Comparison between Ptolemy's text and the second part of Pappus's text.

> *low), it follows that if the distance of one of the two luminaries is given, the distance of the other is also given*, as in Theorem 12, if the distance of the moon is given and the diameters of the sun and moon, the distance of the sun is given. Hipparchus *tries by conjecturing the parallax* and the distance of the sun *to demonstrate the distance of the moon*, [but] *with respect to the sun, not only the amount of this parallax, but also whether it shows any parallax at all is altogether doubtful*. For in this way Hipparchus was in doubt about the sun, not only about the amount of its parallax but also about whether it shows any parallax at all. [2] [a] In the first book "On the Sizes and Distances" it is assumed that the earth has the ratio of a point and center of the [sphere of the] sun. [2.b] And, first, [καί ποτε μὲν] by means of [διὰ] the eclipse adduced by him [2.b.1] it is assumed that the sun shows the smallest parallax, [2.b.2] then [ποτὲ δὲ] a greater parallax [μεῖζον]. [2.c] And thus there arose the different ratios of the distances of the moon. [3] For, in book 1 of "On Sizes and Distances" he takes the following observation: an eclipse of the sun, which in the regions round the Hellespont was an exact eclipse of the whole solar disc, such that no part of it was visible, but at Alexandria by Egypt approximately four-fifths of it was eclipsed. [3.a] By means of this he shows in Book 1 that, in units of which the radius of the earth is one, [τὸ μὲν] the least distance of the moon is 71, [τὸ δὲ] and the greatest 83. Hence [τὸ ἄρα] the mean is 77. [3.b] Having shown the foregoing, at the end of the book he says: «In this work we have carried our demonstrations up to this point. But do not suppose that the question of the moon's distance has been thoroughly examined yet. For there remains some matter of investigation in this subject too, by means of which the moon's distance will be shown to be less than what we have just computed». [3.c] Thus Hipparchus himself also admits that he cannot be altogether sure concerning the parallaxes. [3.d] Then, again, he himself in Book 2 of "On Sizes and Distances" shows from many considerations [ἐκ πολλῶν ἀποδείκνυσιν] that, in units of which the radius of the earth is one, [τὸ μὲν] the least distance of the moon is 62, [τὸ δὲ] the mean 67⅓, and [τὸ δὲ] the sun's distance 490. It is clear that [δῆλον δὲ ὅτι καὶ] the greatest distance of the moon will be 72⅔.

Hipparchus's text can be divided in three parts. The first one [1] is mostly a paraphrase of Ptolemy's text (mainly Ptolemy's text [0]). The second one [2] seems to enumerate the different parallaxes assumed in order to justify the fact that Hipparchus arrived up to different distances of the moon. The third one [3] is the most revealing, distinguishing what Hipparchus did in each book of the treatise and providing the different set of values obtained in each one.

Text [1] is not too clear, probably due to the desire of Pappus to paraphrase Ptolemy's text. But, at least, it is useful to confirm what we suspected from Ptolemy's text [0], i.e., that Hipparchus used the eclipse diagram, for Pappus refers explicitly to it as "Theorem 12". Text [2] is very similar to the rest of Ptolemy's texts, and the parallel among the parts is evident. I present them in table 2.

There are, nevertheless, significant differences. While Ptolemy says that Hipparchus supposed *the least perceptible* parallax for calculating the solar distance, Pappus asserts that, in the first book, Hipparchus assumed that *there is no parallax*. While the first part of the text is not exactly parallel, this doesn't mean that there is a contradiction. Ptolemy and Pappus could be talking of different parts of Hipparchus's work. It is possible that Hipparchus, somewhere calculated the smallest perceptible parallax in order to calculate the distance of the sun and, at another, assumed that the sun has no parallax. Then, both authors introduce the reference to the solar eclipse. And both highlight the two parallaxes assumed by Hipparchus. But it seems that they are not exactly the same. While Ptolemy refers to the first one as "no perceptible", Hipparchus says: "smallest". This could mean the same thing: no parallax at all. Something similar happens with the reference to the second parallax: while Ptolemy uses ἱκανὸν, Pappus says that it is greater (μεῖζον) than the previous one. There is no inconsistency here either. We know that a significant parallax is greater than the smallest. Both authors finished mentioning that these two (or three) different parallaxes are responsible for the different values of the lunar distance.

One last detail of this text should be highlighted: it looks as if that all that Pappus is saying in text [2] is contained just in the first book. This seems odd for at least two reasons. First, because text [3] seems to make explicit text [2], but in text [3] book one and two are openly mentioned. Second, because this would imply that already in the first book there were more than one set of lunar distance values. It is also possible that what Pappus pretends to situate in book one is just the assumption of no parallax of text [2.a].[5]

Text [3] is the clearest and most informative of the three. He says that in book one, Hipparchus uses a solar eclipse and he gives us details of the eclipse: it was total in the Hellespont but at Alexandria just four-fifths of it was eclipsed. [3.a] gives us a set of lunar distances. It seems that the first two were obtained by Hipparchus himself, while the third one was provided by Pappus. This is the natural way of interpreting the ἄρα (hence). So, while Hipparchus in book one obtained the minimum and the maximum distance of the moon, Pappus adds the mean one. Because no solar distance is mentioned, it is natural to suppose that in this calculation the sun had an infinite distance and, therefore, to link this calculation with the parallax referred by Ptolemy like no perceptible parallax and by Pappus like the smallest. Moreover, this is the first of the two mentioned, so the order would be also respected.

Text [3.b] quotes textually the end of the first book of Hipparchus in which he says that the question of the lunar distance is not finished as it has been dealt with in book one and that in book two he will find a smaller distance. In text [3.c], Pappus confirms what Ptolemy said: that Hipparchus admits that he is not sure concerning parallaxes.

Text [3.d] offers information about the calculation of book two. Unfortunately, Pappus is not explicit at all in his reference about the method of calculation (he only says that the values have been obtained "from many considerations"). In this case, three values seems to be obtained

5 It should be noted that, in the text that Hipparchus is paraphrasing, Ptolemy uses πρῶτον meaning "in the first place", so maybe there is a corruption in the text that goes from ὑπέθετο πρῶτον ἐν περὶ μεγεθῶν καὶ ἀποστημάτων to ὑπέθετο ἐν τῷ πρώτῳ περὶ μεγεθῶν καὶ ἀποστημάτων.

explicitly by Hipparchus himself and one added by Pappus. Hipparchus obtained the minimum and the mean distance of the moon, and a solar distance (490). Pappus adds that it is clear what would be the greater lunar distance. Because a solar distance is mentioned here, it is natural to associate this calculation with that of the second parallax, called ἱκανὸν by Ptolemy and μεῖζον by Pappus.

Two things about this text are still worth mentioning. First, that the solar distance seems to have been the result (and not an input value) as much as the other two lunar distances mentioned in the text. Second, that the "many considerations" used by Pappus to refer to the method of calculation does not exclude the use of the solar eclipse. Actually, if the parallel between the two parallaxes mentioned both by Ptolemy and Pappus, and the sets of values provided by Pappus are taken seriously, one has to recognize that both sets of values have been obtained using the solar eclipse, for both parallaxes have been used in relation with the solar eclipse. The "many considerations", nevertheless, could suggest that not only the solar eclipse was used, but something else. Of course, this "something else" could be anything, but a good candidate is the lunar eclipse method mentioned both by Ptolemy and Pappus and that until now had no room in our story.

