Mathematical discourse in philosophical authors: Examples from Theon of Smyrna and Cleomedes on mathematical astronomy

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Introduction

Ancient philosophers and other intellectuals often mention the work of mathematicians, although the latter rarely return the favor.[1] The most obvious reason for this stems from the impersonal nature of mathematical discourse, which tends to eschew any discussion of personal, or lived, experience. There seems to be more at stake than this, however, because when mathematicians do mention names they almost always belong to the small group of people who are known to us as mathematicians, or who are known to us through their mathematical works.[2]

In order to be accepted as a member of the group of mathematicians, one must not only have mastered various technical concepts and methods, but must also have learned how to express oneself in a stylized form of Greek prose that has often struck the uninitiated as peculiar.[3] Because of the specialized nature of this type of intellectual activity, in order to gain real mastery it was probably necessary to have studied it from youth, or to have had the time to apply oneself uninterruptedly.[4] Hence, the private nature of ancient education meant that there were many educated individuals who had not mastered, or perhaps even been much exposed to, aspects of ancient mathematical thought and practice that we would regard as rather elementary (Cribiore 2001; Sidoli 2015).

Starting from at least the late Hellenistic period, and especially during the Imperial and Late-Ancient periods, some authors sought to address this situation in a variety of different ways—such as discussing technical topics in more elementary modes, rewriting mathematical arguments so as to be intelligible to a broader audience, or incorporating mathematical material directly into philosophical curricula. None of this resulted in the equivalent of a modern textbook, but the results were works that were meant to have an educational, or at least introductory,

1 In this paper, I use the term *mathematician* to denote a practitioner of those disciplines that used mathematical techniques or that investigated mathematical objects—actual or ideal—such as geometry, mechanics, optics, astronomy and astrology, number science (*arithmetikē*), harmonics, computational methods (*logistikē*), spherics (*sfairikē*), sphere-making (*sfairopoiïa*), sundial theory (*gnōmonikē*), and so on. See Netz (1997, 6–9) for a list of the names of very nearly all of the mathematics and mathematical scholars who are known to us from the Greco-Roman period. In this paper, I will argue that a number of the scholars in this list would not have been counted as mathematicians by their peers.

2 A striking counterexample is Ptolemy, who at the beginning of his *Almagest* mentions, with praise, the name of Aristotle, immediately following which he inverts the latter's epistemological hierarchies and asserts that only mathematics can produce real theoretical knowledge and that studies of nature and the divine stand to learn from mathematics—not the other way around.

3 There is a large literature on mathematical Greek prose, for a quick overview of which see Sidoli (2014, 29).

4 Galen recounts that he studied the mathematical sciences from his father in his youth, and Theon of Smyrna explicitly tells us that extensive study from youth was necessary for competence in mathematics; see Sidoli (2015, 395–396) and Jones (2016, 471).

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function. Examples of this type of treatise include Geminos' *Introduction to the Phenomena*, which guides the uninitiated through the basics of mathematical astronomy; Theon of Smyrna's *Mathematics Useful for Reading Plato*, which introduces a medley of mathematical topics divided into number science, geometry, harmonics and astronomy; Cleomedes' *On the Heavens*, which shows how mathematical approaches can be of service in the philosophy of nature and the cosmos; and Proclus' *Commentary on Euclid's Elements Book I*, which expounds the details of deductive geometry as part of a larger philosophical, indeed, spiritual project.

For historians and scholars of the ancient period, works such as these offer a rich source of material for learning about mathematical methods and results that have not been preserved in mathematical sources. They are also, however, fraught with interpretive difficulties because the goals, educational backgrounds, philosophical outlooks, and technical competencies of the authors of such sources are often different from that of the authors whose work they report. In order to make a coherent attempt to read through our sources to the claims and practices of the reported mathematicians, we must both separate the goals of our sources from those on which they report, and also situate the reported work in a context of the methods and research programs actually found in other ancient mathematical works. This must be done on a case-by-case basis, and in many instances we will not be able to say much with real certainty.

In this paper, I will look at two short examples—taken from Cleomedes and Theon of Smyrna—with the aim of articulating the context of ancient mathematical work from which this material originates. Although I will not argue that a full reconstruction of the underlying mathematical models and methods is possible, I hope to show that when we situate the ideas and methods discussed in these sources within a context of ancient mathematical methods reported in other sources, we can develop a clearer picture of both the ancient mathematics reported and of the ways our sources handled this material.

Cleomedes, On the Heavens *I.7*

In his only known surviving work, *On the Heavens*, Cleomedes—a Stoic philosopher who lived sometime between the middle of the 1st century BCE and the end of the 2nd century CE—seeks to introduce cosmography to students of philosophy who he takes to have only the most rudimentary knowledge of mathematics.[5] Hence, he goes to some lengths to simplify geometrical configurations to the extent that they can be followed from his oral exposition alone, and with no mathematical procedures beyond the rule-of-three.

The passage below is his discussion of a computation of the terrestrial circumference, as 250,000 stades, which he attributes to Eratosthenes, *Heavens* I.7. Cleomedes tells us that Eratosthenes' procedure was geometrical and difficult, so the striking simplicity of the configuration and the computation that he goes on to describe may come as somewhat of a surprise.

Furthermore, there are issues involved in reconciling both the numerical value and the overall simplicity of Cleomedes' account with Heron's claim in *Dioptra* 35 that "Eratosthenes, having worked rather more accurately than others, showed in his book entitled *On the Measurement of the Earth*" that the terrestrial circumference is 252,000 stades (Acerbi and Vitrac 2014, 104). If the mathematical methods implied by Heron's *Dioptra* 35 are any indication of what he means by a

5 See Bowen and Todd (2004, 1–17), for an introduction to this source.

mathematician working carefully,[6] then it is difficult to see how he could have been impressed by something so trivial as the configuration described by Cleomedes.

