BEFORE PYTHAGORAS

THE CULTURE OF OLD BABYLONIAN MATHEMATICS

NUMBERS ON CLAY

Cuneiform tablets originating in second millennium BCE Babylonian scribal schools preserve exercises and calculations recorded by teachers and pupils, ranging from practical arithmetic to problems well beyond everyday applications. *Before Pythagoras: The Culture of Old Babylonian Mathematics* presents an unprecedented grouping of tablets from the first golden age of mathematics, highlighting both classroom training and advanced curiosity-driven mathematics.

Numbers on Clay was written by Alexander Jones, Professor of the History of the Exact Sciences in Antiquity (ISAW).

The exhibition *Before Pythagoras: The Culture of Old Babylonian Mathematics* is curated by Alexander Jones and Christine Proust and is on view at ISAW from November 12th–December 17th, 2010.

Line drawings by Alexander Jones.

Front and back cover: Old Babylonian "hand tablet" illustrating Pythagoras' Theorem and an approximation of the square root of two. Clay, 19th–17th century BCE, Yale Babylonian Collection YBC 7289. Photo by West Semitic Research.

TABLETS

Clay tablets were the principal writing medium in Mesopotamia. Tablets were usually shaped as rectangles of a size convenient for the information that they were intended to hold, though small "hand-tablets" used for practice writing and calculations sometimes had a round, lentil-like shape. One face (the *obverse*) was normally flat, the other (the *reverse*) slightly convex. Each face might be divided into columns. A tablet was usually flipped vertically, with the bottom of the obverse serving as the top of the reverse. Old Babylonian texts were written from left to right; but in a tablet with multiple columns, the first column of the reverse was sometimes at the far right.

CUNEIFORM WRITING AND NUMERALS

The writing instrument was a stylus, normally a reed stalk with a bevelled end. The angle at which the stylus was pressed in the clay resulted in either nail-shaped or chevron-shaped impressions that gave the style of writing its name, *cuneiform* (from the Latin for "wedge shaped"). The systems of cuneiform writing that evolved in the late fourth and third millennia BCE for Sumerian and Akkadian, the two languages of Mesopotamia, were extremely complex, with hundreds of distinct signs. The basic signs for numbers, however, were extremely simple. They were made by repeating two signs, 𒁹 meaning 1, and 𒌋 meaning 10 as many times as needed to make up the desired number. These are all the signs needed to write numbers from 1 to 59:

1	𒁹
2	𒈫
3	𒐈
4	𒐉
5	𒐊
6	𒐋
7	𒐌
8	𒐍
9	𒐎

10	𒌋
20	𒎙
30	𒌍
40	𒐏
50	𒐐

In writing numbers above 10, the sign for the tens was put to the left of the sign for units; for example, 47 is written 𒐏𒐌.

A PLACE-VALUE SYSTEM

Near the end of the third millennium BCE, scribes developed a way of writing numbers that was very convenient for calculations. Multiplication and division were particularly cumbersome when the quantities were expressed in mixed units of measure, like yards, feet, and inches as used today. In the new system only the symbols representing the numbers 1 through 59 (𒁹 through 𒐐𒐎) were used, but any of these symbols could have any of a range of different values from very large to very small, according to a scale based on multiplying or dividing by 60.

Example: 𒐈 could mean 3
or 180 (3x60)
or 10800 (3x60x60)
or $^{1}/_{20}$ (3÷60)
or $^{1}/_{1200}$ (3÷60÷60)

Any quantity could be broken up into a series of such parts, written from left to right in descending order of size. For example $214^{1}/_{4}$ would be broken up into 3 sixties, plus 34, plus 15 sixtieths, and written 𒐈 𒌍𒐉 𒌋𒐊. When we translate this kind of numeral, we write commas between the parts for clarity—"3,34,15" in this example.

As with present-day Hindu-Arabic numerals, these Babylonian numerals operate within a *place-value* system, meaning that the value of a symbol can be larger or smaller depending on its place in the whole numeral. The system is called *sexagesimal* (from the Latin for "sixty") because it is based on counting in groups of 60—in contrast to the Hindu-Arabic system, which is called *decimal* because it is based on groups of 10.

There is another important difference between our decimal numerals and the Babylonian sexagesimal numerals. In the decimal system the last digit of a number, or the last digit before the decimal point, always means units. If necessary we use the symbol "0" to occupy empty places so that the value of every digit is unambiguous. There was no cuneiform sign functioning as a zero until long after the Old Babylonian Period, and there was never a symbol similar to a decimal point to separate a whole number from a fraction. The order of magnitude of a number written in the sexagesimal notation was thus always ambiguous.

In Old Babylonian mathematics, the sexagesimal notation was used for intermediate calculations, or in abstract calculations as in Plimpton 322 (**13** in this exhibition). More than a thousand years later, it provided the number system for Babylonian astronomy, and by a long course of transmission it has come down to us in the form of our division of hours (and angular degrees) into 60 minutes and 3600 seconds.

Several of the tablets in this exhibition consist mostly of numbers, and they can easily be read without any knowledge of the Akkadian or Sumerian language. A good tablet with which to begin reading cuneiform numerals is the multiplication table B6063 (**6**).