From the analysis of both texts, therefore, what we know about Hipparchus´s method could be summarized thus: Hipparchus in *On the Sizes and Distances* tried to calculate the lunar distance by conjecturing the solar distance. He made at least two different calculations, based on two different assumptions and obtained two different sets of values. In the first book, he first calculated the solar distance that follows from the least perceptible parallax, probably for establishing a lower limit for the sun values: because it seems that the solar parallax is not perceptible, one could not use a solar distance smaller than that. After this, by means of the solar eclipse and assuming that there is no parallax, Hipparchus calculated the minimum and maximum lunar distance, obtaining 71 and 83, respectively. At the end of the book, he said that the research was not finished and that in the next book he would find that the lunar distance is actually smaller. In book two, he made a new calculation from many considerations that surely included again the solar eclipse but also almost certainly something else, probably the lunar eclipse method. In this calculation he used a significant or adequate parallax, greater than the previous one and obtained the minimum and mean lunar distances and the solar distance, 62, 67⅓ and 490 respectively.

Swerdlow's proposal is surely an important step for understanding the calculation of book two. He shows us that Hipparchus used the lunar eclipse method and, using as input value the solar distance of 490, he obtained the mean distance of 67⅓. Nevertheless, his account does not explain (a) the role that the solar eclipse surely played in the calculation, (b) the fact that the solar distance seems to be the result of the calculation as much as the minimum and mean lunar distance, and not an input value and, (3) that Hipparchus obtained not just the mean distance, but also the minimum one or, at least, for some reason, he considered important to make explicit the mean distance and not the maximum one.

Toomer's proposal represents also a very significant step forward in our comprehension of the method used by Hipparchus in book one. Nevertheless, he fails to explain (1) why Hipparchus makes explicit not only the minimum but also the maximum distance and I think this is due mainly to (2) his interpretation of the sets of lunar values of book one as a *minimum*. I will try to show that it is actually, an upper limit. This would solve (3) another oddity of his interpretation, i.e., that Hipparchus obtained in book one an upper limit smaller than the lower limit that he will find in book two. Toomer interprets this inconsistency as a sign of the laudable honesty of Hipparchus but at least it seems odd not just that Hipparchus would have published

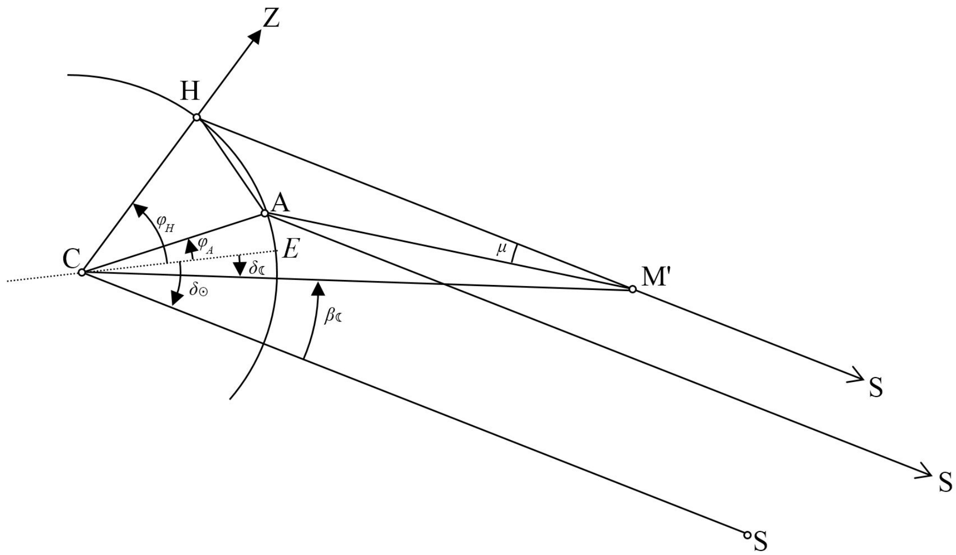


Figure 3. Hipparchus's method for calculating the Moon distance using a solar eclipse. Similar to Figure 2 but with the solar declination ($\delta_{\odot}$) and lunar latitude ($\beta_{☾}$) added.

inconsistent results in a work, but still more so that both Ptolemy and Pappus, who clearly are criticizing Hipparchus, do not mention this obvious inconsistency. As I will show, if we interpret the values of book one as an upper limit, this problem disappears. Finally (4) Toomer (1974:131) seems to justify the necessity of the solar eclipse method because, he says, the lunar eclipse method used in book two is not applicable to an infinite solar distance. But this is not correct, for, as he shows one paragraph above, the lunar eclipse method allows one to find the lower limit of the lunar distance (59.12 *e.r.*), when it is assumed that there is no solar parallax. Therefore, if it is possible—as it is—to obtain both a lower and upper limit for the lunar distance using just the lunar eclipse method and assuming an upper and lower limit for the solar distances, what sense could make for Hipparchus to introduce another method that, besides, produces a set of values inconsistent with the others?

In what follows I will try to make one step further in our comprehension of Hipparchus's calculation. In order to do so, I will start by showing some problems that the geometrical calculation followed by Hipparchus according to Toomer presents and I will offer an alternative approach. Fortunately, this new approach doesn't change Toomer's main achievements: the -189 solar eclipse identified by him is still the eclipse used by Hipparchus. But this new approach would show that the values obtained in book one must be understood as an upper limit and also would allow us to apply the solar eclipse method to finite solar distances. Finally, I will show that this solar eclipse method for finite solar distances could be applied in conjunction with the lunar eclipse method and produce suitable results. I will conclude presenting a new reconstruction of Hipparchus's calculations that, I hope, even if strongly inspired in Toomer's and Swerdlow's significant achievements solves many of the problems found in their reconstructions.

Revisiting Toomer's reconstruction of Hipparchus' procedure (1): The identification of ZHM' and ZCM'

In Figure 3 I have added to Fig. 2 the solar declination and the lunar latitude, which is implicit in Toomer's calculation. So, Angle *ECS* ($\delta_{\odot}$) is the solar declination and angle *SCM*' ($\beta_{☾}$) is the lunar latitude. According to Toomer's calculation, $\delta_{\odot} = -4°$ and $\beta_{☾} = 1°$. The difference between them ($\delta_{\odot} - \beta_{☾}$) is, according to Toomer, the lunar declination ($\delta_{☾} = -3°$).[6]

Let us start analyzing a simplification made by Toomer that could have significant consequences in the final result. As we already said, he assumes that *ZHM*' is equal to ZCM' because the angle HM'C –the difference between them– is so small. But looking at Figure 3, it is easy to see that angle HM'C is equal to $\beta_{☾}$. The simplification doesn't make sense, not only because a difference of 1° could be significant but mainly because, according to Toomer, Hipparchus knew the value. So, it is still true that

8 $$ZCM' = \varphi_h - \delta_{☾}$$

But now

11 $$ZHM' = \varphi_h - \delta_{☾} + \beta_{☾}$$

And $\beta_{☾} - \delta_{☾} +$ is $-\delta_{\odot}$. Consequently:

11.1 $$ZHM' = \varphi_h - \delta_{\odot}$$

While *ZCM*' depends on the lunar declination, *ZHM*' depends on the solar declination. If this correction is applied also to *M*'*HA*, one obtains:

10.1 $$M'HA = ZHA - ZHM' = 90 - \frac{(\varphi_h + \varphi_a)}{2} + \delta_{\odot}$$

With this new equation applied again to eq.(5), *AM*' is now: 69.06 *e.r.* and, therefore, *CM*' approximately 70.06, one earth radius short. This is not terribly bad. The difference is due to the fact that, while *ZHM*' in Toomer's reconstruction depends on $\delta_{☾}$ (-3°), in mine it depends on $\delta_{\odot}$ (-4°). Therefore, If one assumes that $\delta_{\odot} = -3°$, the previous result is restored. And, actually, this assumption is perfectly reasonable, for we know that the solar declination at the moment of the eclipse was between -3 and -4. So, while it is true that it was closer to -4, it is still reasonable to assume that Hipparchus, instead of rounding the value, simply truncated the fractional part, a well attested practice in ancient Greek mathematics. Also, we know that Hipparchus sometimes made mistakes in the determination of the position of the sun, in some cases reaching up to half a degree.[7] So, after all, the simplification introduced by Toomer is not catastrophic, we simply must be ready to modify slightly the value of $\delta_{\odot}$.