Eratosthenes was a younger contemporary of Archimedes, to whom the latter chose to send his *Method*. This, as well as Heron's assessment, is an indication that Eratosthenes was a serious mathematician whose work developed the efforts of other early Hellenistic mathematicians. Hence, we should regard Eratosthenes' lost *On the Measurement of the Earth* as a treatise in the tradition of Aristarchus' *On the Sizes and Distances of the Sun and the Moon*, or Archimedes' *Sand Reckoner*.[7] That is, it probably made use of somewhat crude, observational hypotheses,[8] extensive geometrical modeling using lettered diagrams, proto-trigonometric inequalities between angles and sides of right triangles, ratio manipulation, and some arithmetic operations.[9] None of this, however, is found in Cleomedes' account.

We should consider the possibility that Cleomedes had never read the original source and was less interested in reporting what Eratosthenes actually did than in drawing out certain mathematical principles so as to make his own philosophical points. Cleomedes compares Eratosthenes' procedure with that of Posidonius, from whom he may well have taken both accounts. Although Posidonius was probably interested in arguing against the claims of mathematicians to be able to produce specialized knowledge about the physical world,[10] Cleomedes seems to have had more restricted goals. He was clearly interested in exhibiting a direct cognitive relationship between assumptions and conclusions,[11] and appears to have reworked Eratosthenes' argument so as to make it amenable to such an approach. The goal of Cleomedes' procedure is to show how geometrical assumptions can be combined with sense perceptions to make true assertions about the physical world that go beyond what our senses alone can directly decide.

After describing Posidonius' procedure, Cleomedes turns to Eratosthenes', saying:[12]

> [1] … That of Eratosthenes involves a geometric procedure (*geōmetrikē efodos*),[13] and it is thought to involve something more obscure. But, the assertions by him will be made clear from the following prior suppositions (*proüpothemenos*) of ours.
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> [2.1] Let it here have been assumed by us, first, that Soēnē and Alexandria are situated under the same meridian, [2.2] and [second] that the distance between the cities is 5000 stades, [2.3] and third that the rays sent down from different parts of the sun to different parts of the earth are parallel—just as the geometers assume holds. [2.4]

6 See Sidoli (2005) for a discussion of these mathematical methods. This should be compared against Acerbi and Vitrac (2014, 103–115) for some textual corrections.

7 See Dijksterhuis (1987, 360–373), Berggren and Sidoli (2007), Van Brummelen (2009, 20–32), and Carman (2014) for discussions of the mathematical methods of these sources and their use of hypotheses.

8 That is, claims about observational results that may or may not have been the result of carefully made, carefully recorded observations, but which, in the structure of the mathematical argument, are taken simply as assumptions.

9 See Carman and Evans (2015) for a discussion of the details of what such a research program might have involved.

10 See Bowen (2007) for a discussion of these sorts of jurisdictional disputes in ancient authors, in which Posidonius played a role.

11 See Bowen (2003) and Bowen and Todd (2004, 11–15) for discussions of Cleomedes' interest in the structure of demonstrations.

12 My translation can be compared with that of Gratwick (1995, 179–180) or Bowen and Todd (2004, 81–84). I have tried to preserve those places where Cleomedes' prose seems technically awkward.

13 Bowen (2003) gives a justification for translating *efodos* as *procedure*. See below for a discussion of Cleomedes' characterization of Eratosthenes' procedure as "geometrical."

Fourth, let it be further assumed—as is shown by the geometers—that straight lines that fall on parallel [lines] make the alternate angles equal; [2.5] fifth, that arcs that stand upon equal angles are similar; that is, they have the same proportion and the same ratio to their own circle (*oikeios kuklos*)—which is also shown by the geometers, for whenever arcs stand on equal angles, if any one of them is ten parts of its own circle, all of the rest will be ten parts of their own circles. [2.6] One who has mastered these things would have no difficulties understanding the procedure of Eratosthenes, which is as follows.

[3.1] He says that Soēnē and Alexandria are situated under the same meridian. So, since meridians are great [circles] of those in the cosmos, those lying under them will necessarily be great circles of the earth. [3.2] Hence, however much (*hēlikos*) this procedure shows the circle reaching through Soēnē and Alexandria to be of the earth, so much (*tēlikoutos*) is the great circle of the earth.

[4.1] He states—and it holds—that Soēnē lies under the circle of the summer tropic. So, whenever the sun comes to be in Cancer and brings about the summer tropic,[14] exactly at culmination the gnomons of sundials (*hōrologion*) are necessarily shadowless, because the sun is situated perpendicularly above—and it is said that this is three hundred stades in diameter. [4.2] But, in Alexandria, at the same hour, the gnomons of sundials cast shadows, inasmuch as this city is situated further north than Soēnē.