Units of measure used in the tablets.

Units of length: $^1/_{60}$ UŠ = 1 ninda = 12 cubits = 360 fingers
Units of area: $^1/_{600}$ eše = 1 SAR = 60 gin = 10800 še = 144 square cubits
Units of volume: $^1/_{100}$ iku = 1 SAR = 60 gin = 10800 še = 144 cubic cubits
Units of weight (including silver): 1 ma-na = 60 gin = 10800 še

TRANSLATIONS FROM SELECTED TABLETS

For tablets in which the layout is important, we provide a line-drawing of the tablet with the texts translated into English and into modern numerals, but preserving the sexagesimal notation. In the notes and the object labels we have made some arbitrary assumptions about which place in a series of sexagesimal numerals represents units. Some expressions in the texts have been paraphrased or modernized for clarity.

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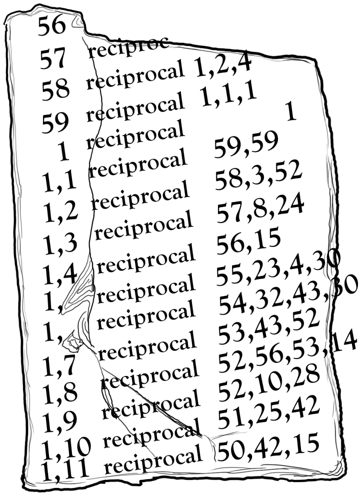
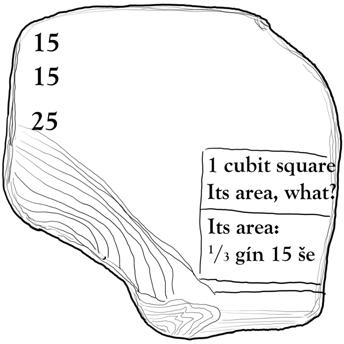
1. Yale Babylonian Collection YBC 10529, obverse.



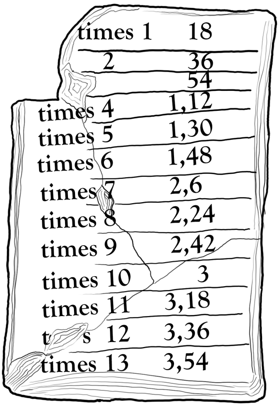
The left-hand column lists whole numbers from 56 through 71. The right-hand column gives each number's reciprocal, sometimes wrapping around the edge of the tablet. Down to the reciprocal of 60 in the fifth line, the first numeral should be interpreted as sixtieths; in the remainder of the table, the first numeral is a sixtieth of a sixtieth. Only the reciprocals of 60 and 64 are exact. There are several errors; for example, the approximate reciprocal of 63 should be 57,8,34.

3. University of Pennsylvania Museum B11318, obverse.

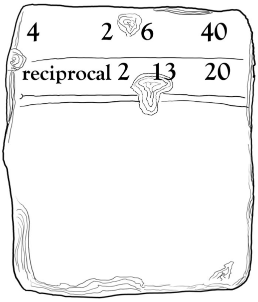


The problem and its solution are written in the lower right corner, and the intermediate calculations in the upper left corner. The problem would have been trivial if the student had been allowed to give the answer in square cubits; but he was required to express the result in the area units gin and še. The student first converted the given length into the larger unit ninda, then used sexagesimal arithmetic to find the area in SAR, and finally converted this result to gin and še. (The two 15s in the upper left are each mistakes for 5.)

6. University of Pennsylvania B6063, obverse.

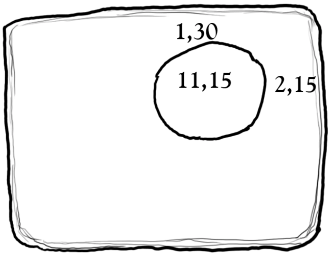


7. University of Pennsylvania 55-21-357, obverse.



The reciprocal of 4,26,40 (i.e., 2/27) is actually 13,30 (i.e., 27/2). One way to find the reciprocal would be to divide 4,26,40 by two, yielding 2,13,30, which is listed in the standard reciprocal tables as the reciprocal of 27. Taking half of 27 gives the correct answer. The student only carried out the first step, but the double ruled line and the label "reciprocal" suggest that he thought he had solved the problem.

8. Yale Babylonian Collection YBC 11120, obverse.



9. Yale Babylonian Collection YBC 7164, selected problems.

(Problem 1) A little canal. Its length is 5 UŠ, its width is 3 cubits, its depth is 3 cubits. A worker's daily load of earth is 10 gin. A worker's daily wages are 6 še of silver. What is the canal's surface area, its volume, the number of workers needed to dig it, and the total cost in silver? Answer: The area is 75 SAR, the volume is 2 iku and 25 SAR, the number of workers is 1290, and the total cost is 2/3 ma-na and 5 gin of silver.