Revisiting Toomer's reconstruction of Hipparchus' procedure (2): The determination of $\beta_{☾}$

Nevertheless, eq.(10.1) implies that *CM*' (the lunar distance) does not depend on the lunar declination ($\delta_{☾}$), but only on the solar declination ($\delta_{\odot}$), which *a priori* seems odd. Is it possible for the

6 In strict sense, this is only true if the ecliptic is parallel to the horizon at the meridian. See appendix.

7 According to Ptolemy (Almagest IV, 11, Toomer 1998: 211-216) this is the reason why he found two different proportions for the *r*/*R* using two different sets of eclipses. See Toomer 1967.

lunar declination, or, equivalently, the lunar latitude to play no role at all in the determination of the lunar distance?

Yes, it is possible, because the moon parallax is already between two points, *A* and *H*: assuming that the sun is at infinite distance, the parallax is μ. This is enough for determining the lunar distance. The difference between $\delta_{\odot}$ and $\delta_{☾}$, i.e., $\beta_{☾}$ is another parallax. And, of course, only one is necessary for obtaining the lunar distance. Therefore, having $\beta_{☾}$ as a datum (obtained from the lunar theory) as in Toomer's approach, the problem is *over-determined*.

For example, it is enough to know that the eclipse was total at the Hellespont and $\beta_{☾}$ at the moment of the eclipse for calculating the lunar distance, ignoring the eclipse magnitude at Alexandria.

So, let us ignore what happens at A and work with the Hellespont and the center of the earth. Applying the law of sines to triangle *CHM*':

12 $$CM' = \frac{CH \cdot \sin CHM'}{\sin HM'C}$$

Now, *CH* is one earth radius, *HM*'*C* is equal to $\beta_{☾}$ and *CHM*' is equal to *ZHM*', known from eq.(15.1). Therefore:

12.1 $$CM' = \frac{\sin(\varphi_h - \delta_{\odot})}{\sin \beta_{☾}}$$

In this case, *CM*' depends exclusively on φ_h, $\beta_{☾}$ and $\delta_{\odot}$. The latitude of Alexandria (φ_a) and the lunar parallax μ play no role at all.

Assuming that $\beta_{☾} = 1°$, as Toomer did, then, *CM*' is 40.51 *e.r.*, around a half of the result previously obtained with the same set of data![8] This shows a serious inconsistency in Toomer's reconstruction for, depending on the geometrical path that one decides to follow, using the same set of data, one obtains different results.

Thus, it can be inferred from the geometrical configuration that, if one wants to use all the data transmitted by Pappus, it is not necessary to use the lunar theory for calculating $\beta_{☾}$. This is the way to block the over-determination. I guess Hipparchus did that, otherwise, part of the data is useless. $\beta_{☾}$ can be easily obtained from eq.(7.1).

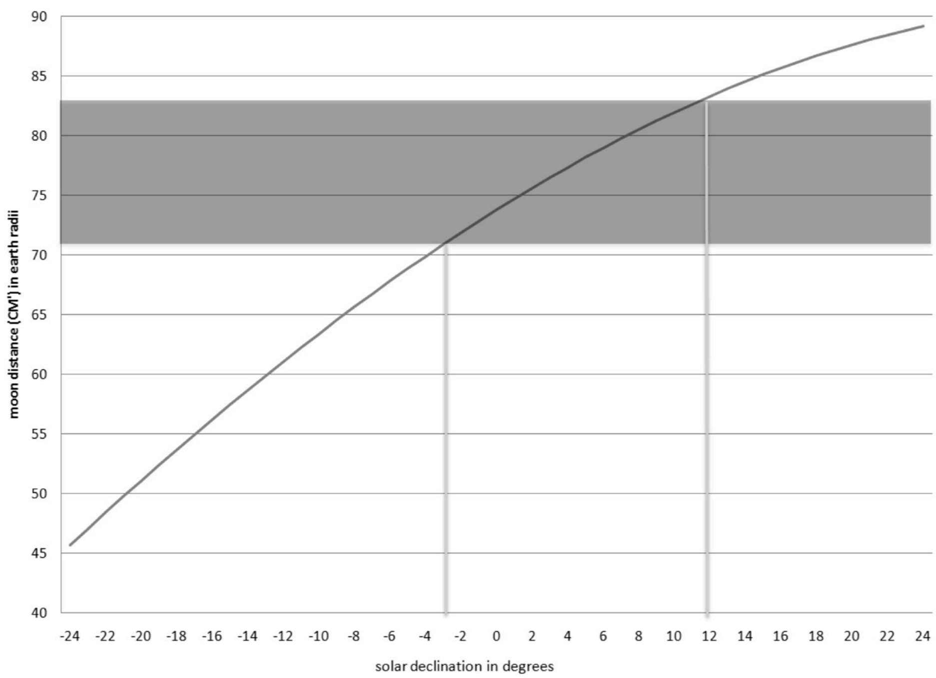
12.2 $$\sin \beta_{☾} = \frac{\sin(\varphi_h - \delta_{\odot})}{CM'}$$

Taking *CM*' = 71 *e.r.*, the corresponding $\beta_{☾}$ is, in this case, 0.57°. Hence, *CM*' is very sensitive to small changes in the $\beta_{☾}$, another reason for obtaining the adequate $\beta_{☾}$ from the figure instead of calculating it from the lunar theory.

Thus, one can calculate *CM*' without reference to $\beta_{☾}$, adding one earth radius to *AM*' (obtained in eq.1).[9] Therefore, eq.(7.2) offers a way to calculate $\beta_{☾}$.

8 If we decide to use Eq. 6 instead of 6.1, *CM*' would be 40,51 *e.r.* So, the difference is not due to the simplification analyzed previously.

9 It is possible to calculate *CM*' in a more precise way Applying the rule of cosine to triangle *CAM*', we obtain that $CM'^2 = AM'^2 + 1 - 2 \cdot AM' \cdot \cos(CAM')$. And $CAM' = 360° - (HAC + HAM')$. *HAC* is obtained in eq.(6) and $HAM' = 180 - (\mu + M'HA)$. Finally, *M*'*HA* is obtained in eq.(10.1). In our analysis we will use this more precise equation.



Graph 1. Lunar distance depending on the Sun declination. Only with solar declinations between -3° and +12° it is possible to obtain values for *CM*' between 71 and 83 e.r.