[5.1] Now, since these cities are situated under a great circle meridian, if we produce an arc from the tip of the shadow of the gnomon around to the base of the gnomon of the sundial in Alexandria, this arc will be a part (*tmēma*) of the greatest of the circles in the bowl [of the sundial],[15] since the bowl of the sundial is situated under a great circle. [5.2] So, if we next imagine (*noeō*) straight lines extended through the earth from each of the gnomons, they will meet at the center of the earth. [5.3] So, since the sundial in Soēnē is located perpendicularly under the sun, if we further imagine a straight line from the sun reaching to the gnomon tip of the sundial, it will be one straight line, reaching from the sun as far as the center of the earth. [5.4] So, if we imagine another straight line from the tip of the shadow of the gnomon to the sun, being produced from the bowl in Alexandria, this [straight line] and the aforesaid straight line will be parallels—that is, they are extending from different parts of the sun to different parts of the earth. [5.5] Now, a straight line reaching from the center of the earth to the gnomon in Alexandria falls on these parallels, hence the alternate angles are made equal—one of which is at the center of the earth, at the meeting of the straight lines that were produced from the sundials to the center of the earth, while the other is at the meeting of the tip of the gnomon in Alexandria and the [straight line] produced from the tip of its shadow to the sun through the point of contact. [5.6] And, upon this [angle] stands the arc produced from the tip of the shadow of the gnomon to its base, while the [arc] produced from Soēnē to Alexandria [stands] upon that at the center of the earth.

14 Literally, "makes (*poieōn*) the summer turn."

15 That is, it will be one of the great circles of the bowl. This expression is not usual in Greek mathematical texts, so I have tried to preserve the literal expression.

> [6.1] Now, the arcs are similar to one another—that is, they stand on equal angles. Therefore (*ara*), the ratio which the [arc] in the bowl has to its own circle, is also that ratio which the [arc] reaching from Soēnē to Alexandria has [to its own circle]. [6.2] But, the [arc] in the bowl is found to be, indeed, a fiftieth proper part (*meros*) of its own circle. So, the distance from Soēnē to Alexandria must also necessarily be a fiftieth proper part of the great circle of the earth—and this [distance] is 5000 stades. [6.3] Therefore (*ara*), the whole circle comes to 25 myriads. Such is the procedure of Eratosthenes.
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> (Todd 1990, 35–37)

Once again, this passage exhibits a number of striking features, when read from the perspective of Greek mathematical sources. The first is the absence of a diagram and letter-names, despite Cleomedes' characterization of the procedure as geometrical. Since diagrams and letter-names are one of the defining characteristics of Greek geometrical prose, it is clear that Cleomedes means something quite specific by calling this procedure *geōmetrikē*. As discussed above, he probably means that the grounds for accepting the claim as true depend on certain hypotheses or knowledge claims that are geometrical, as opposed to physical, or based in sense perception.

Although the deductive language of the passage is fairly standard for philosophical writings, there are peculiarities from a mathematical perspective. For example, the assumptions are set out using impersonal imperatives, while the constructions are imagined using personal verbs, geometrical terminology is used in unusual ways, and there are a number of peculiar expressions that will be discussed in detail below.

Finally, although Cleomedes deliberately structures his account, there is little or no trace of the usual mathematical structure, such as we find in Aristarchus' *On Sizes*, or, less concretely, in Archimedes' *Sand Reckoner*.[16] There is no enunciation of the computational result to be obtained, no exposition of the actual configuration through a lettered diagram, no geometrical constructions, nor any real mathematical argumentation. Instead, Cleomedes has carefully laid out all of the starting points of the reasoning in the beginning, as hypotheses, so that the result can be seen to follow almost immediately from a description of the spatial arrangement of the objects in question. That is, he shows that if certain geometrical propositions are true, they allow us to infer true claims about the world that we cannot perceive, based on the local world that we can perceive, or potentially measure. In order to appreciate Cleomedes procedure, it will be useful to follow his argument in detail.

He begins, in [1], by asserting that the procedure of Eratosthenes is geometrical and difficult, but that he will simplify it, by clearly stating all of its starting points. Bowen and Todd (2004, 78, n. 1), following a suggestion of Gratwick (1995, 178, n. 1), argue that here *geōmetrikos* must be understood as *geodesic*, since Posidonius' procedure also employs geometry. Indeed, modern reconstructions of Posidonius' method often formulate it as parallel to that of Eratosthenes.[17] But these are reconstructions. Cleomedes' account of Posidonius' procedure makes no use of

16 It is well known that Greek propositions rely on structure to convey certain aspects of their deductive force. For discussions of the structure of propositions in the *Elements*, see Mueller (1981, 11–12) and Netz (1999b). Acerbi (2011a, 1–117) bases his discussion of structure on the *Elements* as well, but also treats other Greek mathematical works.

17 For example, Taisbak (1974, 257–259) formulates the two arguments as structurally similar, and provides similar diagrams for each, in which he is followed by Bowen and Todd (2004, 78–85, 181–183).

geometry[18] and it is possible to explicate the argument with a single analemma diagram and no use of the geometrical assumptions that Cleomedes posits for Eratosthenes' procedure.[19]

Whatever our opinion of Posidonius' method, however, it is clear that Cleomedes himself intends us to understand Eratosthenes' procedure as more geometrical than it, because he will go on to preface the latter by discussing five assumptions, three of which he explicitly attributes to "the geometers." Hence, in Cleomedes' opinion, Eratosthenes' procedure relies on a series of assertions and knowledge claims made by a group of specialists—and it is the use of this specialized knowledge, and presumably the use of a specialized way of writing, that marks Eratosthenes' procedure *as geometrical*. Moreover, it is clear from his presentation that Cleomedes intends to simplify this procedure, making it more intelligible to the non-specialist.

The five assumptions that Cleomedes proposes are given in the next paragraph, [2]. The fact that Cleomedes claims these hypotheses as his own is a good indication that he has rearranged the argument. Although some of them, such as [2.3] concerning the rays of the sun, may well have been asserted by Eratosthenes, others, such as [2.4] and [2.5] were most likely assumed without comment, as obvious—just as they would have been by Aristarchus or Archimedes.