(Problem 2) A little canal. Its length is 5 UŠ, its width is 3 cubits, its depth is 3 cubits. For the first cubit of depth, a worker's daily load of earth is 1/3 SAR; for the depth of the next two cubits, a worker's daily load is 10 gin. What length of canal did one man dig per day? Answer: 3 cubits 6 fingers.

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10. Yale Babylonian Collection YBC 4663, selected problems.
Italics indicate additions made to the translation for clarity.

(Obverse, Problem 3) The total cost in silver of digging a trench is 9 gin. Its length is 5 ninda, and its depth is $^1/_2$ ninda. A worker's daily load of earth is 10 gin, and a worker's daily wages are 6 še of silver. What is the canal's width? Solution: Multiply the length and the depth, and you will get 30. Take the reciprocal of the workload, multiply by 30, and you will get 3. Multiply the wages by 3, and you will get 6. Take the reciprocal of 6, and multiply it by 9, the total cost in silver, and you will get its width. $1^1/_2$ ninda is the width. Such is the procedure.

(Reverse, Problem 8) The total cost in silver of digging a trench is 9 gin. The length exceeded the width by 3,30 *(i.e., 3 $^1/_2$)* ninda. Its depth is $^1/_2$ ninda. A worker's daily load of earth is 10 gin, and a worker's daily wages are 6 še of silver. What are the length and the width? Solution: Take the reciprocal of the wages, and multiply by 9, the total cost in silver, and you will get 4,30. Multiply 4,30 by the workload, and you will get 45. Take the reciprocal of $^1/_2$ ninda, and multiply by 45, and you will get 7,30. Take half of the amount by which the length exceeded the width, and you will get 1,45. Make the square of 1,45, and you will get 3,3,45. Add 7,30 to 3,3,45, and you will get 10,33,45. Take its square root, and you will get 3,15. Operate with 3,15 in two ways: add 1,45 to the one, and subtract 1,45 from the other, and you will get the length and the width. 5 ninda is the length, and $1^1/_2$ ninda is the width. Such is the procedure.

11. Yale Babylonian Collection YBC 4713, selected problems.

Italics indicate additions made to the translation for clarity.

(Problem 2) The area equals 1 eše *(i.e., 10,0)*. I multiplied the length by a certain number, and got 2,30. I multiplied the width by a certain number, and got 1,20. The number I multiplied by the length exceeds by 1 the number I multiplied by the width. What are the length and width?

(Problem 3) *Instead of the last condition in Problem 2*: Half the number I multiplied by the length plus 1,30 equals the number I multiplied by the width.

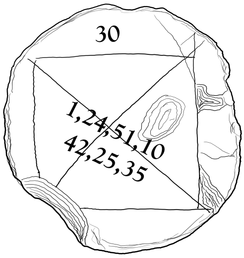
(Problem 4) *Instead of the last condition in Problem 2*: Two-thirds of the number I multiplied by the length plus 40 equals the number I multiplied by the width.

(Problem 5) *Instead of the last condition in Problem 2*: I added one-third of the amount by which the number I multiplied by the length exceeds the number I multiplied by the width, plus the number I multiplied by the length, *and I got* 5,20.

(Problem 6) *In the last condition of Problem 5, instead of adding one-third of the excess of the two numbers,* I multiplied *one-third of the excess* by 2, and I added *the number I multiplied by the length, and I got* 5,40.

(Problem 7) *In the last condition of Problem 5, instead of adding the number I multiplied by the length,* I subtracted the number I multiplied by the length, *and I got* 4,40.

12. Yale Babylonian Collection YBC 7289, obverse.



The approximate decimal equivalent of 1,24,51,10 is 1.41421296 . . .; the actual value of the square root of 2 is 1.41421356. . . .

13. Columbia University Plimpton 322, obverse.

quare of the diagonal torn out and width results	square width	square diagonal	item
,15	1,59	2,49	place 1
,58,14,56,15	56,7	3,12,1	place 2
,41,15,33,45	1,16,41	1,50,49	place 3
,53,10,29,32,52,16	3,31,49	5,9,1	place 4
1,48,54,1,40	1,5	1,37	place
1,47,6,41,40	5,19	8,1	
1,43,11,56,28,26,40	38,11	59,1	place 7
1,41,33,59,3,45	13,19	20,49	place 8
1,38,33,36,36	9,1	12,49	place 9
,35,10,2,28,27,24,26,40	1,22,41	2,16,1	place 10
,33,45	45	1,15	place 11
1,29,21,54,2,15	27,59	48,49	place 12
1,27 ,3,45	7,12,1	4,49	place 13
1,25,48,51,35,6,40	29,31	53,49	place 14
1,23,13,46,40	56	53	place

The column headed square width *gives the length of the shortest leg of a right triangle (or the width of a rectangle); the column headed* square diagonal *gives the hypotenuse of the triangle (or the diagonal of the rectangle). The length of the other leg (or the rectangle's length) was not given on the tablet, but it can be calculated by Pythagoras' Theorem as the square root of the difference of the squares of the other two lengths. For example, in the eleventh row $75^2 - 45^2 = 60^2$, so that the triangle has sides 45, 60, and 75 and is effectively the well-known 3-4-5 Pythagorean triangle. There are six errors in the numbers on the tablet.*