Revisiting Toomer's reconstruction of Hipparchus' procedure (3): The same eclipse

The fact that *CM*' can be determined only using the solar declination renders it even easier to check which one(s) of the eclipses that Toomer found could have been used by Hipparchus. In graph 1 you have *CM*' depending on $\delta_{\odot}$. The graph shows that It is only possible to obtain values for *CM*' between 71 and 83 *e.r.* from eclipses with solar declinations between -3° and +12°. Fortunately, there is only one eclipse that fits the description, the eclipse at -189 March 14. Also fortunate was that this is the same eclipse that Toomer had identified as candidate. One simply needs to assume that $\delta_{\odot}$ was -3° and that $\beta_{☾}$ (if it was used) was around 0.57°.

I will show in the next section that, using this new approach to the problem, it is easier to calculate the lunar distance assuming finite solar distances.

Lunar distance from a finite solar distance

Figure 4 does not suppose any longer that the sun is at an infinite distance. Therefore, line *HS* is not parallel to line *CS*. The moon (*M*) is at the intersection of *HS* and *CM*'. It is easy to see that if the solar distance diminishes, it also diminishes the lunar distance. *CM*' is, consequently, the maximum distance that the moon could reach, assuming that the sun is at an infinite distance. In this new situation, because the sun is not at an infinite distance, μ is not *HM*'*C* but *MAS*, i.e., the angular difference between the sun and moon observed from Alexandria.

Triangles *HM*'*M* and *SCM* are similar. Therefore,

13 $$\frac{CS}{CM} = \frac{HM'}{MM'}$$

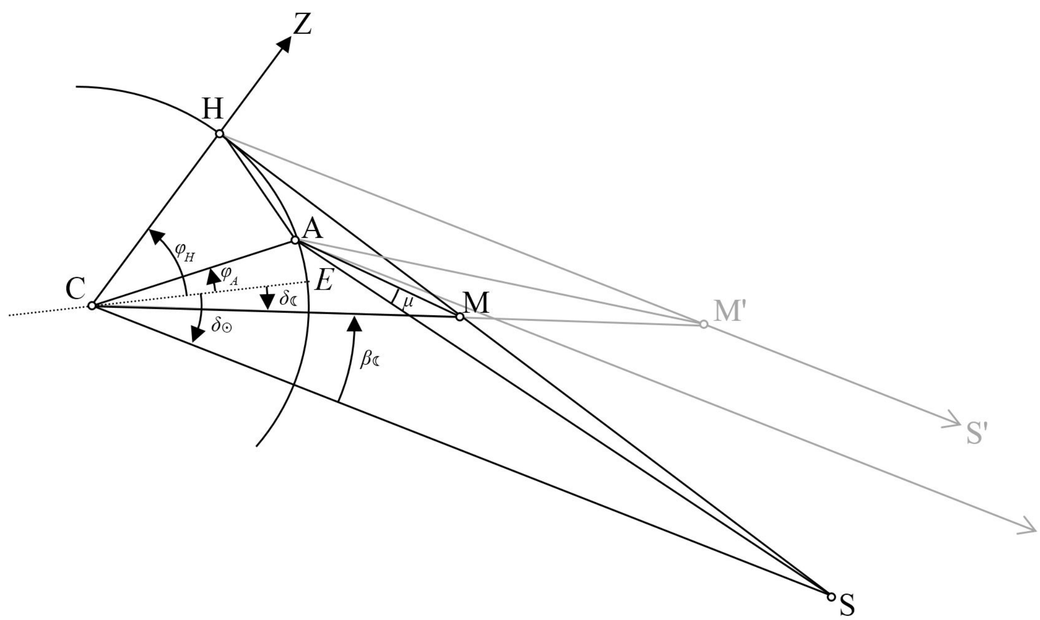


Figure 4. Hipparchus's method for calculating the lunar distance using a solar eclipse, but without assuming that the Sun is at an infinite distance.

And since *MM*' is equal to *CM*' – *CM*,

13.1 $$\frac{CS}{CM} = \frac{HM'}{CM' - CM}$$

which leads to:

13.2 $$CM = \frac{CM'}{\left(\frac{HM'}{CS} + 1\right)}$$

Applying the sine rule to triangle *CHM*' and noting that *CH* is one *e.r.* and *HM*'*C* is equal to $\beta_☾$:

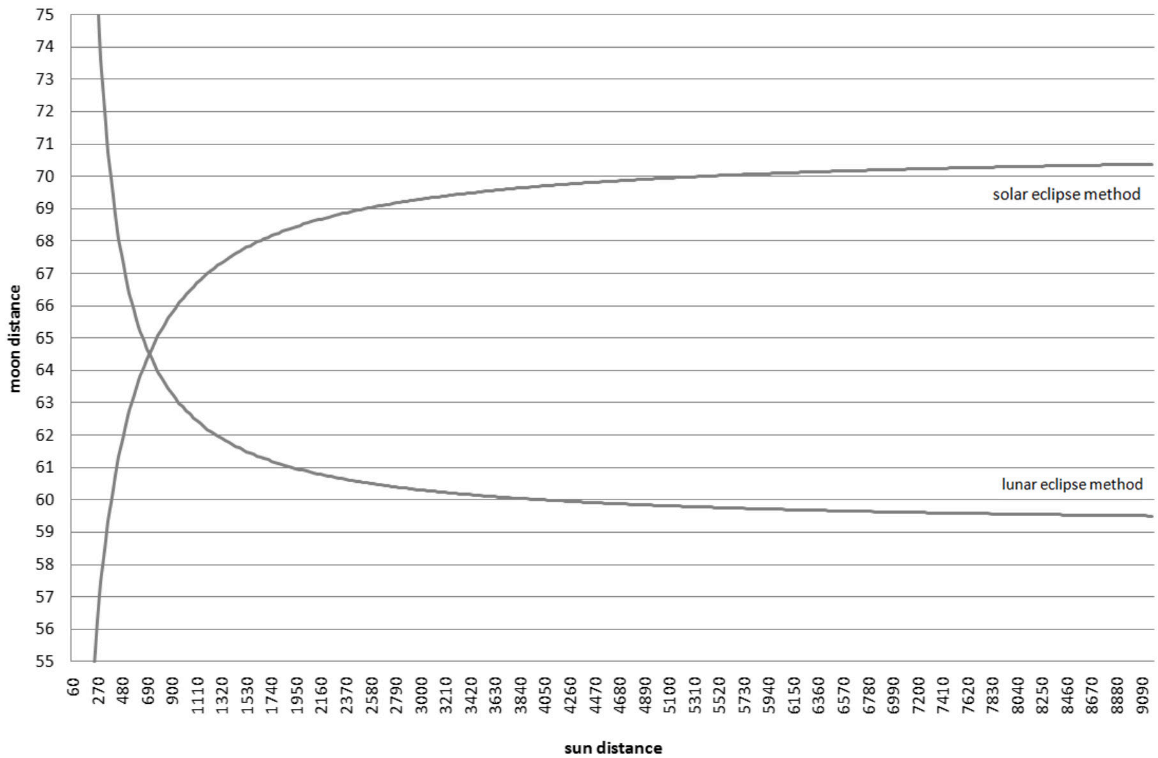
14 $$HM' = \frac{\sin HCM'}{\sin \beta_☾}$$

HCM' is equal to *ZCM*', already obtained at eq.(8) as $\varphi_h - \delta_☾$. Hence, the lunar distance can be obtained from eq.(13.2) knowing the solar distance.