The first two hypotheses, [2.1, 2.2], namely that Soēnē and Alexandria are assumed to lie below a great circle through the celestial poles, and that they should be taken as 5000 stades apart, are essentially the same as those assumed by Posidonius, and Cleomedes apparently did not regard them as geometrical. He presumably thought that they were decidable using empirical methods or were idealizations of empirical claims.[20]

The next three hypotheses are those which Cleomedes claims are either assumed or demonstrated by the geometers. The first of these, [2.3], is that the lines joining different parts of the sun with different parts of the earth must all be taken to be parallel—that is, the lines drawn from any point on the earth to any point on the sun are all parallel. Gratwick (1995, 187–188) argued that this hypothesis must have been muddled by Cleomedes because it does not agree with our understanding of the composite nature of shadows. Furthermore, if the sun is taken to be a body at some definite distance, and of some definite magnitude, as was generally held to be the case, then it would be strictly false that these lines would be parallel. Nevertheless, consideration of Aristarchus' hypotheses in his *On Sizes*,[21] shows that mathematicians did, in fact, make assumptions that they knew to be strictly false, in order to see where the reasoning would lead. If Eratosthenes' approach was anything like that found in Aristarchus' *On Sizes* or Archimedes' *Sand Reckoner*, he would have used observational hypotheses as means to constructing an idealized

18 That is, unless any spatial reasoning must be counted as geometry.

19 For example, we can draw both horizons in the same analemma figure, so that the arc between Canopus on the northern horizon of Rhodes and the southern horizon of Alexandria represents at once the difference in the celestial and the terrestrial latitudes.

20 Bowen (2003, 63—64) claims that Cleomedes took Eratosthenes' value of 250,000 to be better than Posidonius' value of 240,000 because he also asserts it in *On Heavens* I.5 and II.1. This then involves us in having to decide why Posidonius' 5000 stades between Rhodes and Alexandria is a weaker assumption than Eratosthenes' 5000 stades between Alexandria and Soēnē. Another possibility is that Cleomedes took these values to be essentially the same and just asserted the round figure. Since, in either case, a difference of about 200 stades (4%) would give the same value, Cleomedes may well have accepted that, given the crudeness of the procedure, this difference was undecidable—especially since, in [4.1], he seems to say that Soēnē lies in a region of 300 stades that can be *assumed* to lie directly below the summer tropic, so that 200 stades would be well within the margin of error of the hypothesis.

21 See Berggren and Sidoli (2007, 231–234) and Carman (2014, 55–58) for discussions of Aristarchus' use of hypotheses.

geometric model, which would then have become the sole object of his reasoning and computation. Moreover, as Carman and Evans (2015) have argued, the assumption of a sun at an infinite distance may have been used to compute a lower bound for the size of the earth, which could then be compared with an upper bound computed under the equally idealized but contrary assumption of a point-sun at a given, or at least bounded, distance. At any rate, in his presentation, Cleomedes calls on this hypothesis in precisely the way that he states it—as involving the lines joining any point of the earth to any point of the sun.

The next assumption, [2.4], is simply *Elements* I.29—"a straight line falling on parallel straight lines makes the alternate angles equal to one another, the external equal to the opposite and internal, and those on the same side equal to two right angles" (Heiberg 1969–1977, 41). Eratosthenes, however, certainly would not have taken this as a hypothesis in the sense that Cleomedes intends. If Eratosthenes assumed that this were true, he did so because it was the subject of geometrical demonstration and could be assumed without further comment. Whether or not Eratosthenes was thinking of Euclid's text, he certainly would have known that this proposition had been demonstrated. Cleomedes acknowledges as much by pointing out that "the geometers" prove this proposition, but still asserts that he will use it as an explicit assumption of his philosophical argument. That is, he avoids the question of a toolbox of mathematical knowledge and techniques that usually forms the background of any mathematical argument.[22]

The final geometrical assumption, [2.5], is the claim that arcs that subtend equal angles are similar—that is, they compose the same part of, or have the same ratio with, their respective circles. The expression that Cleomedes uses for the circle to which an arc belongs, *oikeios kuklos*, although common in *On the Heavens*, is not standard in Greek mathematical prose. Furthermore, a repetition of the "same proportion" and the "same ratio" is not usual in mathematical authors, and would be strange from a mathematical point of view. Both of these expressions mark this passage as something that a mathematician was unlikely to have actually said. Indeed, I am not aware of any proof of this claim in the elementary geometrical texts, although *Elements* III.def.11 mentions similarity of segments. Nevertheless, this is assumed by Aristarchus in his *On Sizes*—for example in Prop. 7—and, of course, lies at the core of later chord-table trigonometry. The fact that Cleomedes gives a numerical example is an indication that he is thinking of computational work such as we find in *On Sizes*. Eratosthenes may have argued explicitly for this proposition, as part of his development of the proto-trigonometric tradition of Aristarchus and Archimedes, or Cleomedes may simply be using the idea of *showing* loosely.

The subsequent paragraphs develop the logical argument that Cleomedes presents, calling on and applying each of the hypotheses in very nearly the same order as they were presented. Since the procedure itself calls on two further empirical claims, there must be some difference between these and those discussed above for the first two hypotheses, [2.1, 2.2]. As will be discussed below, the two empirical claims presented in the course of the argument are, at least in principle, verifiable with direct sense perception, whereas the distance between the two cities and their location on a single meridian can only be apprehended through a variety of sense perceptions and logical inferences.[23] It is also possible that [2.1] and [2.2] are taken as hypotheses because they are being acknowledged as not strictly true.

After reiterating the first hypothesis, in [3.1], and pointing out that this implies that the two cities lie on a single great circle of the earth, Cleomedes gives a sort of summary of the whole

22 The mathematical toolbox is discussed by Netz (1999a, 216–235), who bases this idea on a research project by Saito (1997, 1998).

23 See the discussion by Bowen (2003), for the distinction between these types of propositions in Cleomedes' work.

procedure, in [3.2]. Namely, if we can determine what part of the great circle of the earth joins the two cities, then we can determine the size of the whole great circle. Cleomedes' way of putting this is vague on two counts. The first is that he does not use the usual language of ratios, or parts, but more general expressions for relating amounts, or quantities (*hēlikos*, *tēlikoutos*)—here intended to imply a use of the computational rule-of-three.[24] The second is that he uses the word *kuklos* in a non-standard way, as a synonym for *perifereia*, denoting both an arc and a whole circumference.

The next paragraph, [4], introduces the first set of empirical considerations that were not assumed from the start. Namely, the claim that around the summer solstice, each day at midday, a gnomon at Soēnē casts no shadow, in [4.1],[25] while one at Alexandria does, in [4.2]. It also asserts that shadows are said to disappear at midday around the solstice "three hundred stades in diameter" (Todd 1990, 36). This ambiguous expression is repeated in two other places in the treatise, neither of which helps us much in understanding its meaning (Todd 1990, 51, 53). The use of *diametros* implies that we are talking about a circle, or a rectangle, but it is unclear how either of these would be defined. Perhaps we are discussing a region defined by midday at around the same time and noon shadowlessness around the summer solstice.[26] Another possibility is that Cleomedes is referring to a band of midday shadowlessness around the latitude of Soēnē. The width of such a region, however, would be measured by the arc of a great circle through the poles, which he elsewhere calls a distance, *diastēma*. At any rate, Cleomedes gives no indication of where in this region he takes Soēnē to be, which is further indication that the starting points of the argument are meant to be understood as loose idealizations of reality.

The bulk of the argument, [5], consists of a description of the spatial arrangement of the various elements of the model, which attempts to do away with the need for a diagram. The first sentence, [5.1], is a way of expressing the idea of passing a cutting plane through the celestial meridian above the two cities, such that it passes through the sundial and gnomon in Alexandria, producing a great circle in the sundial's bowl—which is implicitly taken to be spherical. Hence, an arc from the tip of the shadow to the base of the gnomon—that is, along the shadow—will be a great circle of the dial's face. All of this follows from the fact that a great circle is concentric with its sphere—as shown in Theodosius' *Spherics* I.6, and assumed implicitly in the texts on spherics by Autolycus and Euclid. The next stage of the argument involves a solid configuration, which is signaled by Cleomedes' use of the word *noeō*—a standard term in Greek mathematical prose used to indicate that we are dealing with something that is three dimensional, or not contained in the diagram.[27] Cleomedes, however, uses the term with a personal expression—"we imagine." We imagine lines extended through idealized gnomons in the two cities meeting in the center of the earth, [5.2]. Hence, that of the gnomon in Soēnē will be produced continuously as a ray of the sun, [5.3], and will be parallel to a ray of the sun through the tip of the gnomon at Alexandria, [5.4], so that the continuation of this gnomon falls on these two parallel rays, making equal alternate

24 A passage in Galen's *Art of Medicine* directly relates these expressions to proportion: "For, the spinal [cord] is, in this way, proportional to the many, and just as (*tēlikoutos*) the size of the vertebrae, so much (*hēlikos*) is the spinal cord, and so the whole backbone" (Kühn 1821, 132).

25 Gratwick (1995, 192) claims that Eratosthenes must be referring to the exact moment of midday on the summer solstice, but this is not necessitated by the text, nor was anyone likely to have been able to determine the precise day of summer solstice simply by observing the noon shadow of a gnomon.

26 That is, a region in which we cannot perceive any difference of latitude or longitude.

27 See Netz (1999a, 52–56), Sidoli and Saito (2009, 592, n. 41), and Netz (2009) for discussions of this terminology.

angles, [5.5]. Hence, the angle of the shadow at Alexandria is the same as that at the center of the earth, [5.6]. These passages present the core of the geometric model, and almost all of the mathematical reasoning involved in the procedure that Cleomedes describes.

The final paragraph, [6], asserts the conclusion of the procedure—hence the use of *ara*. Since the arcs stand on equal angles, they are the same part of the circles in which they stand, [6.1]. This is stated, here for the first time, as a ratio, but the expression, although not unknown in general usage, is not common among mathematical authors, who usually assert proportionality as a *sameness* of ratio. We are then told that the shadow in the dial at Alexandria is actually found to be 50′ of the great circle, expressed in the Egypto-Greek fractions that would have been familiar to any educated reader—that is, a unit fraction, or proper part (*meros*).[28] This is another empirical datum, but Cleomedes does not assert it as a hypothesis. Hence, unlike the distance between the two cities, he seems to take it as something directly verifiable, and hence true—just as the location of Soēnē below the summer tropic. Thus, the two given values—one assumed and the other found—can be subjected to the rule-of-three to give the value of the circumference of the earth, 250,000, which is stated as the conclusion of the whole procedure, [6.3].

In this way, Cleomedes rearranged Eratosthenes' claims and arguments so as to present them as part of his overall project of demonstrating Stoic procedures to his audience. Hence, it is difficult to reconstruct Eratosthenes' approach directly from Cleomedes' account. To form a sound idea of Eratosthenes' thought, we need to reconstruct his work in the context of authors such as Aristarchus and Archimedes, in order to see how it could have been sufficiently mathematical as to have struck Heron as having been carefully undertaken.[29] Such a project will, however, have been different both in presentation and conception from that of Cleomedes.