Comparison with the lunar eclipse method

In an interesting sense, the solar eclipse method and the lunar eclipse method are complementary.

As I have already shown, if the lunar apparent radius is $\rho_☾$ and the earth shadow apparent radius is ρ_s, then the lunar eclipse diagram is expressed by the formula:



Graph 2. Lunar distance in function of the Sun distance for the solar eclipse and lunar eclipse method. The lunar eclipse method gives the mean distance of the moon, the solar eclipse method gives the minimum distance of the moon.

4 $$CM = \frac{1}{\sin\left(\rho_{☾} + \rho_s - \sin^{-1}\frac{1}{CS}\right)}$$

A good enough approximation of this formula is:[10]

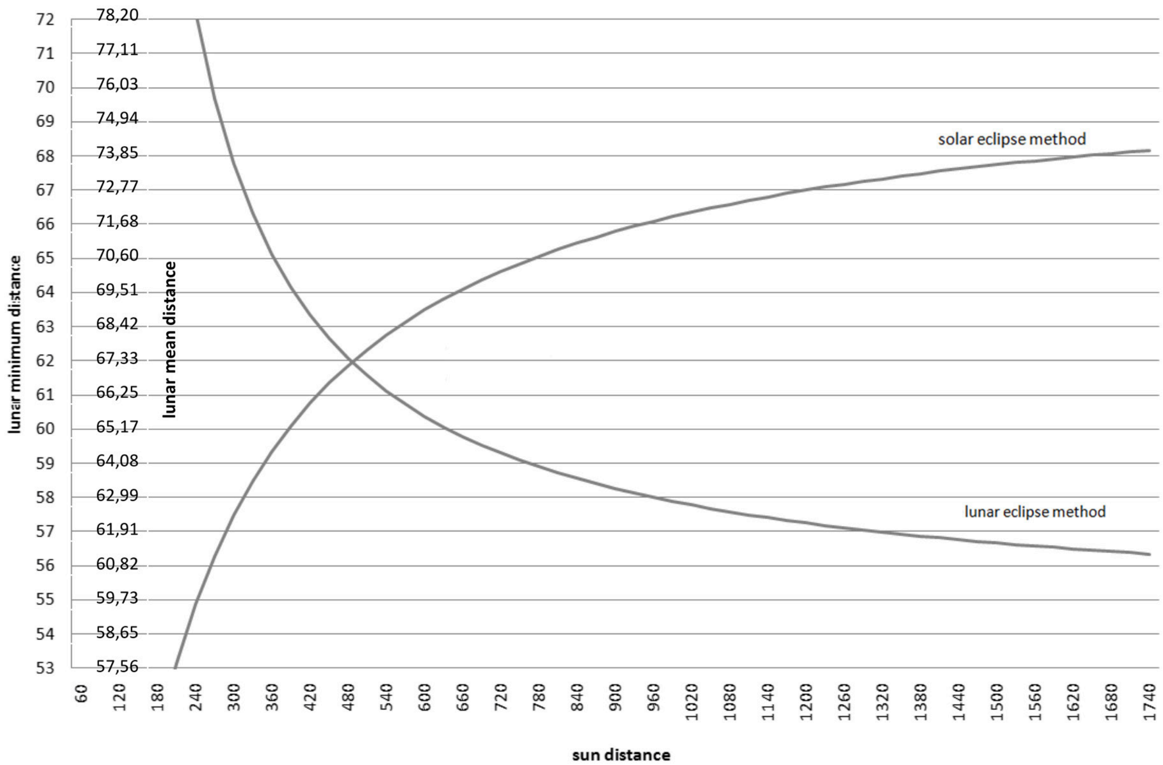
15 $$CM = \frac{1}{\sin(\rho_{☾} + \rho_s) - \cos(\rho_{☾} + \rho_s)\cdot\frac{1}{CS}}$$

Clearly, when *CS* grows, $\cos(\rho_{☾} + \rho_s)\cdot 1/CS$ diminishes and, therefore, $\sin(\rho_{☾} + \rho_s) - \cos(\rho_{☾} + \rho_s)\cdot 1/CS$ also grows, making *CM* to be smaller. So, *CS* and *CM* are inversely proportional. When *CS* is infinite, *CM* would reach its minimum, being

16 $$CM_{min} = \frac{1}{\sin(\rho_{☾} + \rho_s)} \approx 59.12$$

Toomer had already found this result. Exactly the opposite happens with the solar eclipse diagram. It is clear from eq.(13.2) that when *CS* grows, $HM'/CS + 1$ diminishes and, therefore, *CM*

10 I assume that $\sqrt{(1-1/CS^2)} = 1$.



Graph 3. Lunar distance in function of the Sun distance for the solar eclipse and lunar eclipse method. Both methods give the lunar mean distance. They meet at a lunar mean distance of 67⅓ e.r. and the solar distance of around 490 e.r.

grows. The maximum *CM* is obtained when *CS* is infinite. In this case *CM* = *CM*'. For Hipparchus's values (using $\delta_{\odot} = -3$), *CM*' is 70.91 ≈ 71 earth radii.

Therefore, the 71 *e.r.* found by Hipparchus using the solar eclipse method in book one should be understood as an *upper limit* and not a lower limit as Toomer suggested[11]: the lunar distance could not be greater than that.

Graph 2 clearly shows that, when working with both diagrams together, one obtains a maximum and minimum distance for the moon: the maximum around 71 *e.r.* and the minimum around 59 *e.r*. Also, there is one and just one distance of the moon and one and just one distance of the sun at which both diagrams meet. Graph 2 shows that the lunar distance is around 64.5 *e.r.* and the solar distance, 700 *e.r.*[12] But, while the lunar eclipse method gives the mean distance of the moon (because at mean distance sun and moon have the same apparent size), the solar eclipse method gives the minimum distance of the moon (for during the eclipse of -189 the moon was close to its minimum distance and the value corresponds to the minimum distance provided by Pappus). So, the 64.5/700 means nothing.

But let us plot both graphs at the minimum distance. In order to do so, from each result of the lunar eclipse method, which is originally expressed in mean distance, I will subtract the

11 Toomer (1974:135) asserts that it is a lower limit because the geometrical configuration assumes that the eclipse took place at noon. In any other moment of the day, *CM* would be greater. This is not correct, see appendix. But, nevertheless, with respect to the distance of the sun (which is the only relevant respect in this situation) it is an upper limit.

12 These values are curiously close to the distances that Ptolemy offers in the *Canobic Inscription*. (See Jones 2005). But I think that it is no more than a coincidence.

corresponding proportion for converting it to the minimum distance. If *m* is the original result, I will plot ($m - m \cdot r/R$), taking $r/R = 247.5/3122.5$. The new graph looks like Graph 3. The coincidence of both solar and lunar eclipse diagrams now happens when the lunar minimum distance is 62, consequently the mean distance is $67^1/_3$ and the solar distance is around 490. These three values are the results that Pappus says that Hipparchus obtained in his second book!

Therefore, Swerdlow's very smart discovery is just part of the story. It is true that, using the lunar eclipse method and starting with 490 *e.r.* for CS, one finds $67^1/_3$ *e.r.* for the lunar (mean) distance. But it is also true that, using the solar eclipse method and starting with 490 *e.r.* for *CS*, one finds 62 *e.r.* for the lunar (minimum) distance. Either this is an incredible cosmic coincidence or Hipparchus chose this set of values because they fit with both, the lunar and solar eclipse methods at the same time.