Theon of Smyrna, Mathematics Useful for Reading Plato

Theon of Smyrna, a middle Platonist, who can probably be dated to the early part of the 2nd century CE, based on a portrait bust from Smyrna,[30] was a philosopher writing for students of philosophy who wanted to understand Plato, but who had not had much training in mathematics. He only hoped that they should have at least advanced through the "first geometrical elements" (Hiller 1878, 16). Whatever the nature of Theon's own training, he seems never to have developed much understanding of either mathematics or the standard usage of Greek mathematical prose.

Theon took much his material from the philosopher Adrastus, but the technical passages are generally presumed to have originated in the work of a mathematician, such as Hipparchus. In going through Adrastus' discussion of the eccentric and epicyclic solar models, Theon demonstrates that, in the eccentric model, the solar orbit is *given in position and in magnitude*. The property of being *given* was fundamental in theoretical Greek mathematics, and Euclid devoted his *Data* to developing theories of different modes of being given. In broad strokes, an object was said to be given if it were present at the beginning of the mathematical discourse, introduced by the mathematician, or produced from either of these in a determinate way.[31]

28 See Bernard, Proust and Ross (2014, 38–51) and Sidoli (2015) for discussions of mathematics education in the Greco-Roman world. Here and below, I use n' for $1/n$, as is standard in scholarship on Greek mathematical sources. In scholarship on Egyptian sources such parts are usually denoted $\bar{n}$.

29 See Carman and Evans (2015) for an investigation along such lines.

30 Musei Capitolini inv. 529; see Richter (1965, vol. 3, 285).

31 See Acerbi (2011b) for an overview of the way the term is used in Greek mathematical texts and Sidoli (2018)

Theon's argument proceeds by taking the degree position of the solar apogee and the ratio of solar eccentricity as given—but it does not do so in a straightforward way. The solar apogee is taken to be Gem 5,2′°—that is, 65;30°. Theon does not say how this value is derived but it is the same as that in the solar model attributed to Hipparchus by Ptolemy.[32] The ratio of the distance from the earth to the center of the sun's orbit compared to the radius of the sun's orbit is taken to be 1 : 24, as was apparently known "through the treatise *On Sizes and Distances*," probably by Hipparchus (Hiller 1878, 158).

In Ptolemy's account of Hipparchus' procedure, these two values are derived through chord-table trigonometric computation on the basis of assumed observations of season lengths. The following passage of Theon's presentation appears instead to argue that we can use the derived parameters of the model to show that the position and size of the solar orbit is given—that is, determined in place and in size. In fact, however, this passage also includes numbers relating to season length—the sums of the length of spring plus summer and of autumn plus winter. These numbers alone, however, are not sufficient to determine the parameters that Hipparchus derived. The passage we are interested in reads as follows:

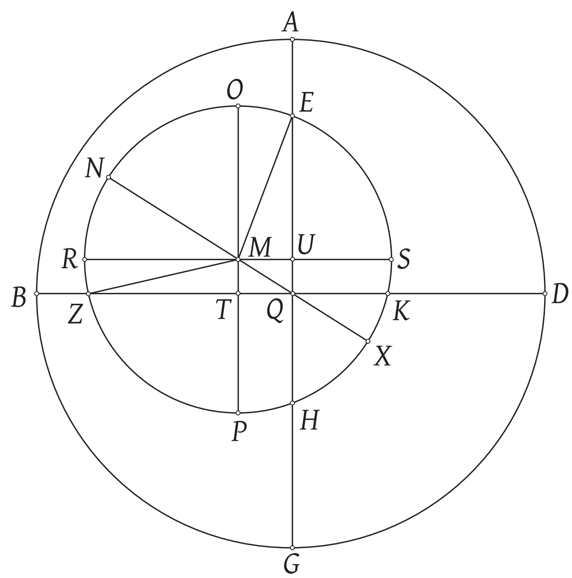

Figure 1. Reconstructed diagram for Theon's eccentric solar model, following Martin (1849, Descriptio III).

> [1] The circle *EZHK* is found *given in position and in magnitude*. [2] For through *M* let parallels to *AG*, *BD* be produced perpendicular to one another, *OP*, *RS*; and let *ZM*, *ME* be joined. [3] Then, it is clear that, the circle *EZHK* being divided into 365,4′ days, arc *EZH* is 187 of such days, and *HKE* is 178,4′ days. [4] So (*ara*), each one of the pairs *EO*, *PH* and *RZ*, *SK* are equal, but the original arcs *SP*, *PR*, *RO*, and *OS* are equal to 91,4′,16′ of such [days]. [5] So (*ara*), the given angle *OMN* will be equal to *QMT*, and likewise,

for a discussion of the concept of given in Greek mathematics.

32 Ptolemy's presentation of Hipparchus' solar model, with which this passage should be compared, is found in *Almagest* III.4.

angle *RMN* is equal to angle *UMQ*.[33] [6] So (*ara*), the ratio *MT* to *MQ*, or rather (*toutesti*) *MT* to *TQ*, will be [given]. [7] So (*ara*), triangle *MTQ* is *given in form*.[34] [8] And the center of the cosmos, *Q*, to each of the points *N* and *X* is given, for one defines the greatest distance, and the other the least; and *QM* is between the center of the cosmos and [the center] of the solar circle. [9] So (*ara*), the circle *EZHK* is *given in position and magnitude*, since it is found through the treatise *On Sizes and Distances* that the ratio *QM* to *MN* is nearly one to 24.

(Hiller 1878, 157–158)

There are a number of conspicuous features of this passage that give us pause in categorizing it as normal mathematical prose. The first arises only when we look at the manuscript source for this passage, *Marc. gr.* 303. The diagram for this passage, like many of the diagrams for this text in the manuscript, appears to have been drawn as a sort of afterthought (folio 12r, Figure 2). It was squeezed into the bottom margin, where it was later partially trimmed off. It is so poorly drawn that it is unlikely that the text could have been understood on the basis of this diagram alone. Diagrams in mathematical texts are sometimes poorly drawn,[35] but those accompanying Theon's treatise in *Marc. gr.* 303 are particularly inept. Nevertheless, the use of letter-names in the text makes it clear that it was meant to be read with a diagram, and we may presume that the diagram that Theon originally produced was correct. Hence, it seems that the copyists and readers of this treatise thought of it as part of a philosophical tradition and were not much concerned with mathematical details, and the corruption of the diagram was probably due to the accidents of transmission.

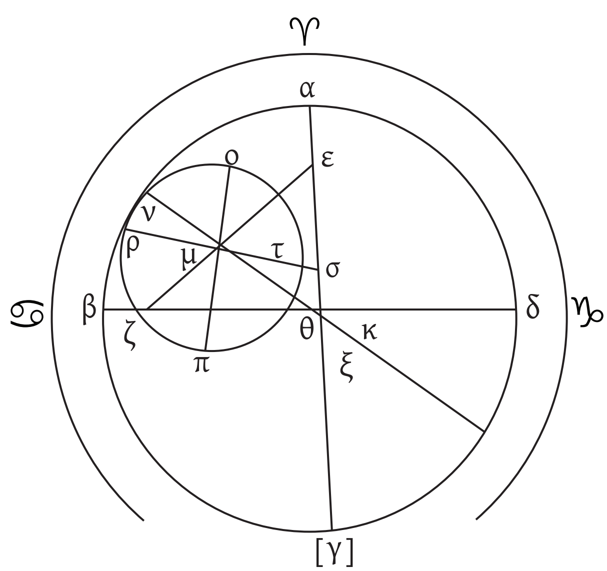

Figure 2. Diagram for Theon's eccentric solar model in *Marc. gr.* 303, f. 12r.

The next conspicuous feature of this passage is its peculiar use of mathematical prose. The passage appears to have been written by someone who was uninterested in following common

33 *Elements* I.15.

34 *Data* def.3.

35 See Saito and Sidoli (2012) for an overview discussion of the diagrams in the manuscripts of Greek mathematics.

mathematical usage and perhaps did not fully understand the direction and force of the original argument. While each of his statements is valid, why this is so is often not clear from Theon's exposition. For example, the particle *ara* usually denotes strict deductive force from the forgoing argument and is translated with *therefore*.[36] For Theon, however, it introduces nearly every statement and rarely indicates strict logical dependence; hence, I have translated with *so*, indicating a merely temporal transition. Another example is Theon's use of *toutesti*, which I have translated with *or rather* in sentence [6]. This expression usually indicates strict equality, or sameness. Theon, however, uses it to indicate a given ratio which is not the same, but which is given for the same reason.

The final feature that marks this passage as non-mathematical is its overall lack of structure. Structure is one of the most conspicuous features of Greek mathematical prose—it is structure that tells us what we are given and what we wish to show or to do, where we are in course of the argument, what we have done and what remains to do.[37] The lack of structure in Theon's account makes it difficult to understand what he thinks he is doing and how he intends to do it. In order to understand Theon's passage, we must read it in the context of coherent works of Greek mathematics, such as Euclid's *Data* and Ptolemy's *Almagest*. These works give us a sense of the meaning of *given* with which Theon is working, and the underlying computational practice, which he ignores.

The opening sentence, [1], states the claim to be shown—namely, that a certain circle is fixed in place and in size. Then, Theon begins his treatment of the problem, in [2], by assuming, without comment, that the center of the sun's eccentric orbit is located at *M* and producing lines through this point parallel to the lines joining the earth with the cardinal points of the ecliptic. In these sentences, he uses the usual expressions of geometric constructions.

In [3], Theon divides a solar year of $365{,}4'^{d}$ into two parts such that $\mathbf{Arc}(EZH) = 187^{d}$ and $\mathbf{Arc}(HKE) = 178{,}4'^{d}$.[38] Theon seems to imply that these latter numbers follow as a matter of course from the year length. In fact, however, they come from season lengths that Hipparchus claims to have observed. According to Ptolemy, in *Almagest* III.4, Hipparchus derived the parameters for his solar model, $e : R = QM : MN$ and $\lambda_A = \mathbf{Ang}(OMN) = \mathbf{Ang}(AQN)$, under the assumption that the interval from the spring equinox to the summer solstice is $94{,}2'^{d}$ while that from the summer solstice to the autumnal equinox is $92{,}2'^{d}$. Theon quotes these season lengths a few pages earlier in his treatise (Hiller 1878, 152–154), but the fact that he does not give the spring and summer separately here indicates that he may have been unaware that a division of the year into 187^{d} and $178{,}4'^{d}$ is insufficient for the determination of the model. Moreover, it is not clear from Theon's presentation how he intends the double season lengths that he asserts to be related to the rest of the argument.

In [4], the geometry of the figure is used to infer that $\mathbf{Arc}(EO) = \mathbf{Arc}(PH)$, so $\mathbf{Arc}(RZ) = \mathbf{Arc}(SK)$, and that $\mathbf{Arc}(SP) = \mathbf{Arc}(PR) = \mathbf{Arc}(RO) = \mathbf{Arc}(OS)$ are each $365, 4'^{d} \div 4 = 91, 4', 16'^{d}$, which is again written in the usual form for Egypto-Greek fractions, proper parts. Theon's use of *ara*, however, indicates that he took all this to be implied, somehow, from the double season lengths—which is not the case.