Hipparchus had two equations (lunar and solar eclipse methods) with two variables (*CS* and *CM*). Instead of conjecturing one of the values, he *solved* the system of equations. So, while in book one he established the upper and lower limit of each equation, in book two he solved the system of equations. I think that this reconstruction offers a much more coherent procedure.

The analytical solution consists in equating eq.(13.2) for the solar eclipse diagram and eq.(15) for the lunar eclipse diagram. Nevertheless, while eq.(13.2) gives the minimum distance of the moon, eq.(15) gives the mean one. So, in order to obtain the minimum distance from eq.(15), one has to multiply it by 1/ (1-*r*/*R*). Therefore:

15.1 $$CM_{min} = \frac{\left(1 - \frac{r}{R}\right)}{\sin(\rho_{☾} + \rho_s) - \cos(\rho_{☾} + \rho_s) \cdot \frac{1}{CS}}$$

Now, from eq.(15.1) and eq.(13.2):

17 $$\frac{CM'}{\left(\frac{HM'}{CS} + 1\right)} = \frac{\left(1 - \frac{r}{R}\right)}{\sin(\rho_{☾} + \rho_s) - \cos(\rho_{☾} + \rho_s) \cdot \frac{1}{CS}}$$

And for *CS*:

18 $$CS = \frac{HM' \cdot \left(1 - \frac{r}{R}\right) + \cos(\rho_{☾} + \rho_s) \cdot CM'}{CM' \cdot \sin(\rho_{☾} + \rho_s) - 1 + \frac{r}{R}} = 486.16 \approx 490$$

And, applying this solar value in eq. (13.2), *CM* is 61.97 for the minimum distance of the moon and in eq.(15), *CM* is 67.3 for the mean distance of the moon.

Of course, it is not necessary for Hipparchus to solve the system of equations, it would not be hard to find the result by trial and error, knowing the limits for *CM*.

Conclusion

Let me finish summarizing what I think was Hipparchus's procedure and how it solves the problems found in both Toomer's and Swerdlow's proposals. Hipparchus procedure consists in establishing first an upper and lower limit for the lunar distance assuming that the sun has no

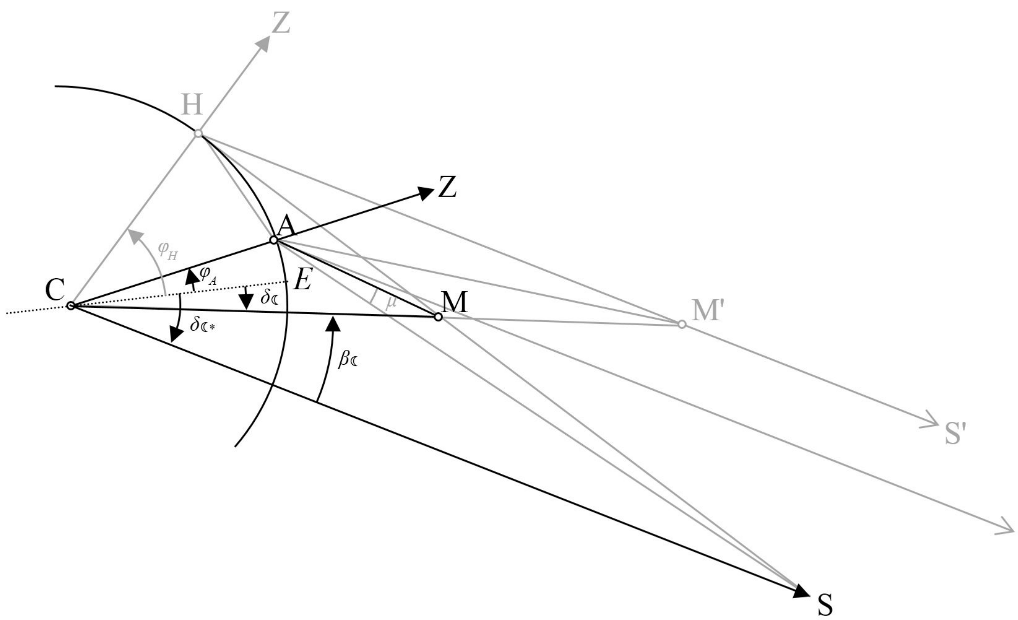


Figure 5. Hipparchus's method for calculating the Moondistance using a solar eclipse, (without assuming that the Sun is at aninfinite distance) used by Ptolemy for calculating the lunar parallax.

parallax: he first applied the solar eclipse method to the eclipse identified by Toomer and found that the upper limit for the lunar distance is 71 *e.r.* But this is the upper limit of the minimum distance (for, as we said, at the moment of the eclipse the moon was close to its minimum distance). Therefore, Hipparchus, using his ratio r/R naturally calculates also the maximum distance, 83 *e.r.*, so that he can offer the upper limit in an absolute sense: the moon could not be farther than 83 *e.r*. This explains why Hipparchus explicitly mentioned in book one the minimum (71) and the maximum (83), but not the mean one. The minimum because it happens to be the result of the calculation, and the maximum because he is suggesting an upper limit. I agree with Toomer than, even if we do not have textual evidence, we can suppose that Hipparchus probably calculated later the lower limit of the lunar distance that follows again from assuming that the sun is at infinite distance but now from the lunar eclipse method. He found 59.12 *e.r.* and he probably offered also the corresponding minimum distance (54.4 *e.r.*). I would like to suggest that both limit calculations have been performed in book one, while he reserved book two for the solution of the system of equations. Pappus' quotation at the end of book one, however, suggests that Hipparchus still didn't calculate the lower limit for he says that the value that he will find in book two is smaller than and not between the values found in book one So, probably book one is entirely devoted to the solar eclipse method and how to find the upper limit. Then, in book two he introduces the lunar eclipse method and, first, he calculated the lower limit and then he solved the system of equations finding at the same time that the only set of values is 62 for the minimum distance in the solar eclipse method, 67⅓ for the mean distance in the lunar eclipse method and 490 for the solar distance that follows, obviously, from both methods. These, again, are the three values that Hipparchus mentioned as the final result in book two. I think, therefore, that this last set must not be understood as a limiting case, but as the final result that is between the upper (83) and lower (54.4) limits. Needless to say that Hipparchu's development for solving the system of equations should have been as complex as to make Pappus prefer to say simply "from many considerations" than to try to explain it in a few words.

This account explains at the same time all the things that were left unaccounted for in Swerdlow's account: the role that the solar eclipse method played in the calculation of book two, the fact that the solar distance (490) is not an input value but a result obtained together with the minimum and mean lunar distances, and the reason why Hipparchus mentions both the minimum and the mean distance (but not the maximum) as the results of book two. It also explains why in book one, Hipparchus mentioned the minimum (71) and the maximum (83), but not the mean distance and it shows that this set of values must be interpreted as an upper limit and, therefore, it solves the inconsistency in Hipparchus's sets of values that Toomer's account implied. Finally, it explains the convenience of using both methods. I have already mentioned that, if one wants to find and upper and lower limit, it would be enough to apply the lunar eclipse method to a maximum and minimum solar distance. There is no need of the solar eclipse method. But if one wants to find a particular set of values, both methods have to be combined, rendering also necessary the use of the solar eclipse method.

The solar eclipse method in Ptolemy.

As far as we know, Ptolemy did not use the solar eclipse method at all but calculated the lunar distance measuring its parallax and then applied this value to the lunar eclipse method for finding the solar distance.

There are obvious reasons why Ptolemy could decide not to use the solar eclipse method. First of all, he could have realized that the input values are not too trustworthy: mainly exactly at what latitude of the Hellespont region the eclipse was total and what exactly was the eclipse magnitude at Alexandria. But, I think he discarded the solar eclipse method for a stronger and simpler reason: as I have already shown, if one can calculate the lunar latitude, it is an over-determined method. Ptolemy was able to calculate the lunar latitude, therefore, he did not need any datum from the Hellespont, it was enough with just one angle measured from Alexandria.

Figure 5 represents again the solar eclipse method but leaving in black just the part usable by Ptolemy. In his method, Ptolemy measured angle *ZAM* (i.e., the angular distance between the moon and the zenith at Alexandria) obtaining 50;55° and calculated angle *ZCM* (=49;48°) knowing the latitude of Alexandria (φ_A = 30;58°), the lunar latitude ($\beta_{☾}$= 4;59°) and the declination of the ecliptic at the moon longitude ($\delta^*_{\odot}$ = -23;49°). Knowing both angles it is easy to calculate *CMA* for it is the difference of both angles (1;7°). The problem is solved, one can easy calculate *CM*. For example, applying the rule of sine to the triangle *CAM*, knowing that *CAM* = 180 – *ZAM* and that, of course, *CA* = 1 *e.r.*. Then:

19 $$CM = \frac{\sin(180 - ZAM)}{\sin CMA} = 39;49\ er$$

Ptolemy obtains 39;45 *er*.

So, in a sense, the method used by Ptolemy for calculating the lunar parallax could be understood as a simplification of the solar eclipse method, once one realizes that the method is over-determined. So, even if Ptolemy didn't use the solar eclipse method, he could have been inspired by it for calculating the lunar distance.

There is another fact that could indicate that Ptolemy had in mind the solar eclipse method when he measured the lunar distance. The observation for measuring angle *ZAM* was made by Ptolemy himself in Alexandria 50 minutes after noon of 135 October 1st. It happens that the interval between the solar eclipse used by Hipparchus (-189 March 14) and Ptolemy's observation is 118.542 days which is exactly 18 Saros cycles (or 6 Exeligmos cycles) plus 6 days. Ptolemy's observation is close to a quarter moon, 6 days after the new moon at which, according to the Exeligmos period, a solar eclipse with similar characteristics of that observed by Hipparchus should have taken place. So allow me to suggest a hypothesis that I know that is very speculative and, at any event, impossible to check, but so attractive that I cannot resist offering it. The previous full moon of Ptolemy's observation, i.e., 135 September 25, was a spectacular opportunity (actually, the only one during the active life of Ptolemy) to check Hipparchus's premises in order to use the solar eclipse method. Because he had not yet calculated the lunar distance and parallax, he could not know whether the solar eclipse would be visible at Alexandria before observing it. But the eclipse was not visible at all at Alexandria, because it went too south. Therefore, Ptolemy decided to abandon this method and 6 days later, observed the moon for calculating its distance.

Appendix: assumptions

In all our calculations I simplified the geometrical situation assuming, like Toomer, two suppositions: 1) that the eclipse took place at noon and 2) that at the moment of the eclipse, the ecliptic was perpendicular to the vertical plane. The first assumption allows us to add and subtract declinations and geographical latitudes, as well as to identify the sun altitude with the geographical latitude of the place minus the solar declination. The second assumption, assuming that the Sun and moon are in the same vertical plane, allows as to obtain the angle μ just as the difference between the eclipse magnitudes from these two places (and the sun's apparent size).

Both suppositions are assumed by Ptolemy in his calculation of the lunar distance but, in order to minimize the effect, he explicitly chooses an observation in which the moon is close to these conditions. Of course, solar eclipses are really infrequent, and it would be still rarer for records to exist of a solar eclipse seen from two different places on the same meridian, therefore, Hipparchus could not have followed Ptolemy's strategy: he would have had to simply assume these conditions even if the difference compared to the real eclipse was not truly negligible, or he could try to work out the more complicated geometry assuming different planes. We suppose with Toomer that this would have been beyond Hipparchus's ability. Therefore, the aim of this appendix is just to complete the geometrical analysis.

In this appendix I will analyze the effect that making these two assumptions produces in the value of the distance of the moon. In the first place I will obtain a formula for calculating the lunar distance using as input: the magnitude of the eclipse from two different places, the zenith distance of the Sun, and the angles of the ecliptic with the horizon at H and A (these latter angles can to a very good approximation be found using only the time and the longitude of the Sun). The key point is to obtain a formula in which the lunar theory plays no role, as in eq.(5) if, as I suggested, angle M'HA is obtained from eq.(10.1), i.e., using just the geographical latitudes and the declination of the sun. Then, I will show the equivalence between this new formula and eq.(5), if we assume the two suppositions. Finally, we will apply this formula to analyze the effect that our assumptions produce on the distance of the sun.

1) The formula

For a solar eclipse the magnitude as seen by an observer at a site with geographical latitude φ is

20 $$m = \frac{r_S + r_M - \gamma}{2r_S}$$

or

21 $$\gamma = r_S + r_M - 2r_S m$$

where γ is the distance between the center of the Sun and the center of the apparent (topocentric) Moon as seen by the observer at the moment of maximum obscuration. Note that γ, being a distance, is always positive.

On the other hand, to a very good approximation,

22 $$\beta' = \beta + \pi_\beta \approx \gamma \cdot \mathrm{sgn}(\beta') = \gamma'$$

where β and β' are the true and apparent latitudes of the Moon and sgn(β') is the sign (+ if the obscuration is north of the ecliptic, – if it is south) of β', so γ', unlike γ, is signed. The lunar parallax in altitude π_h is related to π_β by

23 $$\pi_h = -\frac{\pi_\beta}{\sin\psi} = \frac{\beta - \gamma'}{\sin\psi}$$

where ψ is the angle of the ecliptic to the vertical direction at the apparent Moon. This angle is tabulated in Almagest II 13. Since the various parallaxes and both β and β' are small at an eclipse, we have to a very good approximation

24 $$\sin\pi_M = \frac{\sin\pi_h}{\sin\zeta'_M}$$

Then the geocentric distance to the Moon is

25 $$\begin{aligned} D_M &= \frac{1}{\sin\pi_M} \\ &= \frac{\sin\zeta'_M}{\sin\pi_h} \\ &= -\frac{\sin\zeta'_M \sin\psi}{\sin\pi_\beta} \\ &\approx \frac{\sin\zeta'_M \sin\psi}{\sin(\beta - \gamma')} \\ &\approx \frac{\sin\zeta'_M \sin\psi}{\beta - \gamma'} \end{aligned}$$

Now in this expression β is a geocentric value while ζ'_M, γ, and γ' are topocentric (dependent on the latitude of the observer). Thus we may write for the two locations Hellespont (*H*) and Alexandria (*A*),

26 $$(\beta - \gamma'_H)D_M = \sin\zeta'^H_M \sin\psi_H \approx \sin\zeta^H_S \sin\psi_H$$

27 $$(\beta - \gamma'_A)D_M = \sin\zeta'^H_A \sin\psi_A \approx \sin\zeta^A_S \sin\psi_A$$

where we can to a very good approximation write $\zeta'_M = \zeta_S$ since we are at an eclipse. Then subtracting we can eliminate β and get

28 $$D_M = \frac{(\sin\zeta_S^H \cdot \sin\psi_H - \sin\zeta_S^A \cdot \sin\psi_A)}{(\gamma'_A - \gamma'_H)}$$

29 $$D_M = \frac{(\sin\zeta_S^H \cdot \sin\psi_H - \sin\zeta_S^A \cdot \sin\psi_A)}{(\gamma'_A - \gamma'_H)}$$

For completeness, to calculate ψ first calculate the longitude Λ of the ecliptic rising at the moment of the eclipse using

30 $$\tan\Lambda = \frac{\cos\theta}{-\sin\varepsilon\tan\varphi - \cos\varepsilon\sin\theta}$$

and then

31 $$\cos\psi = \frac{-1}{\tan\zeta_S \tan(\Lambda - \lambda_S)}$$

2) the reduction to the other formula

I will show that when we assume that $\psi = 1$, and that the eclipse take place at meridian, both equations eq.(5) and eq.(20) are equal.