Theon next states, in [5], that $\mathbf{Ang}(OMN)$ is given. This is so because, in the previous discussion, he has remarked that the solar apogee is Gem $5, 2'^{\circ}$, so that $\mathbf{Ang}(AQN) = \mathbf{Ang}(OMN) =$

36 See discussions of *ara* by Mugler (1958, 82–83) and Acerbi (2012, 173–174).

37 For discussions of the importance of structure in Greek prose see Netz (1999a, chaps. 4 and 5) and Acerbi (2011a).

38 I use the following abbreviations: $\mathbf{Arc}(a)$ for arc a, $\mathbf{Ang}(b)$ for angle b, and $\mathbf{T}(c)$ for triangle c.

65; 30°. Hence, all three angles of $\mathbf{T}(MTQ)$ are given in degrees. Again, the lack of structure makes it difficult to see that this is being taken as a given in the argument.

Theon then claims, in [6], that $MT : MQ$ and $MT : TQ$ are both given. From a purely geometric perspective, these would follow as a result of *Data* 40, which shows that a triangle which has three given angles, is *given in form*, and the definition of given in form, *Data* def.3, which states that a figure that is given in form has its angles given, and the ratios of its sides given. Theon will go on, however, in [7], to state that $\mathbf{T}(MTQ)$ is *given in form*, so it is unclear, again, how he understands the progression of the deduction.

Whatever the intended order of the reasoning, *Data* 40 is a purely geometric argument that relies on constructing a similar triangle and gives us no means of stating the ratio of the sides of a triangle as a pair of numbers. That is, *Data* 40 provides no way of treating the ratios of a triangle *given in form* other than by laying out a set of line segments. What is required here, however, is some method, presumably by means of a chord table, of using the values of the angles in $\mathbf{T}(MTQ)$ to derive the ratios of the sides as values, or relations between values. When Theon states, in [7], that $\mathbf{T}(MTQ)$ is *given in form* he means the same thing that Ptolemy would have meant if he had used that expression—that is, its angles and the ratios of its sides are both geometrically contractable and are also expressible by determinate numerical values for the purposes of calculation.[39]

The rest of the passage is again muddled but the overall sense is clear. Sentence [8] asserts that $NQ : XQ$ is given because, as [9] states, $MQ : MN = e : R = 1 : 24$. We can flesh this out by noting that if $MQ : MN$ is given, then *Data* 5 implies that $QX : MN$ is given, while *Data* 6 implies that $NQ : MN$ is given. Finally, by *Data* 8, $NQ : XQ$ is given.[40]

Again, 1 : 24 is one of the parameters of the Hipparchus' solar model, and Theon is assuming it as given. The position and magnitude of circle *EZHK* is then given in relation to *Q* by the fact that the two ratios $MT : TQ$ and $MQ : MN$ are both given.[41] Because the fact that the ratio $MQ : MN$ is given involves expressing it as a relation of two values, we should understand the fact that $MT : TQ$ is given in the same way.

It is difficult to reconstruct the meaning or purpose of this argument because we do not have any sources that provide us with examples of the pre-Ptolemaic chord-table trigonometric practice that would have been found in Adrastus' sources.[42] In order to understand this passage, it is necessary to read it in the context of the extant work of mathematicians like Euclid and Ptolemy. It is possible that Theon is trying to construct an argument of his own to the effect that taking the numerical parameters of the model as fixed implies that the geometrical model is given—that is, determinate and knowable. Or, more likely, he may be attempting to familiarize students of philosophy with the language of givens used by mathematicians.

39 See Sidoli (2018, 387–391) and Sidoli (2020) for discussions of arguments by givens in Ptolemy's *Almagest* and *Analemma*.

40 That is, $NQ : XQ = 25 : 23$, although these values are not mentioned by either Theon or Ptolemy.

41 Strictly speaking we would probably say that the relative magnitude of the circle is given, but such expressions are not found in ancient sources.

42 The overall incoherence of this passage is further evidence that Theon of Smyrna cannot have been the man that Ptolemy refers to as "Theon the mathematician." This point has already been argued by Martin (1849, 8–10) and Jones (2015, 2016, 468, n. 11; 76, n. 2).

Conclusion

As this reading of these two texts has confirmed, neither Cleomedes' *On the Heavens* nor Theon's *Mathematics Useful for Reading Plato* are treatises of mathematics, although they contain mathematical topics. Neither of these authors had the inclination, or perhaps the competence, to express himself in the manner adopted by the mathematicians. Nevertheless, as treatises about mathematical subjects they are valuable to us in giving evidence for mathematical traditions for which we might otherwise have had little evidence.

Cleomedes' discussion of Eratosthenes' mathematical approach makes it clear that the latter continued the traditions of Aristarchus and Archimedes in his investigations of the size of the earth. That is, he started with hypotheses that involved idealizations of observational claims, some of which gave rise to a geometric model and others of which produced numerical starting points, then he applied elementary geometry, ratio manipulations and computations to produce numerical values, and probably bounds, for something beyond the purview of our senses—namely, the size of the earth. This was probably one of the final chapters in the proto-trigonometric work of the early Hellenistic period.

Theon's discussion of Hipparchus' solar model gives us further evidence for the blending of computational procedures and justificatory practices that we find in the work of Heron and Ptolemy, in the Imperial period. The fact that this language is associated with Hipparchus gives us reason to believe that these kinds of arguments were already being made by Hipparchus and the mathematicians who followed him. Hence, we can take this as one of the opening episodes in the development of the chord-table trigonometry of the late Hellenistic period.

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