The equation for obtaining *AM*' is:

5 $$AM' = \frac{AH \cdot \sin M'HA}{\sin\mu}$$

We will first compare both denominators and then both numerators. Let me start with the denominator. At the meridian we know that:

32 $$(\gamma'_A - \gamma'_H) = \mu$$

Actually,

33 $$\gamma'_A = r_s + r_m - 2r_s m_A$$

and we know that $r_s = r_m = 360/(650\cdot 2) = 0.2769$ and the magnitude m_A at *A* was 4/5. So,

34 $$\gamma'_A = 2r_s - 2r_s \cdot 0.8 = 0.4\, r_s \approx 0.11$$

We also know that γ' at H is 0 because m_H at *H* was 1. Therefore,

35 $$(\gamma'_A - \gamma'_H) = \gamma'_A = \mu$$

So, the difference between the denominator of both equations is just the difference between $\sin(\mu)$ and μ, which is really small, being μ also very small.

Now, let me compare the numerators. Because both ψ are 1, we have:

36 $$(\sin\zeta_S^H \cdot \sin\psi_H - \sin\zeta_S^A \cdot \sin\psi_A) = (\sin\zeta_S^H - \sin\zeta_S^A)$$

Now, according to figure 3:

37 $$\zeta_S^H = ZHM' = \varphi_H - \delta_S$$

38 $$\zeta_S^A = ZAS = \varphi_A - \delta_S$$

Therefore,

39 $$(\sin\zeta_S^H - \sin\zeta_S^A) = (\sin(\varphi_H - \delta_S) - \sin(\varphi_A - \delta_S))$$

Applying some trigonometric identities, we arrived at the following equation:

40 $$(\sin(\varphi_H - \delta_S) - \sin(\varphi_A - \delta_S)) = 2 \cdot \sin\left(\frac{\varphi_H - \varphi_A}{2}\right) \cdot \cos\left(\frac{\varphi_H + \varphi_A}{2} - \delta_S\right)$$

Now, from eq.(7), we know that

41 $$2 \cdot \sin\left(\frac{\varphi_H - \varphi_A}{2}\right) = AH$$

and, from eq.(10.1), we know that:

42 $$\cos\left(\frac{\varphi_H + \varphi_A}{2} - \delta_S\right) = \sin\left(90 - \frac{\varphi_H + \varphi_A}{2} + \delta_S\right) = \sin M'HA$$

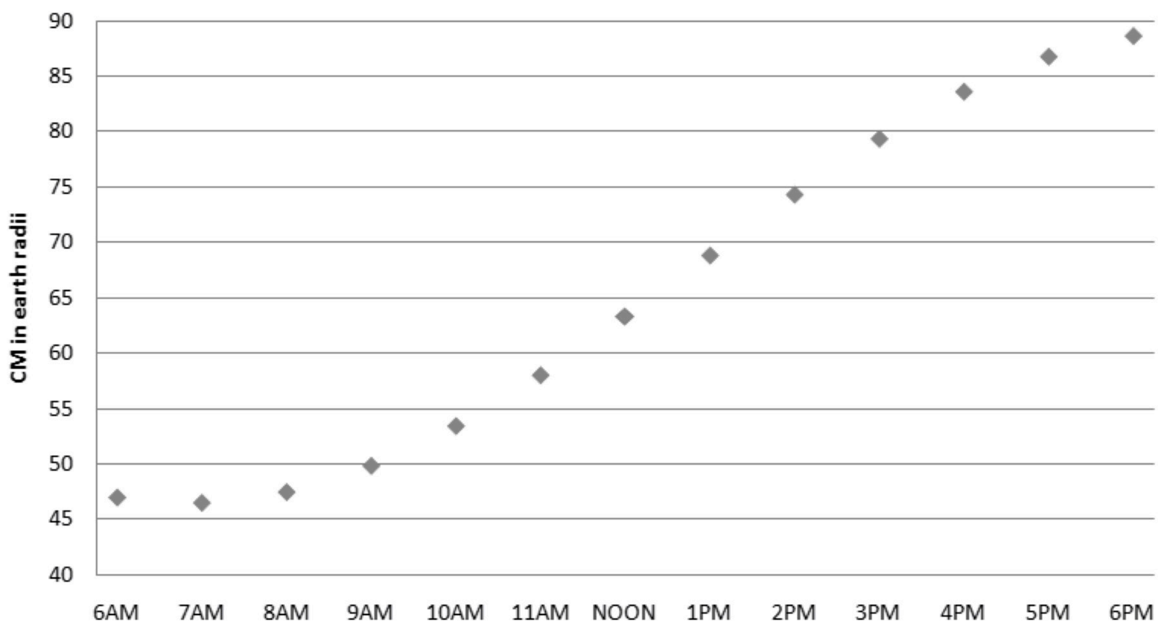
Therefore,

43 $$(\sin\zeta_S^H - \sin\zeta_S^A) = AH \cdot \sin M'HA$$

We have shown, therefore, that if we assume that $\psi = 1$ and the eclipse took place at noon, then $D_M = AM'$, leaving aside the small difference between $\sin(\mu)$ and μ. So, curiously, D_M gives us the distance from Alexandria, not from the center of the earth. This is due to the fact that for obtaining D_M, we used the approximation $\zeta_{mp} = \zeta_s$. In that case D_M from *A* and D_M from *C* are equal. So, if we add 1 earth radius to D_M, we will have exactly our *CM*' (assuming that *CM*' = *AM*'+1).

3) the effect on CM

In the next graph I will show the variation of CM' assuming that the eclipse took place at different times of the day of the eclipse. Therefore, using the new equation, I will keep constant the magnitudes of the eclipse from A and H (and use the Hipparchian value of the apparent size of the sun and moon), but I will take for ζ_s (from H and A) and for ψ (for H and A), the value that corresponds to this time of the day. In this way we can simulate the value for the lunar distance



Graph 4. Variation of *CM*' assuming that the eclipse took place at different times of the day of the eclipse.

that Hipparchus what would have obtained if he had assumed that the eclipse took place at different times.

The value at noon is not 71 because we are not assuming that ψ is 0 and, also, because of the small differences between the real values and the values we suppose Hipparchus assumed for the declination of the Sun and the geographical latitudes of Alexandria and the Hellespont. Nevertheless, the graph is useful for noting two things: first that, again, because the eclipse took place at morning, the noon value could be understood as an upper limit, and not a lower limit, as Toomer (1974:135) proposed; second, that the error due to the assumptions assumed is really significant, on the order of about 15 earth radii (this is the difference between 9 AM and noon).

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